Controlling regularized conservation laws via Entropy–Entropy flux pairs

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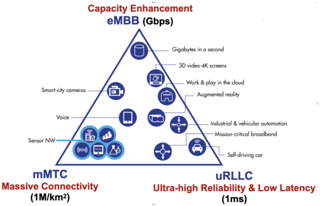
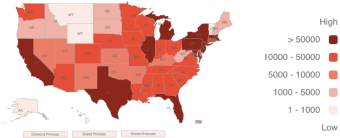
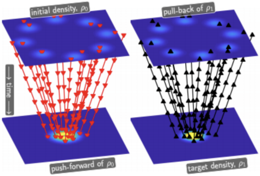
Mean field control



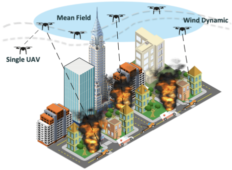
5G communication



COVID 19 pandemic control

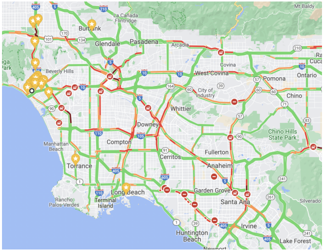


AI

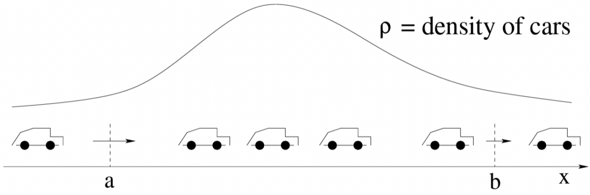


UAV

Traffic in LA



Mean-field PDE models



t= time, $\quad x$= space variable along road, $\quad \rho = \rho(t,x)$ = density of cars

flux: = [number of cars crossing the point x per unit time]

= [density] $\times$ [velocity] $= \rho \cdot v \qquad v = V(\rho)$

$$\rho_t + \big[\rho\, V(\rho)\big]_x = 0$$

Regularized scalar conservation laws

Consider

$$\partial_t u + \nabla \cdot f(u) = \beta \nabla \cdot (A(u) \nabla u),$$

where the known variable is $u = u(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $f \colon \mathbb{R}^d \to \mathbb{R}^d$ is a flux function, $A(u) \in \mathbb{R}^{d \times d}$ is a positive semi-definite matrix function, and $\beta > 0$ is a viscosity constant.

The PDE is known as a regularized scalar conservation law.

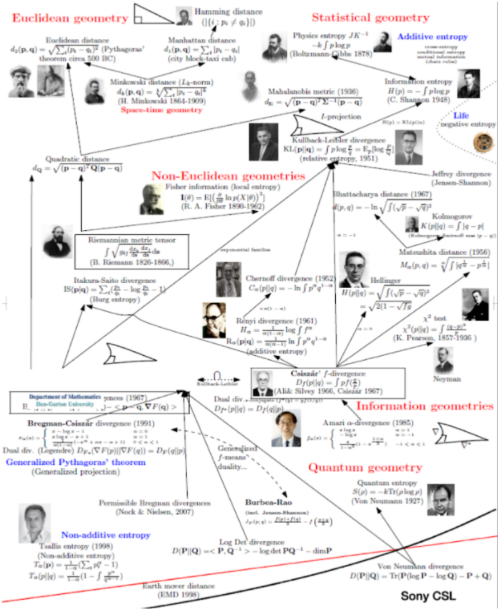
Conservation laws and generalized optimal transport

Based on recent thanksgiving holiday studies 2019–2021, we are working on *conservation laws, generalized optimal transport and mean field control problems*.

As is known in literature, there are actively joint studies to work on entropy, Fisher information, and transportation. Nowadays, these connections have applications in mean field games, information theory, AI, quantum computing, computational graphics, computational physics and Bayesian inverse problems.

In this talk, we connect Lax's entropy-entropy flux in conservation laws with optimal transport type metric spaces. Following this connection, we further design variational discretizations for conservation laws and mean field control of conservation laws. This includes a new and efficient method for developing unconditionally stable in **time implicit approximations** to regularized conservation laws and many other initial value problems.

Entropy, Information, transportation



Heat equation and Entropy dissipation

Consider the heat equation in $\mathbb{R}^d$ by

$$\frac{\partial u(t,x)}{\partial t} = \nabla \cdot (\nabla u(t,x)) = \Delta u(t,x).$$

Consider the negative Boltzmann-Shannon entropy given by

$$\mathcal{H}(u) = -\int_{\mathbb{R}^d} u(x)(\log u(x) - 1)dx.$$

Along the time evolution of heat equation, the following dissipation relation holds:

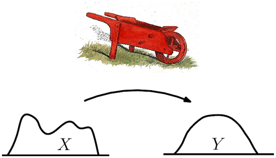
$$\frac{d}{dt}\mathcal{H}(u(t,\cdot)) = \int_{\mathbb{R}^d} \|\nabla_x \log u(t,x)\|^2 u(t,x)dx = \mathcal{I}(u(t,\cdot)),$$

where $\mathcal{I}(u)$ is the Fisher information functional.

