Contact Invariants of $\mathbb{Q}$-Gorenstein Toric Contact Manifolds and the Ehrhart (quasi-) Polynomials of their Toric Diagrams

(joint with L. Macarini and M. Moreira)

arXiv: 2202.0044

Plan:

Ⅰ $\mathbb{Q}$-Gorenstein Toric Contact Manifolds

Ⅱ Contact Betti numbers

Ⅲ Ehrhart (quasi-) polynomials

Ⅳ Main Result

Ⅴ Other Stuff

(I) $\mathbb{Q}$-Gorenstein Toric Contact Manifolds

- $(M^{2n+1}, \xi) \circlearrowleft \mathbb{T}^{n+1}$, i.e. $S(M)^{2(n+1)} \circlearrowleft \mathbb{T}^{n+1}$ is a toric symplectic cone

- Constructed as contact reductions of $(S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}, \xi_{std}) \circlearrowleft \mathbb{T}^d$ by $\mathbb{K} := \ker(\beta : \mathbb{T}^d \twoheadrightarrow \mathbb{T}^{n+1})$

- Determined by $\nu_j \in \mathbb{Z}^{n+1}$, $j = 1, \ldots, d$, which are also the defining normals of

$$\boxed{\text{moment cone } C \equiv \text{image of moment map } \mu : S(M)^{2n+2} \longrightarrow \mathbb{L}^*_{\mathrm{ie}}\mathbb{T}^{n+1} \cong \mathbb{R}^{n+1}}$$

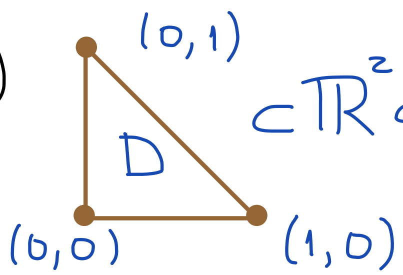
- $\mathbb{Q}$-Gorenstein, i.e. $m \cdot c_1(\xi) = 0$, implies w.l.o.g.

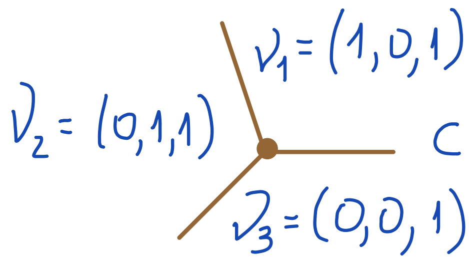
$$\nu_j := (\tilde{\nu}_j, m) \in \mathbb{Z}^{n+1} \text{ w/ } \tilde{\nu}_j \in \mathbb{Z}^n.$$

- <u>Smoothness</u> requires that each facet of $\boxed{D := \mathrm{conv}(v_1, \dots, v_d) \subset \mathbb{R}^n,\ v_i := \tilde{v}_i/m}$ is $\mathrm{Aff}(n,\mathbb{Z})$-equivalent to $\mathrm{conv}(e_1/m, \dots, e_n/m)$ with $\{e_1, \dots, e_n\}$ = standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$, i.e. $\boxed{D \subset \mathbb{R}^n \text{ is a (rational) toric diagram}}$

- <u>Notation:</u> $D \rightsquigarrow (M_D, \xi_D)$

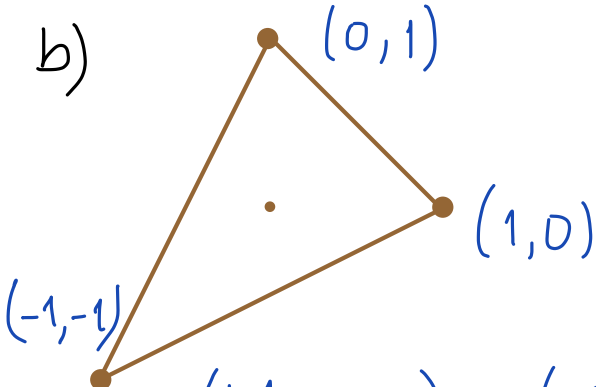
- <u>Examples</u>

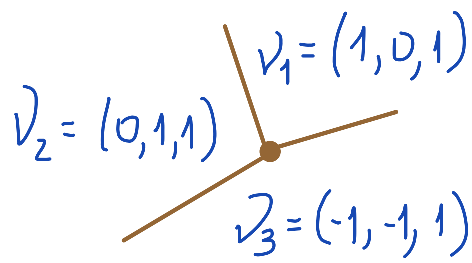
a) 

