Classical Bulk-Boundary Correspondences via Factorization Algebras

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TQFT Club Seminar

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Goal & Overview

Goal: To use the language of factorization algebras to generalize the insight that led Kontsevich to use two-dim$^{\underline{l}}$ field theory to quantize Poisson mflds.

Outline:
I] Deformation Quantization

II] Factorization Algebras

III] Shifted Poisson structures for FAs

IV] Shifted Poisson Structures and Variants of the Batalin-Vilkovisky Formalism

V] Main Theorem

Deformation Quantization, I

Let $(A, \cdot, \{,\})$ be a Poisson algebra.

comm. prod.

Poisson bracket

A <u>deformation quantization</u> of A is an assoc., $\hbar$-linear algebra structure

$$\star : A[[\hbar]] \otimes A[[\hbar]] \longrightarrow A[[\hbar]]$$

s.t. $f, g \in A$

1) $f \star g = f \cdot g + \hbar(\ldots)$

2) $f \star g - g \star f = \hbar\{f, g\} + \hbar^2(\ldots)$

Deformation Quantization, II

Ex: $A = \mathrm{Sym}(\mathfrak{g})$ (Lie alg.)

- comm. prod. is sym alg. prod.
- Poisson bracket is, on generators of A, Lie bracket of $\mathfrak{g}$.

A = functions on $\mathfrak{g}^*$ ← Poisson mfld.

Set: $U_\hbar(\mathfrak{g}) = F(\mathfrak{g} \oplus \mathbb{R}\langle\hbar\rangle) \big/ \big/ \langle \hbar \text{ central}, \; x \otimes y - y \otimes x = \hbar [x, y] \rangle$

By PBW Thm, $U_\hbar(\mathfrak{g}) \cong A[[\hbar]]$, and alg. str. on $U_\hbar(\mathfrak{g})$ induces a def. quant. of A.

Thm [Kontsevich]: If $A = C^\infty(M)$ (Poisson mfld), then $\exists$ def. quant. of A.

Cattaneo-Felder: The product on $A[[\hbar]]$ arises from computation of correlation functions in a $\boxed{2D}$ TQFT.

Factorization Algebras

Defn: Let M be a top^l space. A factorization algebra on M is

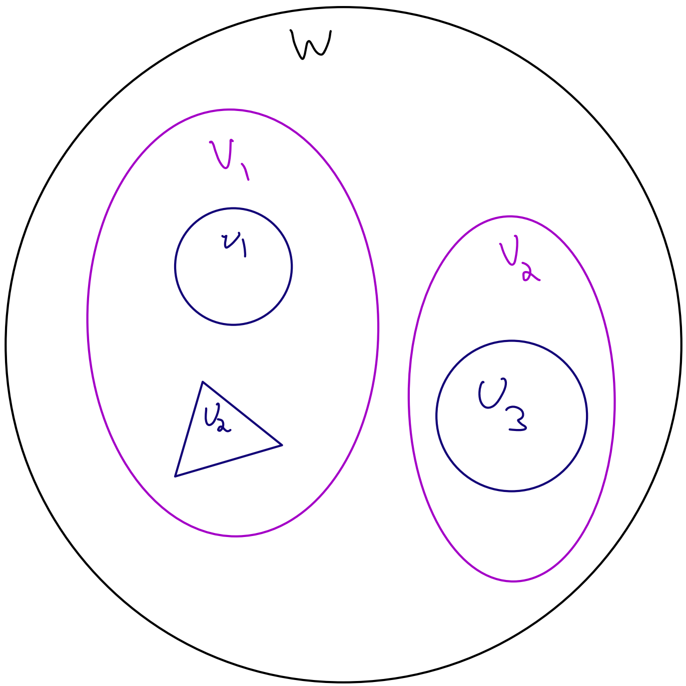
1) For each open subset $U \subseteq M$, a cochain complex $\mathcal{F}(U)$

2) For each collection $(U_1, \ldots, U_k ; V)$ of open subsets s.t. U_i are pairwise disj. subsets of V

$$m^V_{U_1, \ldots, U_k} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(V)$$

s.t.
- a nat^l assoc. cond.
- ...