There is a formulation behind this relation, namely

- The heat flow is a gradient descent flow of entropy in optimal transport metric.

Optimal transport



What is the optimal way to move or transport the mountain with shape X, density $u^0(x)$ to another shape Y with density $u^1(y)$?

Literature

The optimal transport problem was first introduced by Monge in 1781 and relaxed by Kantorovich (Nobel prize) in 1940. It defines a distance in the space of probability distributions, named optimal transport, Wasserstein distance, or Earth Mover's distance.

- Mapping/Monge-Ampére equation: Gangbo, Brenier, et.al;
- Gradient flows: Otto, Villani, Ambrosio, Gigli, Savare, Carillo, Mielke, et.al;
- Hamiltonian flows: Compressible Euler equations, Potential mean field games, Schrodinger bridge problems, Schrodinger equations: Benamou, Brenier, Lions, Georgiou, Nelson, Lafferty, et.al.
- Numerical OT and MFG: Benamou, Nurbekyan, Oberman, Osher, Achdou, et.al.

We first review the classical optimal transport and its relations with entropy. We next design generalized optimal transport problems and mean field controls for conservation laws with entropy-entropy flux pairs.

Entropy dissipation = Lyapunov methods

The gradient flow of the negative entropy

$$-\mathcal{H}(u) = \int_{\mathbb{R}^d} u(x) \log u(x) dx,$$

w.r.t. optimal transport metric distance satisfies

$$\frac{\partial u}{\partial t} = \nabla \cdot (u \nabla \log u) = \Delta u.$$

Here the major trick is that

$$u \nabla \log u = \nabla u.$$

In this way, one can study the entropy dissipation by

$$-\frac{d}{dt}\mathcal{H}(u) = -\int_{\mathbb{R}^d} \log u \nabla \cdot (u \nabla \log u) dx = \int_{\mathbb{R}^d} \|\nabla \log u\|^2 u dx.$$

Lyapunov method induced calculus

Informally speaking, Wasserstein-Otto metric refers to the following bilinear form:

$$\langle \dot{u}_1, \mathcal{G}(u)\dot{u}_2 \rangle = \int (\dot{u}_1, (-\Delta_u)^{-1}\dot{u}_2)dx.$$

In other words, denote

$$\Delta_u = \nabla \cdot (u\nabla),$$

and $\dot{u}_i = -\Delta_u \phi_i = -\nabla \cdot (u\nabla\phi_i), \quad i = 1, 2$, then

$$\langle \phi_1, \mathcal{G}(u)^{-1}\phi_2 \rangle = \langle \phi_1, -\nabla \cdot (u\nabla)\phi_2 \rangle = \int (\nabla\Phi_1, \nabla\Phi_2)udx,$$

where $u \in \mathcal{P}(\Omega)$, $\dot{u}_i$ is the tangent vector in $\mathcal{P}(\Omega)$ with

$$\int \dot{u}_i dx = 0,$$

and $\phi_i \in C^\infty(\Omega)$ are cotangent vectors in $\mathcal{P}(\Omega)$ at the point u. Here $\nabla\cdot$, ∇ are standard divergence and gradient operators in Ω.

Gradient flows

The Wasserstein gradient flow of an energy functional $\mathcal{F}(u)$ leads to

$$\begin{aligned}\partial_t u =& - \mathcal{G}(u)^{-1}\frac{\delta}{\delta u}\mathcal{F}(u)\\ =& \nabla \cdot (u \nabla \frac{\delta}{\delta u}\mathcal{F}(u)).\end{aligned}$$

- If $\mathcal{F}(u) = \int F(x)u(x)dx$, then

$$\partial_t u = \nabla \cdot (u \nabla F(x)).$$

- If $\mathcal{F}(u) = \int u(x) \log u(x) dx$, then

$$\partial_t u = \nabla \cdot (u \nabla \log u) = \Delta u.$$

Hamiltonian flows

Consider the Lagrangian functional

$$\mathcal{L}(u, \partial_t u) = \frac{1}{2}\int \Big(\partial_t u, (-\nabla\cdot(u\nabla))^{-1}\partial_t u\Big)dx - \mathcal{F}(u).$$

By the Legendre transform,

$$\mathcal{H}(u, \phi) = \sup_{\partial_t u}\int \partial_t u\phi dx - \mathcal{L}(u, \partial_t u).$$

And the Hamiltonian system follows

$$\partial_t u = \frac{\delta}{\delta\phi}\mathcal{H}(u, \phi), \quad \partial_t\phi = -\frac{\delta}{\delta u}\mathcal{H}(u, \phi),$$

where $\frac{\delta}{\delta u}$, $\frac{\delta}{\delta\phi}$ are L^2 first variation operators w.r.t. u, ϕ, respectively and the density Hamiltonian forms

$$\mathcal{H}(u, \phi) = \frac{1}{2}\int \|\nabla\phi\|^2 u dx + \mathcal{F}(u).$$

Here u is the “density” state variable and ϕ is the “density” moment variable.

Examples: Optimal transport and mean field games

Thus, the Hamiltonian flow satisfies

$$\begin{cases} \partial_t u + \nabla \cdot (u \nabla \phi) = 0 \\ \partial_t \phi + \frac{1}{2} \|\nabla \phi\|^2 = -\frac{\delta}{\delta u} \mathcal{F}(u). \end{cases}$$

This is a well known dynamic, which is studied in potential mean field games, optimal transport and PDE.