$\subset \mathbb{R}^2 \subset \mathrm{Lie}\,\mathbb{T}^3 \rightsquigarrow$ 

$\subset \mathrm{Lie}^*\,\mathbb{T}^3$

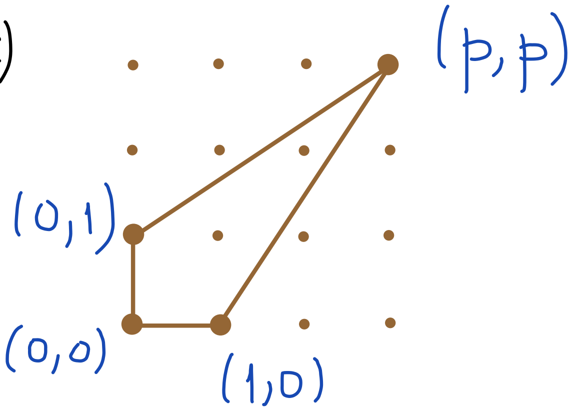
$$(M_D, \xi_D) = (S^5, \xi_{std}), \qquad \boxed{m = 1}$$

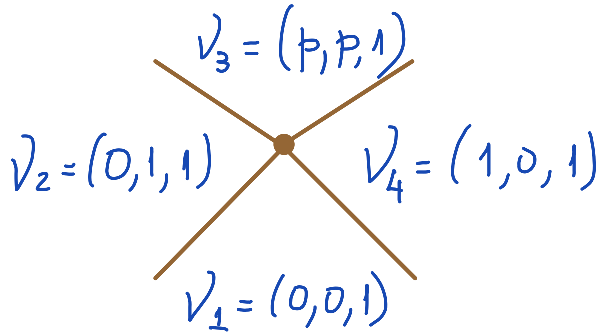
b) 

$\rightsquigarrow$ 

$$(M_D, \xi_D) = (S^5/\mathbb{Z}_3, \xi_{std}) = L^5_3(1,1,1) \qquad \boxed{m = 1}$$

c)



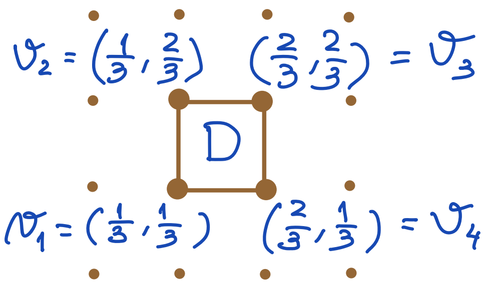


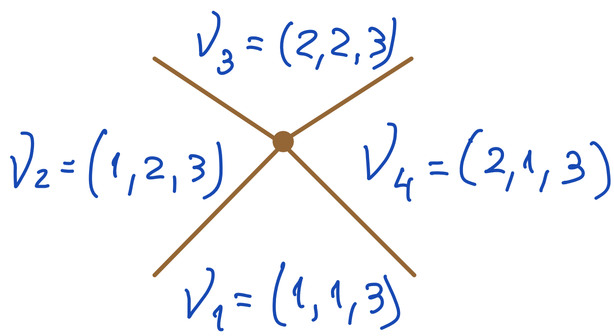
$$(M_D, \xi_D) = (S^2 \times S^3, \xi_p), \quad p \in \mathbb{N}, \qquad \boxed{m = 1}$$

<u>Note</u>: $p = 1$ gives unit co-sphere bundle of S^3 $\cong$ preq. of $(S^2 \times S^2, \sigma \times \sigma)$, $\int_{S^2} \sigma = 2\pi$

d) $(M_D, \xi_D) = (S^2 \times S^3, \xi_1)/\mathbb{Z}_3$ $\cong$ preq. of $(S^2 \times S^2, 3\sigma \times 3\sigma)$

$$\boxed{m = 3}$$





(II) Contact Betti numbers

- [Martelli - Sparks - Yau, 2006]

$\mathrm{int}(D) \subset \mathbb{R}^n$ parametrizes (normalized) toric Reeb vector fields: $v = (r_1, \dots, r_n) \in \mathrm{int}(D)$

$\rightsquigarrow V = (r_1, \dots, r_n, 1) \in \mathrm{Lie}(\mathbb{T}^{n+1}) \rightsquigarrow R_V$

- [A. - Macarini, A. - Macarini - Moreira]

R_V non-deg. $\Longleftrightarrow$ $\{r_1, \dots, r_n, 1\}$ $\mathbb{Q}$-indep.

