$\boxed{\text{Groups of area preserving homeomorphisms of surfaces and their simplicity}}$

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$\mathrm{Homeo}_0(S^n, \mathrm{vol})$ = volume preserving homeos of S^n.

Theorem (Fathi, 80) This group is simple for $n \geq 3$

[no non-trivial proper normal subgroup]

What about: $\boxed{\mathcal{G} = \mathrm{Homeo}_0(S^2, \mathrm{area})}$?

Remark: all the cousins of $\mathcal{G}$ are simple!

- $\mathrm{Homeo}_0(S^n)$ (Ulam, Von Neumann)
- $\mathrm{Diff}_0^\infty(S^n)$ (Epstein, Herman, Mather) 70's

- $\mathrm{Diff}_0(S^n, \mathrm{Vol})$, $n \geq 3$ (Thurston 70's)
  $n = 2$ (Banyaga 78)
- $\mathrm{Homeo}_0(S^n, \mathrm{Vol})$, $n \geq 3$ (Fathi 80)

<u>Theorem</u> (CG-H-S, 20) $\boxed{\mathcal{G} \text{ is not simple}}$.

On other surfaces, Fathi constructed the mass flow homomorphism $\rho : \mathrm{Homeo}_0(\Sigma, \text{area}) \longrightarrow H_1(\Sigma)/\Gamma$

<u>Theorem</u> (CG-H-M-S-S, 21) For every compact surface (possibly with boundary), $\ker \rho$ is not simple.

<u>Remark</u>: ideas from the 70's (Epstein, Thurston) yield:

- every normal subgroup of $\mathcal{G}$ contains $[\mathcal{G}, \mathcal{G}]$

  Subgroup generated by commutators

  Thus: $\boxed{[\mathcal{G}, \mathcal{G}] \neq \mathcal{G}}$

- Moreover: $\boxed{[\mathcal{G}, \mathcal{G}] \text{ simple}}$

Remark • The first th. uses Periodic Floer homology (PFH), due to Hutchings

• The second th. uses another variant of Floer theory (constructions uses ideas from Heegaard Floer homology of Ozsváth - Szabó)

$\rightsquigarrow$ less sophisticated.

Commutator length and quasimorphisms

Let G perfect group $([G,G]=G)$

Commutator length: $g \in G$

$$\boxed{cl(g) = \inf \{ k \mid g \text{ is a product of } k \text{ commutators} \}}$$

(Important object in geometric group theory)

Quasimorphisms

$\mu : G \longrightarrow \mathbb{R}$ is called a quasimorphism if $\exists D \geq 0$, $\forall f, g \in G$, $\boxed{|\mu(fg) - \mu(f) - \mu(g)| \leq D}$

μ is homogeneous if $\mu(f^n) = n\mu(f)$.

Remark: if μ q.m, then $\tilde{\mu}(f) = \lim_{k \to \infty} \frac{1}{k} \mu(f^k)$ is a h.q.m

Exercise: Let μ be a h.q.m.

Then $\quad cl(g) \geq \frac{1}{D}\mu(g)$

Examples:

- Many examples on finite type groups (e.g. Brooks quasi-morphisms)
- Poincaré rotation number on $\widetilde{\mathrm{Homeo}}_0(S^1)$
- Entov-Polterovich (2003) on $\mathrm{Diff}_0(S^2, \text{area})$
- Gambaudo-Ghys (2003) on $\mathrm{Homeo}(\Sigma, \text{area})$ for $\Sigma \neq S^2$

Th (Tsuboi 2006) $cl \leq 1$ on $\mathrm{Homeo}_0(S^2)$

$\rightsquigarrow$ No quasimorphism!

$$\boxed{\mathcal{G} = \mathrm{Homeo}_0(S^2, \text{area}) \quad ?}$$

Theorem (G-H-M-S-S, 21) The space of h.q.m on $\mathcal{G}$ is ∞-dimensional.