Optimal transport type formulaisms

We are currently working on the "universe" of generalized optimal transport and mean field games.

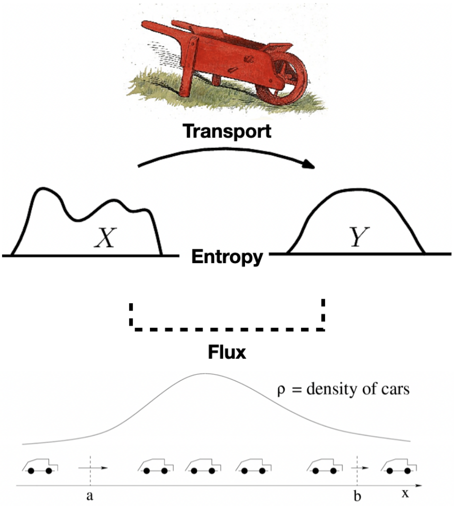
- AI theory, inference, and computations;
- Quantum computing;
- Algorithms of mean-field MCMC algorithms, and numerical PDEs.

In this lecture, we point out that there are optimal transport type formalisms to study the control of conservation laws and to design variational numerical schemes.

Related works

- Godunov (1970);
- Lax–Friedrichs (1971);
- Brenier (2018);
- Osher, Shu, Harten, van Leer: Monotone schemes, TVD schemes, ENO, WENO, et.al.

Main topic: Controlling conservation laws



Entropy-Entropy flux pairs

Consider

$$\partial_t u(t,x) + \nabla_x \cdot f(u) = 0.$$

where

$$u\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d,$$

and

$$f\colon \mathbb{R}^d \to \mathbb{R}^d.$$

Definition (Entropy-entropy flux pair for systems (Lax))

We call (G, Ψ) an entropy-entropy flux pair for the above conservation law system if there exists a convex function $G\colon \mathbb{R}^d \to \mathbb{R}$, and $\Psi\colon \mathbb{R}^d \to \mathbb{R}$, such that

$$\Psi'(u) = G'(u) f'(u).$$

And an entropy solution satisfies

$$\partial_t G(u) + \nabla_x \cdot \Psi(u) \leq 0.$$

This is trivial for scalar conservation laws, where $d = 1$.

Lyapunov methods: Entropy-Entropy flux

Denote

$$\mathcal{G}(u) = \int_\Omega G(u)dx.$$

In this case,

$$\begin{aligned}\int_\Omega G'(u)\nabla \cdot f(u)dx &= \int_\Omega G'(u)(f'(u), \nabla u)dx \\ &= \int_\Omega (\Psi'(u), \nabla u)dx \\ &= \int_\Omega \nabla \cdot \Psi(u)dx = 0,\end{aligned}$$

where we apply the fact that $f'(u)G'(u) = \Psi'(u)$ and $\int_\Omega \nabla \cdot \Psi(u)dx = 0$.

Lyapunov methods: Viscosity

In addition,

$$\int_\Omega G'(u)\nabla\cdot(A(u)\nabla u)dx = -\int_\Omega \big(\nabla G'(u), A(u)\nabla u\big)dx$$
$$= -\int_\Omega \big(\nabla G'(u), A(u)G''(u)^{-1}\nabla G'(u)\big)dx,$$

where we apply

$$\nabla G'(u) = G''(u)\nabla u,$$

in the last equality. We require that

$$A(u)G''(u)^{-1} \succeq 0,$$

and A is nonnegative definite, $G''(u) > 0$.

Entropy-Entropy flux-Fisher information dissipation

Hence we know that

$$\begin{aligned}\partial_t \mathcal{G}(u) =& \int_\Omega G'(u) \cdot \partial_t u dx \\ =& -\beta \int_\Omega \big(\nabla G'(u), A(u) G''(u)^{-1} \nabla G'(u)\big) dx \leq 0.\end{aligned}$$

This implies that $\mathcal{G}(u)$ is a Lyapunov functional for regularized conservation laws.

Entropy-Entropy flux-transport metrics

Definition (Entropy-entropy flux-metric condition)

We call (G, Ψ) an entropy-entropy flux pair-metric for conservation laws if there exists a convex function $G\colon \mathbb{R}^d \to \mathbb{R}$, and $\Psi\colon \mathbb{R}^d \to \mathbb{R}$, such that

$$\Psi'(u) = G'(u) f'(u),$$

and

$$A(u) G''(u)^{-1} \text{ is symmetric positive semi-definite.}$$

Lyapunov methods induced transport metrics

Under the entropy-entropy flux-metric condition, there is a metric operator for regularized conservation laws. Define the space of function u by

$$\mathcal{M} = \Big\{ u \in C^\infty(\Omega) \colon \int_\Omega u(x)dx = \text{constant} \Big\}.$$

The tangent space of $\mathcal{M}(u)$ at point u satisfies

$$T_u\mathcal{M} = \Big\{ \sigma \in C^\infty(\Omega) \colon \int_\Omega \sigma(x)dx = 0 \Big\}.$$