In that case:

(i) simple closed R_V-orbits $\overset{1:1}{\longleftrightarrow}$ facets of D

$\gamma_1, \dots, \gamma_c$

(ii) $\deg(\gamma_\ell^N) := \mu_{CZ}(\gamma_\ell^N) + n - 2 = 2(j + a)$

for any $\ell = 1, \dots, c$, $N \in \mathbb{N}$,

and for some $j \in \left(\frac{1}{m}\mathbb{Z}\right) \cap]-1, 0]$, $a \in \mathbb{N}_0$

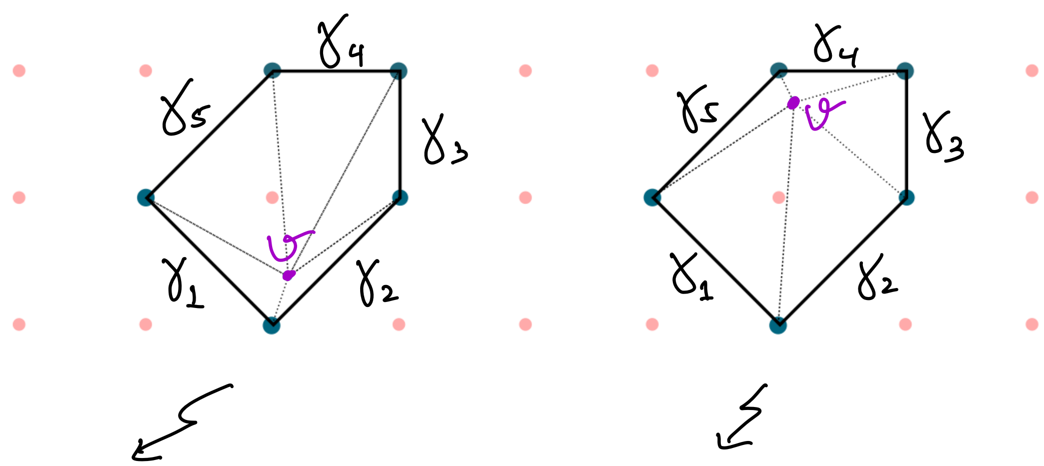
(iii) $cb_{\frac{k}{m}}(D,\nu) := \#\{$ closed R_ν-orbits with degree $= \frac{k}{m}$ $\}$

should be a contact invariant of (M_D, ξ_D)

$= \operatorname{rank} HC_{\frac{k}{m}}(M_D, \xi_D) \equiv$ contact Betti $\#$

In particular, each $cb_{\frac{k}{m}}(D,\nu)$ should be independent of ν and give rise to the contact Betti numbers $cb_*(D)$.

(iv) General idea for toric combinatorial proof:



γ_1 and γ_2 with much bigger degrees than γ_3, γ_4 and γ_5

γ_4 and γ_5 with much bigger degrees than γ_1, γ_2 and γ_3

III Ehrhart (quasi-) polynomials

- $L_D(t) := \#\left(D \cap \frac{1}{t}\mathbb{Z}^n\right)$, $t \in \mathbb{N}$

- Properties:

(i) L_D is a quasi-polynomial of degree n and period m, i.e.

$$L_D(t) = \sum_{k=0}^{n} c_k(D,t)\, t^k, \quad \text{with}$$

$c_k(D, t+m) = c_k(D,t)$, $\forall t \in \mathbb{N}$, $k \in \{0, \dots, n\}$

(ii) $c_0(D,0) = 1$ and $c_n(D,t) = \mathrm{vol}(D)$, $\forall t \in \mathbb{N}$.

(ii) [Stanley] The coefficients of $L_D(t)$ in the quasi-polynomial basis

$$\left\{ \binom{\frac{t-k}{m} + n}{n}, \ k = 0, \dots, m(n+1)-1, \ t \equiv k \bmod m \right\}$$

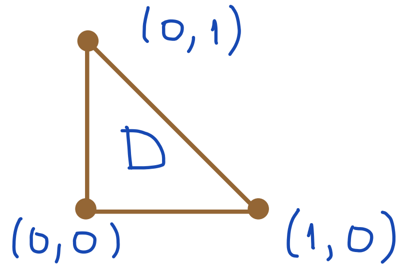
are non-negative integers, i.e.

$$L_D(t) = \sum_{\substack{k \in [0,\, m(n+1)-1] \\ k \equiv t \bmod m}} \delta_k(D) \binom{\frac{t-k}{m}+n}{n}, \quad \text{w/ } \underline{\delta_k(D) \in \mathbb{N}_0}.$$

Note: define $\delta_k = 0$ for $k<0$ and $k > m(n+1)-1$.