Corollary: cl is not bounded on $[\mathcal{G}, \mathcal{G}]$.

in fact we show: for every C^0-neighborhood of id, scl is unbounded.

scl = stable commutator length

(not true for $\Sigma_{g\geq 1}$ by Bowden-Hensel-Webb 2019)

Ideas for the proof of non-simplicity

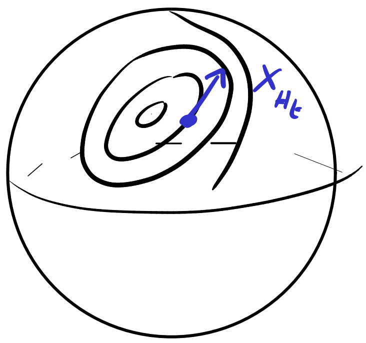
Fact 1 Area preserving diffeos of S^2 are Hamiltonian

i.e constructed via the recipe:

$H = (H_t)_{t\in[0,1]} : S^2 \longrightarrow \mathbb{R}$ Hamiltonian

$\rightsquigarrow X_{H_t} = \mathrm{Rot}_{\frac{\pi}{2}} \nabla H_t$ Hamiltonian vector field

$\rightsquigarrow$ flow ϕ_H^t



$$\boxed{\mathrm{Diff}(S^2, \text{area}) = \mathrm{Ham}(S^2)}$$

$$\| \quad \{ \phi_H^1 \mid H : [0,1]\times S^2 \longrightarrow \mathbb{R} \}$$

Fact 2: $\mathrm{Homeo}_0(S^2, \text{area}) = C^0$-closure of $\mathrm{Ham}(S^2)$

Finite energy homeomorphisms

Def: $\psi \in G$ is a finite energy homeomorphism if $\exists H_i$ such that:

- $\phi_{H_i}^1 \xrightarrow[i\to\infty]{C^0} \psi$
- $\|H_i\| = \int_0^1 (\max H_{i,t} - \min H_{i,t})\, dt$ bounded

"Hofer energy of H_i"

Denote $\mathcal{F} = \{ \varphi \text{ finite energy} \}$

(This def is a variant of a def of Oh-Müller 2006)

Prop: $\mathcal{F} \triangleleft \mathcal{G}$

Proof: If $\varphi = \lim \phi^1_{H_i}$ and $\varphi' = \lim \phi^1_{H_i'}$

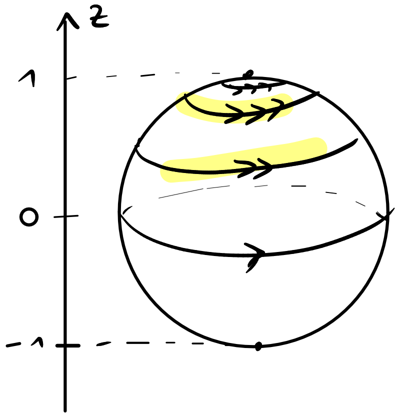
then $\varphi \circ \varphi' = \lim \phi^1_{H_i} \circ \phi^1_{H_i'}$

$$= \lim \phi^1_{H_i \# H_i'}$$

where $H_i \# H_i' := H_i(t,x) + H_i'\left(t, (\phi^t_{H_i})^{-1}(x)\right)$ $\square$

$\neq$?

Infinite twists



S^2, $\omega = \frac{1}{4\pi} d\theta \wedge dz$

$H(\theta, z) = \frac{1}{2} h(z)$, $h: [-1,1) \longrightarrow \mathbb{R}$,

$h(z) = 0$ for $z \leq 0$.

$\tau := \phi^1_H(\theta, z) = (\theta + h'(z), z)$

extends to an element of $\mathcal{G}$

Lemma If $\int_{-1}^1 h = \infty$, then $\tau \notin \mathcal{F}$

"infinite twists"

Tool: A Lagrangian Floer homology on symmetric products C-G-H-M-S-S

inspired by previous works by Osváth - Szabó and Polterovich - Shelukhin

Floer homology

Let (M, ω) symplectic manifold

Let $L \subset M$ Lagrangian submanifold

$(\dim L = \frac{1}{2} \dim M, \ \omega|_L = 0)$

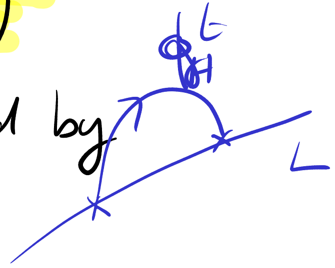
Assume L is connected and monotone

$(\exists \lambda, \forall u \in \pi_2(M, L), \omega(u) = \lambda \mu(u))$

Let H Hamiltonian on M s.t. $\phi_H^1(L) \pitchfork L$.