Denote an elliptic operator $L_{\mathcal{C}} \colon C^\infty(\Omega) \to C^\infty(\mathbb{R})$ by

$$L_{\mathcal{C}}(u) = -\nabla \cdot (A(u)G''(u)^{-1}\nabla).$$

Lyapunov methods induced transport metrics

Definition (Metric)

The inner product $\mathbf{g}(u)\colon T_u\mathcal{M} \times T_u\mathcal{M} \to \mathbb{R}$ is given below.

$$\begin{aligned}
\mathbf{g}(u)(\sigma_1, \sigma_2) &= \int_\Omega (\Phi_1, L_{\mathcal{C}}(u)\Phi_2)dx \\
&= -\int_\Omega \Phi_1 \nabla \cdot (A(u)G''(u)^{-1}\nabla\Phi_2)dx \\
&= \int_\Omega (\nabla\Phi_1, A(u)G''(u)^{-1}\nabla\Phi_2)dx \\
&= \int_\Omega \sigma_1\Phi_2 dx = \int_\Omega \sigma_2\Phi_1 dx,
\end{aligned}$$

where $\Phi_i \in C^\infty(\Omega)$ satisfies

$$\sigma_i = -\nabla \cdot (A(u)G''(u)^{-1}\nabla\Phi_i), \quad i = 1, 2.$$

Entropy induced Gradient flows

Proposition (Gradient flow)

Given an energy functional $\mathcal{E}\colon \mathcal{M} \to \mathbb{R}$, the gradient flow of $\mathcal{E}$ in $(\mathcal{M}, \mathbf{g})$ satisfies

$$\partial_t u = \nabla \cdot (A(u)G''(u)^{-1}\nabla \frac{\delta}{\delta u}\mathcal{E}(u)).$$

If $\mathcal{E}(u) = \mathcal{G}(u) = \int_\Omega G(u)dx$, then the above gradient flow satisfies

$$\partial_t u = \nabla \cdot (A(u)\nabla u).$$

Flux-gradient flows

Definition (Flux–gradient flow)

Given an energy functional $\mathcal{E}\colon \mathcal{M} \to \mathbb{R}$, consider a class of PDE

$$\partial_t u + \nabla_x \cdot f_1(x,u) = \beta \nabla_x \cdot (A(u) G''(u)^{-1} \nabla_x \frac{\delta}{\delta u} \mathcal{E}(u)), \tag{1}$$

where $f_1\colon \Omega \times \mathbb{R}^1 \to \mathbb{R}^n$ is a flux function satisfying

$$\int_\Omega f_1(x,u) \cdot \nabla_x \frac{\delta}{\delta u(x)} \mathcal{E}(u) dx = 0.$$

Example

If $\mathcal{E}(u) = \mathcal{G}(u) = \int_\Omega G(u)dx$ and $f_1(x,u) = f(u)$, then equation (1) forms regularized conservation laws.

$$\begin{aligned}
\partial_t u + \nabla_x \cdot f(u) =& \beta \nabla_x \cdot (A(u) G''(u)^{-1} \nabla_x \frac{\delta}{\delta u} \mathcal{G}(u)) \\
=& \beta \nabla_x \cdot (A(u) G''(u)^{-1} \nabla_x G'(u)) \\
=& \beta \nabla_x \cdot (A(u) G''(u)^{-1} G''(u) \nabla_x u) \\
=& \beta \nabla_x \cdot (A(u) \nabla u),
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \nabla_x \frac{\delta}{\delta u} \mathcal{G}(u) \cdot f(u) dx =& \int_\Omega \nabla_x G'(u) \cdot f(u) dx \\
=& - \int_\Omega G'(u) \cdot \Big(\nabla_x \cdot f(u) \Big) dx \\
=& - \int_\Omega \nabla \cdot \Psi(u) dx \\
=& 0.
\end{aligned}$$

Controlling flux-gradient flows

Given smooth functionals $\mathcal{F}$, $\mathcal{H}\colon \mathcal{M} \to \mathbb{R}$, consider a variational problem

$$\inf_{u,v,u_1} \int_0^1 \Big[\int_\Omega \frac{1}{2}\big(v, A(u)G''(u)^{-1}v\big)dx - \mathcal{F}(u)\Big]dt + \mathcal{H}(u_1),$$

where the infimum is taken among variables $v\colon [0,1] \times \Omega \to \mathbb{R}^n$, $u\colon [0,1] \times \Omega \to \mathbb{R}$, and $u_1 : \Omega \to \mathbb{R}$ satisfying

$$\partial_t u + \nabla \cdot f(u) + \nabla \cdot (A(u)G''(u)^{-1}v) = \beta \nabla \cdot (A(u)\nabla u),$$

and

$$u(0,x) = u_0(x).$$

Primal-dual conservation laws

Proposition (Hamiltonian flows of conservation laws)