- Examples

a) $D = \operatorname{conv}\{0, e_1, \dots, e_n\} \subset \mathbb{R}^n$

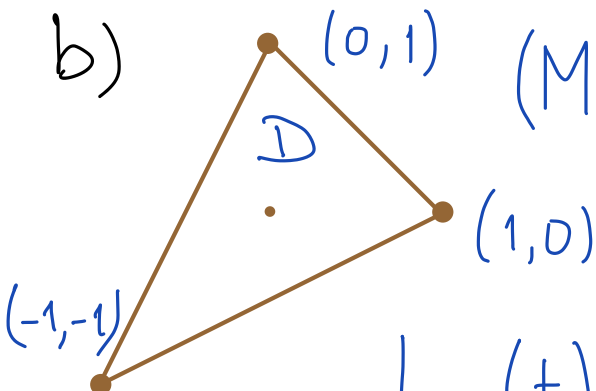


$$(M_D, \xi_D) = (S^{2n+1}, \xi_{std}), \qquad \boxed{m=1}$$

$$L_D(t) = \frac{(t+1)\cdots(t+n)}{n!} = \binom{t+n}{n}$$

$$\Longrightarrow \delta_k(D) = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

b)



$$(M_D, \xi_D) = (S^5/\mathbb{Z}_3, \xi_{std}) = L_3^5(1,1,1)$$

$$\boxed{m=1}$$

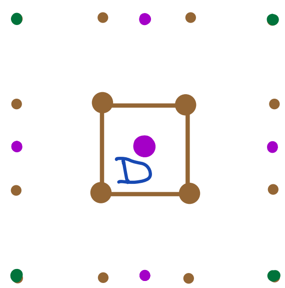
$$L_D(t) = \frac{1}{2}(3t^2+3t+2)$$

$$= \delta_0(D)\binom{t+2}{2} + \delta_1(D)\binom{t+1}{2} + \delta_2(D)\binom{t}{2}$$

$$= \delta_0(D)\,\frac{(t+2)(t+1)}{2} + \delta_1(D)\,\frac{(t+1)t}{2} + \delta_2(D)\,\frac{t(t-1)}{2}$$

with $\delta_0(D) = \delta_1(D) = \delta_2(D) = 1$

c) $(M_D, \xi_D) = (S^2 \times S^3, \xi_1)/\mathbb{Z}_3 \cong$ preq. of $(S^2 \times S^2, 3\sigma \times 3\sigma)$



$\boxed{m = 3}$

$$L_D(t) = \begin{cases} \dfrac{(t-1)^2}{9}, & t \equiv 1 \mod 3 \\[2mm] \dfrac{(t+1)^2}{9}, & t \equiv 2 \mod 3 \\[2mm] \dfrac{(t+3)^2}{9}, & t \equiv 3 \mod 3 \end{cases}$$

$$= \begin{cases} \delta_1 \dfrac{\left(\frac{t-1}{3}+2\right)\left(\frac{t-1}{3}+1\right)}{2} + \delta_4 \dfrac{\left(\frac{t-4}{3}+2\right)\left(\frac{t-4}{3}+1\right)}{2} + \delta_7 \dfrac{\left(\frac{t-7}{3}+2\right)\left(\frac{t-7}{3}+1\right)}{2} \\[3mm] \delta_2 \dfrac{\left(\frac{t-2}{3}+2\right)\left(\frac{t-2}{3}+1\right)}{2} + \delta_5 \dfrac{\left(\frac{t-5}{3}+2\right)\left(\frac{t-5}{3}+1\right)}{2} + \delta_8 \dfrac{\left(\frac{t-8}{3}+2\right)\left(\frac{t-8}{3}+1\right)}{2} \\[3mm] \delta_0 \dfrac{\left(\frac{t-0}{3}+2\right)\left(\frac{t-0}{3}+1\right)}{2} + \delta_3 \dfrac{\left(\frac{t-3}{3}+2\right)\left(\frac{t-3}{3}+1\right)}{2} + \delta_6 \dfrac{\left(\frac{t-6}{3}+2\right)\left(\frac{t-6}{3}+1\right)}{2} \end{cases}$$