A Picture of Associativity



$$
\begin{tikzcd}
 & \mathcal{F}(V_1)\otimes\mathcal{F}(V_2) \arrow[dr, "m^{W}_{V_1,V_2}"] \arrow[d, phantom, "\circlearrowright"] & \\
\mathcal{F}(U_1)\otimes\mathcal{F}(U_2)\otimes\mathcal{F}(U_3) \arrow[ur, "m^{V_1}_{U_1,U_2}\otimes m^{V_2}_{U_3}"] \arrow[rr, "m^{W}_{U_1,\dots,U_3}"] & & \mathcal{F}(W)
\end{tikzcd}
$$

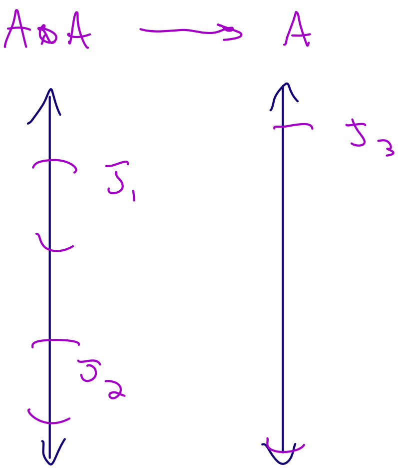
An Instructive Example

Let $A = \mathrm{Sym}(a_y)$ (more generally, A a unital, assoc. algebra)

There is a FA $\mathcal{F}_{a_y,1}$ on $\mathbb{R}$, defined as follows

Let $\mathbb{R} \supseteq U = \bigsqcup_{i \in I} J_i$ interval , $\mathcal{F}_{a_y,1}(U) = \bigotimes_{i \in I} A$

Structure maps



0) $m^{J}_{\emptyset} : \mathbb{R} \to A$ is unit map

1) $m^{J_2}_{J_1} : A \to A$ is identity

2) $m^{J_3}_{J_1, J_2} : A \otimes A \to A$ is mult.

Shifted Poisson Structures on Factorization Algebras

Question: How does one encode Poisson structure on $\mathrm{Sym}(\mathfrak{g})$ in language of FAs?

Problem: The mult. map $\mu : A \otimes A \to A$ is not a map of Poisson algebras, so the structure maps of $\mathcal{F}_{\mathfrak{g},1}$ are not compatible w/ Poisson str. on A.

Sol^n: Replace $\mathcal{F}_{\mathfrak{g},1}$ by $\tilde{\mathcal{F}}_{\mathfrak{g},1}(U) := \mathrm{Sym}\left(\Omega^\bullet_c(U) \otimes \mathfrak{g}[1]\right)$

$$m^V_{U_1,\dots,U_k} : \tilde{\mathcal{F}}_{\mathfrak{g},1}(U_1) \otimes \cdots \otimes \tilde{\mathcal{F}}_{\mathfrak{g},1}(U_k) \xrightarrow{\text{ext. by } 0} \tilde{\mathcal{F}}_{\mathfrak{g},1}(V)^{\otimes k} \xrightarrow{\text{mult. in Sym. alg.}} \tilde{\mathcal{F}}_{\mathfrak{g},1}(V)$$

Properties of $\tilde{\mathcal{F}}_{g,1}$

1) There is a q.i. of FAs

$$\tilde{\tilde{\mathcal{F}}}_{g,1} \to \tilde{\mathcal{F}}_{g,1}$$

2) $\tilde{\tilde{\mathcal{F}}}_{g,1}$ has a Poisson bracket of cohom. deg $\underline{+1}$ which $\underline{\text{is}}$ compatible with $\check{m}_{v_1,\ldots,v_k}$ ($\hat{\mathcal{F}}_{g,1}$ is a $\mathbb{P}_0$-FA)

Generalizations of $\tilde{\mathcal{F}}_{\mathfrak{g},1}$

a) Replace $\mathbb{R}$ by $\mathbb{R}^n$ $\rightsquigarrow$ $\tilde{\mathcal{F}}_{\mathfrak{g},n}$ a $\mathbb{P}_0$-FA on $\mathbb{R}^n$

b) Replace $\mathfrak{g}^*$ by V, a general formal Poisson manifold $\rightsquigarrow$ $\tilde{\mathcal{F}}_V$ a $\mathbb{P}_0$-FA on $\mathbb{R}$

c) If $\mathfrak{g}$ has an inv$^{\underline{t}}$ pairing K, then can define a $\mathbb{P}_0$-FA $\tilde{\mathcal{F}}_{\mathfrak{g},\mathbb{C},K}$ on $\mathbb{C}$.

$\mathbb{P}_n$-algebra: Poisson alg. where bracket has deg. $1-n$

$H_\bullet(E_n) = \mathbb{P}_n \quad n \geq 2$

Abstract Theory

Thm [Lurie, Safronov]

Classical Field-Theoretic Origins for $\mathbb{P}_0$-FAs

Note: $\mathbb{P}_0$-structures arise on FAs of observables of classical field theories in BV formalism & its variants.