The critical point system of the variational problem is given below. There exists a function $\Phi\colon [0,1]\times\Omega\to\mathbb{R}$, such that

$$v(t,x)=\nabla\Phi(t,x),$$

and

$$\begin{cases}\partial_t u+\nabla\cdot f(u)+\nabla\cdot(A(u)G''(u)^{-1}\nabla\Phi)=\beta\nabla\cdot(A(u)\nabla u),\\ \partial_t\Phi+(\nabla\Phi,f'(u))+\dfrac{1}{2}(\nabla\Phi,(A(u)G''(u)^{-1})'\nabla\Phi)+\dfrac{\delta}{\delta u}\mathcal{F}(u)\\ =-\beta\nabla\cdot(A(u)\nabla\Phi)+\beta(\nabla\Phi,A'(u)\nabla u).\end{cases}\tag{2}$$

Here $'$ represents the derivative w.r.t. variable u. The initial and terminal time conditions satisfy

$$u(0,x)=u_0(x),\quad \frac{\delta}{\delta u_1}\mathcal{H}(u_1)+\Phi(1,x)=0.$$

Hamiltonian flows of conservation laws

Proposition

The primal-dual conservation law system (2) has the following Hamiltonian flow formulation.

$$\partial_t u = \frac{\delta}{\delta \Phi}\mathcal{H}_{\mathcal{G}}(u,\Phi), \quad \partial_t \Phi = -\frac{\delta}{\delta u}\mathcal{H}_{\mathcal{G}}(u,\Phi),$$

where we define the Hamiltonian functional $\mathcal{H}_{\mathcal{G}} \colon \mathcal{M} \times C^\infty(\Omega) \to \mathbb{R}$ by

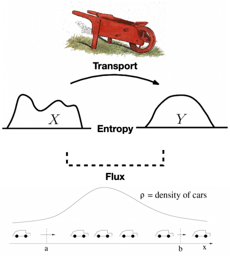
$$\begin{aligned}\mathcal{H}_{\mathcal{G}}(u,\Phi) = \int_\Omega \Big[\frac{1}{2}(\nabla\Phi, A(u)G''(u)^{-1}\nabla\Phi) + (\nabla\Phi, f(u)) \\ - \beta(\nabla\Phi, A(u)\nabla u)\Big] dx + \mathcal{F}(u).\end{aligned}$$

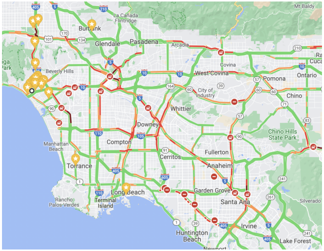
In other words, the Hamiltonian functional $\mathcal{H}_{\mathcal{G}}(u,\Phi)$ is conserved along Hamiltonian dynamics:

$$\frac{d}{dt}\mathcal{H}_{\mathcal{G}}(u,\Phi) = 0.$$

Modeling: Controlling traffic flows

Position control. The unknown variable u in traffic flows represents the density function of cars (particles) in a given spatial domain. Here, the background dynamics of u is the classical traffic flow. The control variable is the velocity for enforcing each car's velocity in addition to its background traffic flow dynamics. The goal is to control the "total enforced kinetic energy" of all cars, in which individual cars can determine their velocities through both noise and traffic flow interactions.





Controlling Traffic flow problems

Consider

$$\inf_{u,v,u_1} \int_0^1 \Big[\int_\Omega \frac{1}{2}\|v(t,x)\|^2 u(t,x)dx - \mathcal{F}(u)\Big] dt + \mathcal{H}(u_1), \tag{3a}$$

such that

$$0 \leq u(t,x) \leq 1, \quad \text{for all } t \in [0,1],$$

and

$$\partial_t u(t,x) + \nabla\cdot\big(u(t,x)(1-u(t,x))\big) + \nabla\cdot\big(u(t,x)v(t,x)\big) = \beta\Delta u(t,x), \tag{3b}$$

with

$$u(0,x) = u_0(x).$$

Controlling traffic flow dynamics

There exists a scalar function Φ, such that

$$v(t,x) = \nabla\Phi(t,x),$$

and

$$\begin{cases} \partial_t u + \nabla \cdot \big(u(1-u)\big) + \nabla \cdot \big(u\nabla\Phi\big) = \beta\Delta u, \\ \partial_t \Phi + \big(1-2u, \nabla\Phi\big) + \dfrac{1}{2}\|\nabla\Phi\|^2 + \dfrac{\delta}{\delta u}\mathcal{F}(u) = -\beta\Delta\Phi. \end{cases}$$

The Primal-dual Algorithms

The classical primal-dual hybrid gradient algorithms (PDHG) solves the following saddle point problem

$$\min_z \max_p \langle Kz, p \rangle_{L^2} + g(z) - h^*(p).$$

When the operator K is nonlinear, we apply its extension with

$$K(z) \approx K(\bar{z}) + \nabla K(\bar{z})(z - \bar{z}).$$

Here, the extension of the PDHG scheme is as follows

$$\begin{aligned}
z^{n+1} &= \operatorname*{argmin}_z \langle z, [\nabla K(z^n)]^T \bar{p}^n \rangle_{L^2} + g(z) + \frac{1}{2\tau} \|z - z^n\|^2_{L^2}, \\
p^{n+1} &= \operatorname*{argmax}_p \langle K(z^{n+1}), p \rangle_{L^2} - h^*(p) - \frac{1}{2\sigma} \|p - p^n\|^2_{L^2}, \\
\bar{p}^{n+1} &= 2p^{n+1} - p^n.
\end{aligned}$$