with $\delta_1 = 0$, $\delta_4 = \delta_7 = 1$

$\delta_2 = \delta_5 = 1$, $\delta_8 = 0$

$\delta_0 = \delta_3 = 1$, $\delta_6 = 0$

Ⅳ Main Result

Thm. [A. - Macarini - Moreira]

$$cb_{2j}(D,\nu) - cb_{2(j-1)}(D,\nu) = \delta_{m(n-j)}(D)$$

i.e. $cb_*(D)$ are combinatorially given by

$$\boxed{cb_{2(j+a)}(D) = \sum_{k=0}^{a} \delta_{m(n-j-k)}(D), \quad j \in \tfrac{1}{m}\mathbb{Z} \cap]-1,0], \ a \in \mathbb{N}_0.}$$

Cor.

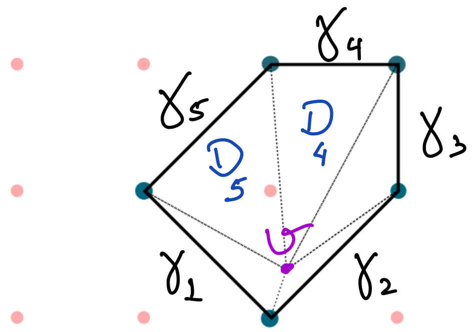
(i) For each $j \in \frac{1}{m}\mathbb{Z} \cap]-1,0]$, the sequence $\{cb_{2(j+a)}(D)\}_{a\in\mathbb{Z}}$ is <u>non-decreasing</u> and

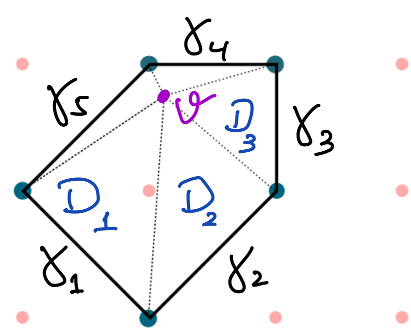
$$\boxed{cb_{2(j+a)}(D) = n!\,\mathrm{vol}(mD), \ \forall a \geq n.}$$

(ii) $$\boxed{cb_0(D) = \delta_{mn} = \#(\mathrm{int}(mD) \cap \mathbb{Z}^n)}$$

"Proof":

Lemma: Given $n, S \in \mathbb{N}$, the number of solutions of $\sum_{i=1}^{n} m_i = S$, with $m_i \in \mathbb{N}_0$, is given by $\binom{S+n-1}{n-1}$.





$$\Rightarrow L_{\mathrm{int}\, D_\ell}(t) = \sum_{N \geq 1} \binom{\frac{t}{m} - \frac{1}{2}\deg \gamma_\ell^N + n - 2}{n-1}$$

$$= \sum_{j \in \frac{1}{m}\mathbb{Z}} \left(\#\{N \geq 1 : \deg \gamma_\ell^N = 2j\}\right) \binom{\frac{t}{m} - j + n - 2}{n-1}$$

$$\Rightarrow L_{\mathrm{int}\, D}(t) = \sum_{j \in 1/m\,\mathbb{Z}} cb_{2j}(D, v) \binom{\frac{t}{m} - j + n - 2}{n-1}$$

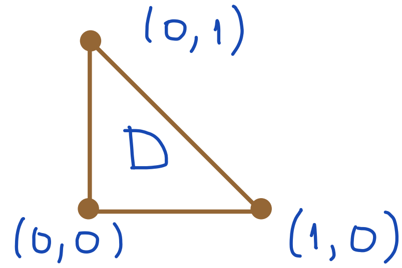
$$\| \leftarrow \text{Ehrhart reciprocity}$$

$$(-1)^n L_D(-t)$$

"Q.E.D."