$\rightsquigarrow$ Floer homology $HF(H; L)$

(chain complex formally generated by orbits of H from L to L)



Symmetric Product

Let $d \geq 1$

$$\boxed{\mathrm{Sym}^d S^2 = S^2 \times \dots \times S^2 / \mathfrak{S}_d}$$

symmetric group

Fact: $\mathrm{Sym}^d S^2 \simeq \mathbb{CP}^d$

homeomorphism
smooth in the complement
of $\Delta = \{(z_1, \dots, z_d) \mid \underset{i \neq j}{z_i = z_j}\}$

- ω area form on S^2

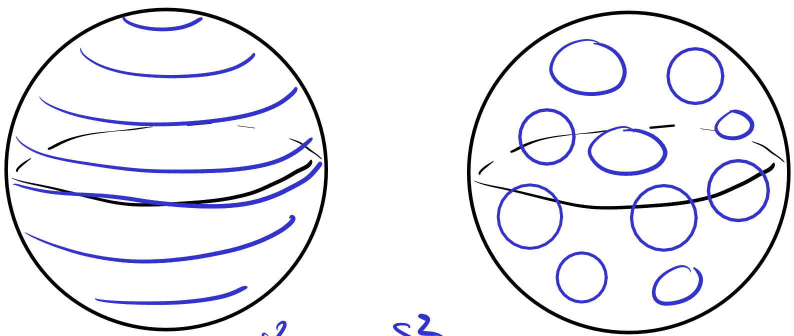
  $\omega \oplus \dots \oplus \omega$ symplectic form on $S^2 \times \dots \times S^2$
  induces a form $\mathrm{Sym}^d \omega$ in $\mathrm{Sym}^d S^2$ (singular along Δ)

- H Hamiltonian on S^2

  $H_t(x_1) + \dots + H_t(x_d)$ induces $\mathrm{Sym}^d H$ on $\mathrm{Sym}^d S^2$
  (singular along Δ)

- $\underline{L} = L_1 \cup \dots \cup L_d$ disjoint simple curves in S^2

<u>ex:</u>



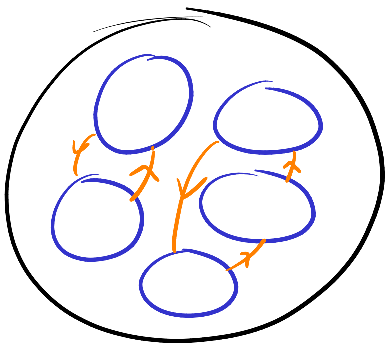
$L_1 \times \dots \times L_d \subset S^2 \times \dots \times S^2$ induces smooth Lagrangian
torus $\mathrm{Sym}\,\underline{L}$ in $(\mathbb{CP}^d, \mathrm{Sym}^d \omega)$

For every neighborhood V of Δ, one can smooth
out $\mathrm{Sym}^d \omega$ in V (Perutz) $\rightsquigarrow \omega_V$
Also smooth $\mathrm{Sym}^d H$ out $\rightsquigarrow H_V$

Theorem Assume the connected components of $S^2 \setminus \underline{L}$ all have the same area $\frac{1}{d+1}$. Then $\mathrm{Sym}\,\underline{L}$ is monotone in $(\mathbb{CP}^d, \omega_V)$.
Moreover $HF(H_V, \mathrm{Sym}\,\underline{L}, \omega_V)$ is well-defined non trivial and does not depend on the smoothings

Denote it $\boxed{HF(H, \underline{L})}$

Generators are collections of Hamiltonian chords



By a min-max idea we extract "spectral invariants"
(Viterbo, Oh, Leclercq - Zapolsky)

$\rightsquigarrow \ell(H, \underline{L}) \in \mathbb{R}$

Prop (i) $\quad |\ell(H, \underline{L}) - \ell(H, \underline{L}')| \leq \|H\|$
(Hofer energy)