Poisson BV Theories on M

BV Theories on M

Bulk-Boundary Systems on M

$\partial M \neq \phi$, possibly

All examples we've discussed

$\partial M = \phi$

Butson-Yoo

Costello-Gwilliam

$\leftarrow$ Quantum Versions $\rightarrow$

[R]

$\mathbb{P}_0$-FA Obs^{cl} on M

Family Tree: Variants of the (perturbative) BV Formalism

Variant	Space of Fields	EOM + sym. encoded in ho-str.?	Action Fcnl?	EOM derived from Action?
"Vanilla"	dg-mfld	✓	✓	✓
Poisson BV	dg-mfld	✓	✗	✗
Bulk-boundary systems	dg-mfld	✓	✓	✓, once bdy cond. imposed

There is a "functor"

$$\{\text{PBV theories on } N^{4}\} \xrightarrow[\text{"Universal"}]{BB} \{\text{BBS on } N \times \mathbb{R}_{\geq 0}\}$$

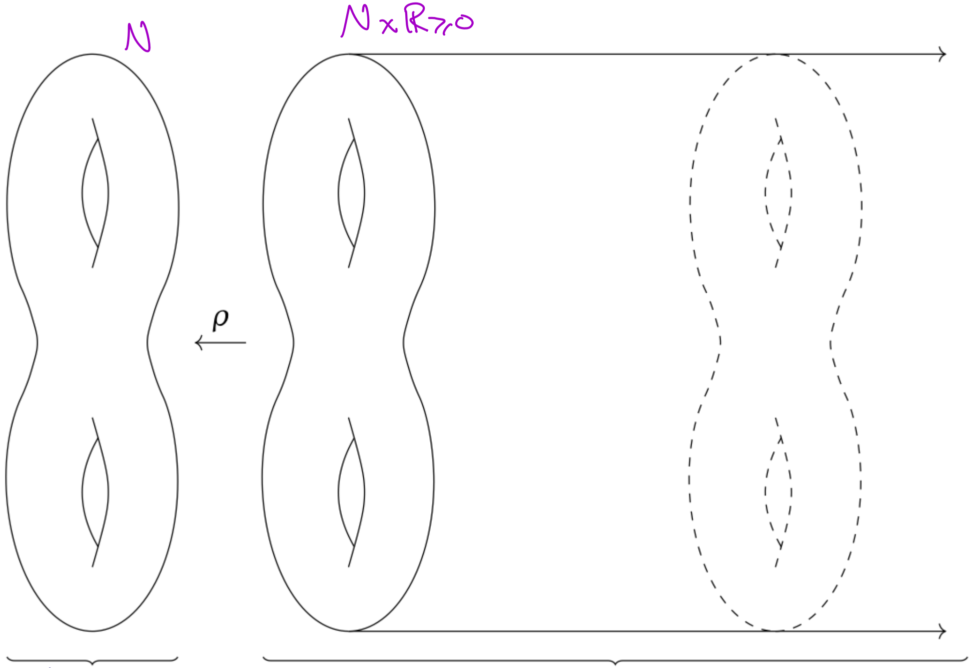
$$\mathcal{T} \longmapsto Z(\mathcal{T})$$

$$\tilde{\mathcal{T}}_{\mathfrak{g},n} \longmapsto \text{BF thy on } \mathbb{H}^{n+1}$$

$$\tilde{\mathcal{T}}_{V} \longmapsto \text{Poisson sigma model on } \mathbb{H}^{2}$$

$$\tilde{\mathcal{T}}_{\mathfrak{g},c} \longmapsto \text{CS/chiral WZW on } \mathbb{H}^{3}$$

Main Theorem



$\mathrm{Obs}^{cl}_{\mathcal{J}}, \ \rho_* \mathrm{Obs}^{cl}_{\mathcal{Z}(\mathcal{J})}$ $\qquad\qquad$ $\mathrm{Obs}^{cl}_{\mathcal{Z}(\mathcal{J})}$

<u>Thm [R]</u> ("Moral" Version): There is an equivalence of $\mathbb{P}_0$-FAs on N

$$\boxed{\rho_* \mathrm{Obs}^{cl}_{\mathcal{Z}(\mathcal{J})} \longrightarrow \mathrm{Obs}^{cl}_{\mathcal{J}}}$$

Deformation Quantization of PBV Theories

Theorem suggests the following for quantizing PBV theories:

What's "Universal" About the Universal Bulk-Boundary System?

Thm [Thomas]: If A is an E_n-algebra, then $Z(A) = HH^{\bullet}(A)$ is the universal E_{n+1}-algebra appearing in a Swiss-Cheese pair (B, A) over A.

Conj: $\mathrm{Obs}^{cl}_{Z(\mathcal{F})}$ is final amongst P_0-FAs on $N \times \mathbb{R}_{\geq 0}$ $\tilde{\mathcal{F}}$

s.t.

1) $p_* \tilde{\mathcal{F}} \simeq \mathrm{Obs}^{cl}_{\mathcal{F}}$

2) $\tilde{\mathcal{F}}$ is "top'l" along $\mathbb{R}_{\geq 0}$

"Honest" Version of Main Theorem