Set

$$\begin{aligned}
z = (u, m), &\quad p = \Phi, \\
K\left((u,m)\right) &= \partial_t u + \nabla \cdot f(u) + \nabla \cdot m - \beta \Delta u, \\
g\left((u,m)\right) &= \int_0^1 \left(\int_\Omega \frac{\|m\|^2}{2u} + \mathbf{1}_{[0,1]}(u) dx - \mathcal{F}(u) \right) dt, \\
h(Kz) &= \begin{cases} 0 & \text{if } Kz = 0 \\ +\infty & \text{else} \end{cases} .
\end{aligned}$$

Solve the Control Problem with Primal-dual Algorithms

We rewrite the varational problem as follows:

$$\inf_{u,m} \sup_{\Phi} \quad \mathcal{L}(u,m,\Phi), \quad \text{with } u(0,x)=u_0(x),\ \Phi(1,x)=-\frac{\delta}{\delta u(1,x)}\mathcal{H}(u),$$

$$\mathcal{L}(u,m,\Phi)=\int_0^1 \left(\int_\Omega \frac{\|m\|^2}{2u}+\mathbf{1}_{[0,1]}(u)dx-\mathcal{F}(u)\right)dt$$

$$+\int_0^1\int_\Omega \Phi\left(\partial_t u+\nabla\cdot f(u)+\nabla\cdot m-\beta\Delta u\right)dxdt,$$

and $f(u)=u(1-u)$ is the traffic flux function. We denote the indicator function by $\mathbf{1}_A(x)=\begin{cases}+\infty & x\notin A\\ 0 & x\in A\end{cases}$.

Algorithm

Algorithm :PDHG for the conservation law control system

While $k <$ Maximal number of iteration

$$\left(u^{(k+1)}, m^{(k+1)}\right) = \mathrm{argmin}_u \;\; \mathcal{L}(u, m, \bar{\Phi}^{(k)}) + \frac{1}{2\tau}\|(u,m) - (u^{(k)}, m^{(k)})\|^2_{L^2};$$

$$\Phi^{(k+1)} = \mathrm{argmax}_\Phi \;\; \mathcal{L}(u^{(k+1)}, m^{(k+1)}, \Phi) - \frac{1}{2\sigma}\|\Phi - \Phi^{(k)}\|^2_{H_1^2};$$

$$\bar{\Phi}^{(k+1)} = 2\Phi^{(k+1)} - \Phi^{(k)};$$

Monotone schemes

Definition

For $p, q \in \mathbb{N}$, a scheme

$$u_j^{k+1} = G(u_{j-p-1}^k, ..., u_{j+q}^k)$$

is called a *monotone scheme* if G is a monotonically nondecreasing function of each argument.

Lax–Friedrichs Scheme. For discretization $\Delta t, \Delta x$ in time and space, denote $u_j^k = u(k\Delta t, j\Delta x)$, then the Lax–Friedrichs scheme is as follows

$$u_j^{k+1} = u_j^k - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^k) - f(u_{j-1}^k)\right) + (\beta + c\Delta x)\frac{\Delta t}{(\Delta x)^2}\left(u_{j+1}^k - 2u_j^k + u_{j-1}^k\right).$$

To guarantee that the above scheme is monotone, we need:

$$1 - 2(\beta + c\Delta x)\frac{\Delta t}{(\Delta x)^2} \geq 0, \quad -\frac{\Delta t}{2\Delta x}|f'(u)| + (\beta + c\Delta x)\frac{\Delta t}{(\Delta x)^2} \geq 0.$$

As we want the scheme works when $\beta \to 0$, the restriction on c and space–time stepsizes can be simplified as follows:

$$c \geq \frac{1}{2}|f'(u)|, \quad (\Delta x)^2 \geq 2(\beta + c\Delta x)\Delta t.$$

The first inequality suggests the artificial viscosity we need to add. The second one impose a strong restriction on the stepsize in time when $\beta > 0$.

Discretization of the control problem

We consider the control problem of scalar conservation law defined in $[0, b] \times [0, 1]$ with periodic boundary condition on the spatial domain. Given $N_x, N_t > 0$, we have $\Delta x = \frac{b}{N_x}$, $\Delta t = \frac{1}{N_t}$. For $x_i = i\Delta x, t_l = l\Delta t$, define

$$
\begin{aligned}
u_i^l &= u(t_l, x_i) && 1 \le i \le N_x, 0 \le l \le N_t, \\
m_{1,i}^l &= (m_{x_1}(t_l, x_i))^+ && 1 \le i \le N_x, 0 \le l \le N_t - 1, \\
m_{2,i}^l &= -(m_{x_1}(t_l, x_i))^- && 1 \le i \le N_x, 0 \le l \le N_t - 1, \\
\Phi_i^l &= \Phi(t_l, x_i) && 1 \le i \le N_x, 0 \le l \le N_t, \\
\Phi_i^{Nt} &= -\frac{\delta}{\delta u(1, x_i)} \mathcal{H}(u_i^{N_t}) && 1 \le i \le N_x,
\end{aligned}
$$

where $u^+ := \max(u, 0)$ and $u^- = u^+ - u$. Note here $m_{1,i}^l \in \mathbb{R}_+, m_{2,i}^l \in \mathbb{R}_-$. Denote

$$
\begin{aligned}
(Du)_i &:= \frac{u_{i+1} - u_i}{\Delta x} \\
[Du]_i &:= \left((Du)_i \, , (Du)_{i-1} \right) \\
\widehat{[Du]}_i &= \left((Du)_i^+ \, , -(Du)_{i-1}^- \right) \\
Lap(u)_i &= \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}
\end{aligned}
$$