(ii) $\displaystyle \frac{1}{d} \sum_{i=1}^{d} \int_0^1 \min_{L_i} H_t \, dt \leq \ell(H, \underline{L}) \leq \frac{1}{d} \sum_{i=1}^{d} \int_0^1 \max_{L_i} H_t \, dt$

(in particular, if $H = 0$ on $\underline{L}$, then $\ell(H, \underline{L}) = 0$)

Let $\eta^{(H)}_{\underline{L}, \underline{L}'} := \ell(H, \underline{L}) - \ell(H, \underline{L}')$

Key property

Prop $\eta_{\underline{L},\underline{L}'}(H)$ depends only on ϕ_H^1 and $\eta_{\underline{L},\underline{L}'} : Ham(S^2) \longrightarrow \mathbb{R}$ is C^0-continuous and extends to $\mathcal{G}$.

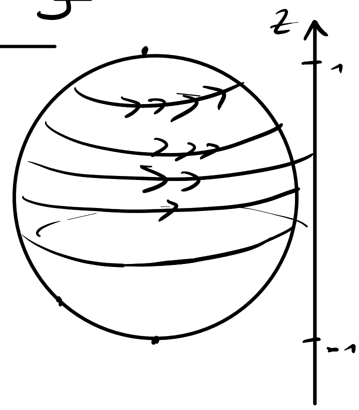
By (i), $\boxed{\text{if } \phi \in \mathcal{F}, \text{ then } |\eta_{\underline{L},\underline{L}'}(\phi)| < \text{Cste}}$ independently of $\underline{L}, \underline{L}'$

Proof that the infinite twist $\tau \notin \mathcal{F}$

Recall: $\tau(\theta, z) = (\theta + h'(z), z)$

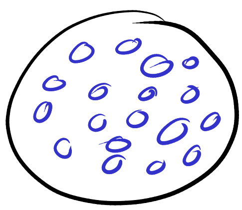
generated by $H = \frac{1}{2} h(z)$

with $\int_{-1}^{1} h = \infty$



We can write $\tau = \lim_{C^0} \phi^1_{H_i}$ where $H_i = \frac{1}{2} h_i(z)$, smooth

$\int H_i \to \infty$ and $h_i(z) = 0$ for $z \leq 0$.

Take $\underline{L} =$  $\underline{L}' =$ 

Then: $\eta_{\underline{L},\underline{L}'}(\tau) \underset{C^0 \text{ continuity}}{\approx} \eta_{\underline{L},\underline{L}'}(\phi^1_{H_i}) \underset{(ii)}{=} \ell(H_i, \underline{L}) \underset{(ii)}{\approx} \int H_i \approx \infty$

Thus $\tau \notin \mathcal{F}$ $\square$

Constructing quasimorphisms

Let $\underline{L}, \underline{L}'$ be configurations with $d \neq d'$ comp.

Define $\boxed{\mu_{d,d'}(\phi) = \lim_{k \to \infty} \frac{1}{k} \eta_{\underline{L},\underline{L}'}(\phi^k)}$

Theorem (CG-H-M-SS) $\mu_{d,d'}$ is a quasimorphism on $Ham(S^2)$. It does not depend on $\underline{L}, \underline{L}'$. It is C^0-continuous and extends to $\mathcal{G}$.

Idea: Recall:

$$\eta_{\underline{L},\underline{L}'}(\phi_H^1) = \ell(H, \underline{L}) - \ell(H, \underline{L}')$$

$$\ell(H, \underline{L}) = \ell(\mathrm{Sym}^d H, \mathrm{Sym}\, \underline{L})$$

$\subseteq \mathbb{C}P^d$

General fact: lag spectral invariants $\leq$ Ham. spectral invariants

$$\text{so } \ell(H, \underline{L}) \leq c(\mathrm{Sym}^d H)$$

Fact (Entov Polterovich): in $\mathbb{C}P^d$, c is a quasimorphism.

$\rightsquigarrow$ allows to conclude that the homogenization of ℓ is a q.m

In fact: "$\boxed{\mu_{d,d'}(\phi) = \mu_{EP}(\mathrm{Sym}^d \phi) - \mu_{EP}(\mathrm{Sym}^{d'} \phi)}$"

Obrigado!