Discretization of the control problem

The first conservation law equation adapted from the Lax–Friedrichs scheme satisfies

$$\frac{\left(u_i^{l+1}-u_i^l\right)}{\Delta t}+\frac{\left(f(u_{i+1}^{l+1})-f(u_{i-1}^{l+1})\right)}{2\Delta x}+(Dm)_{1,i-1}^l+(Dm)_{2,i}^l=(\beta+c\Delta x)Lap(u)_i^{l+1}.$$

Following the discretization of the conservation law, the discrete saddle point problem has the following form:

$$\min_{u,m}\max_{\Phi} L(u,m,\Phi),$$

where

$$\begin{aligned}L(u,m,\Phi)&=\Delta x\Delta t\sum_{i,l}\frac{(m_{1,i}^{l-1})^2+(m_{2,i}^{l-1})^2}{2u_i^l}+\mathbf{1}_{[0,1]}(u_i^l)-\Delta t\sum_l\mathcal{F}(u^l)+\Delta x\sum_i\mathcal{H}(u_i^{N_t})\\&\quad+\Delta x\Delta t\sum_{i,l}\Phi_i^l\Big(\frac{u_i^{l+1}-u_i^l}{\Delta t}+\frac{f(u_{i+1}^{l+1})-f(u_{i-1}^{l+1})}{2\Delta x}+(Dm)_{1,i-1}^l+(Dm)_{2,i}^l\\&\quad-(\beta+c\Delta x)Lap(u)_i^{l+1}\Big).\end{aligned}$$

Discretization of the control problem

By taking the first order derivative of u_i^l, we automatically get the implicit finite difference scheme for the dual equation of Φ that is backward in time:

$$\frac{1}{\Delta t}\left(\Phi_i^{l+1}-\Phi_i^l\right)+\frac{(\Phi_{i+1}^l-\Phi_{i-1}^l)}{2\Delta x}f'(u_i^l)+\frac{1}{2}\|[\widehat{D\Phi}]_i^l\|^2+\frac{\delta\mathcal{F}(u_i^l)}{\delta u}=-(\beta+c\Delta x)Lap(\Phi)_i^l.$$

The discrete form of the Hamiltonian functional at $t=t_l$ takes the form

$$H_{\mathcal{G}}(u,\Phi)=\sum_i\left(\frac{1}{2}\|[\widehat{D\Phi}]_i^l\|^2u_i^l+\frac{(\Phi_{i+1}^l-\Phi_{i-1}^l)}{2\Delta x}f(u_i^l)+(\beta+c\Delta x)u_i^lLap(\Phi)_i^l\right)-\mathcal{F}(u^l).$$

- When $f(u)=0$, the above discretization reduces to the finite difference scheme for the mean-field game system.
- When $\mathcal{F}=0$, $\mathcal{H}=c$ for some constant c, the variational problem becomes classical conservation laws with initial data. No control will be enforced on the density function u.
- Our algorithm provides an alternative way to solve the nonlinear conservation law with implicit discretization in time.

Example 1: Traffic flows

We consider the traffic flow equation

$$\partial_t u + \partial_x f(u) = 0, \quad u(0,x) = \begin{cases} 0.8, & 1 \le x \le 2 \\ 0 & \text{else} \end{cases},$$

where $f(u) = \frac{1}{2}u(1-u), \mathcal{F} = 0, \mathcal{H} = 0, \beta = 0, k = 0.5.$

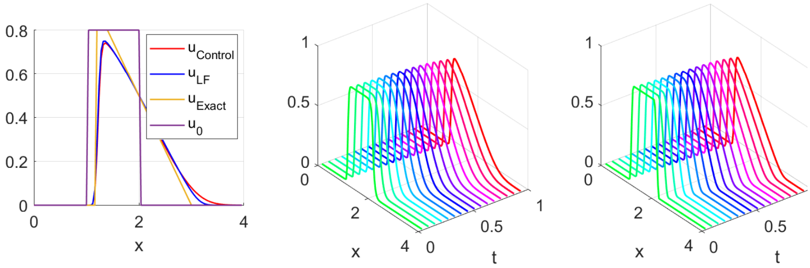


Figure: Left: a comparison with the exact entropy solution at $t = 1$; middle: the numerical solution via solving the control problem; right: the numerical solution to the conservation law using Lax–Friedrichs scheme.

Example 2: Traffic flows transport problems

We consider the traffic flow equation with $f(u) = u(1-u)$, $\beta = 0.1$, $\mathcal{F} = 0$, $c = 0.5$. The final cost functional

$$\mathcal{H}(u(1,\cdot)) = \mu \int_\Omega u(1,x) \log(\frac{u(1,x)}{u_1}) dx, \mu = 1.$$

We set

$$u_0 = 0.001 + 0.9e^{-10(x-2)^2},$$
$$u_1 = 0.001 + 0.45e^{-10(x-1)^2} + 0.45e^{-10(x-3)^2}.$$

We also compare the result from the control of conservation law with a mean-field game problem, i.e., $f = 0$.

Example 2: Traffic flows transport problems

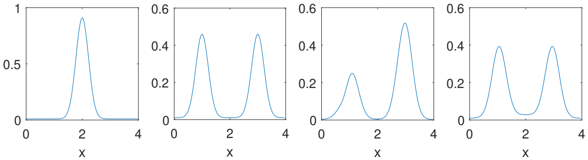


Figure: From left to right: initial configurations of u_0, u_1, solution $u(1,x)$ for the control of conservation law, solution $u(1,x)$ for the mean-field game problem.

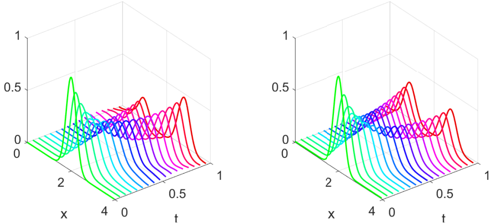


Figure: Left: solution $u(t,x)$ for the problem of controlling the conservation law; right: solution $u(t,x)$ for the mean-field game problem.

Example 3: Traffic control

Consider the traffic flow equation with $f(u) = u(1-u)$, $\beta = 10^{-3}$,$\mathcal{F} = -\alpha \int_\Omega u \log(u) dx, \alpha \geq 0$. The final cost functional $\mathcal{H}(u(1,\cdot)) = \int_\Omega u(1,x) g(x) dx$. The initial density and final cost function are as follows

$$u_0(x) = \begin{cases} 0.4 & 0.5 \leq x \leq 1.5 \\ 10^{-3} & \text{else} \end{cases}, \qquad g(x) = -0.1 \sin(2\pi x).$$

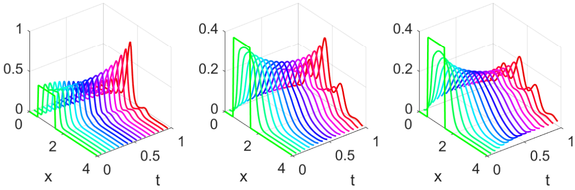


Figure: Solution $u(t,x)$ for the problem of controlling the conservation law. From left to right: $\alpha = 0, 0.5, 1$.

Example 3: Traffic control

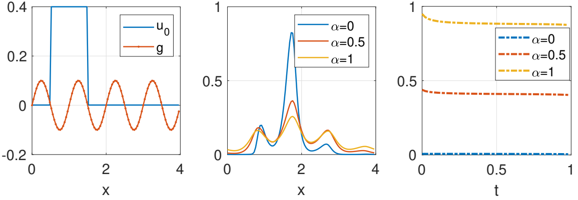


Figure: Left: boundary conditions for the control problems u_0. Middle: solution $u(1,x)$ for the problem of controlling the conservation law. Right: the numerical Hamiltonian $H_{\mathcal{G}}(u,\Phi)$.

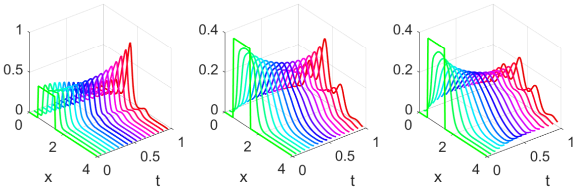


Figure: Solution $u(t,x)$ for the problem of controlling the conservation law. From left to right: $\alpha = 0, 0.5, 1$.

Extending to systems of conservation law

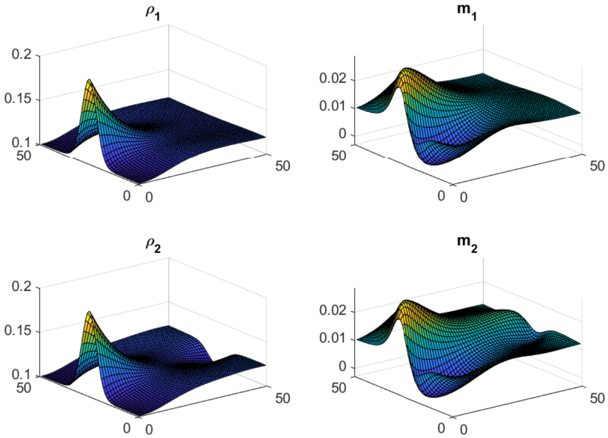


Figure: Control 1D compressible Navier-Stokes equations. Top: no control. Bottom: with control.

Road Ahead: Entropy, Information, transportation, conservation laws

- Conservation law induced mean-field games;
- Conservation law enhanced sampling and AI optimization algorithms;
- Variational numerical schemes for conservation laws;
- Conservation law Quantum computing;
- Entropy dissipation of conservation law equations in generalized optimal transport space.