- ## Examples

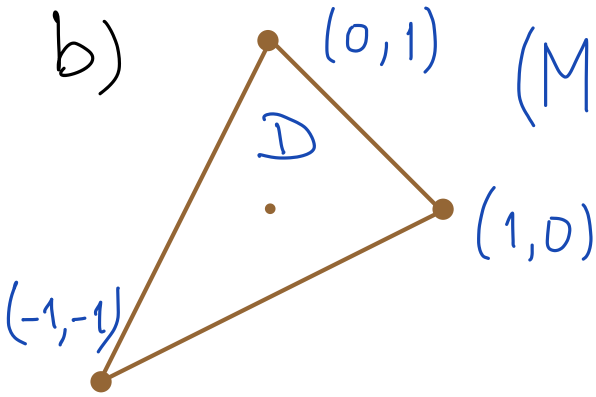
a) $D = \operatorname{conv}\{0, e_1, \dots, e_n\} \subset \mathbb{R}^n$



$$(M_D, \xi_D) = (S^{2n+1}, \xi_{std}), \qquad \boxed{m = 1}$$

$$\delta_k(D) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow cb_*(S^{2n+1}) = \begin{cases} 1, & * = 2k \geq 2n \\ 0, & \text{otherwise} \end{cases}$$

b)



$$(M_D, \xi_D) = (S^5/\mathbb{Z}_3, \xi_{std}) = L^5_3(1,1,1)$$

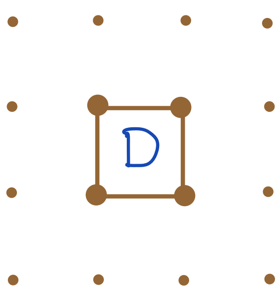
$$\boxed{m = 1}$$

$\delta_0(D) = \delta_1(D) = \delta_2(D) = 1$

$\delta_k(D) = 0$ otherwise

$$\Rightarrow cb_*(L^5_3(1,1,1)) = \begin{cases} 1, & * = 0 \\ 2, & * = 2 \\ 3, & * = 2k \geq 4 \\ 0, & \text{otherwise} \end{cases}$$

c) $(M_D, \xi_D) = (S^2 \times S^3, \xi_1)/\mathbb{Z}_3 \cong$ preq. of $(S^2 \times S^2, 3\sigma \times 3\sigma)$



$$\boxed{m = 3}$$

$$\delta_4 = \delta_7 = \delta_2 = \delta_5 = \delta_0 = \delta_3 = 1$$

$$\delta_1 = \delta_8 = \delta_6 = 0$$

$* =$	$-\frac{8}{3}$	$-\frac{6}{3}$	$-\frac{4}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{6}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	$\frac{12}{3}$
$cb_* =$	0	0	0	1	0	1	2	1	2	2	2

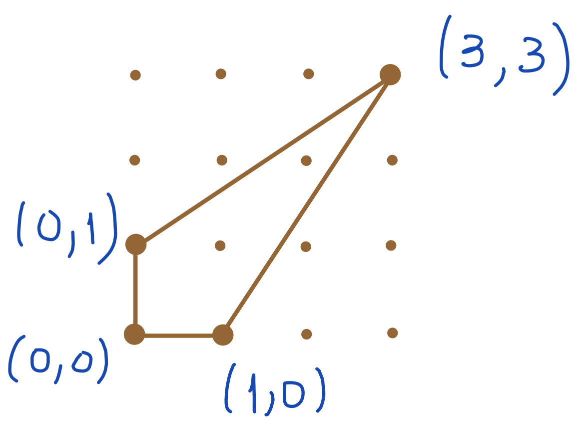
$+\delta_8$ $+\delta_7$ $+\delta_6$ $+\delta_5$ $+\delta_4$ $+\delta_3$ $+\delta_2$ $+\delta_1$ $+\delta_0$

Ⓥ Other Stuff

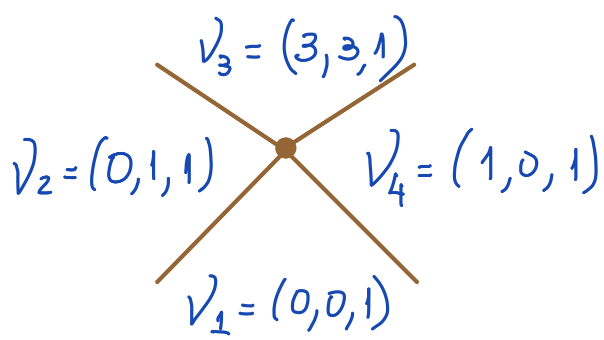
- Triangulation $\mathcal{T}$ of D

  $\rightsquigarrow$ (partial) resolution $X_{\mathcal{T}}$ of the toric isolated singularity at vertex of $S(M_D)$

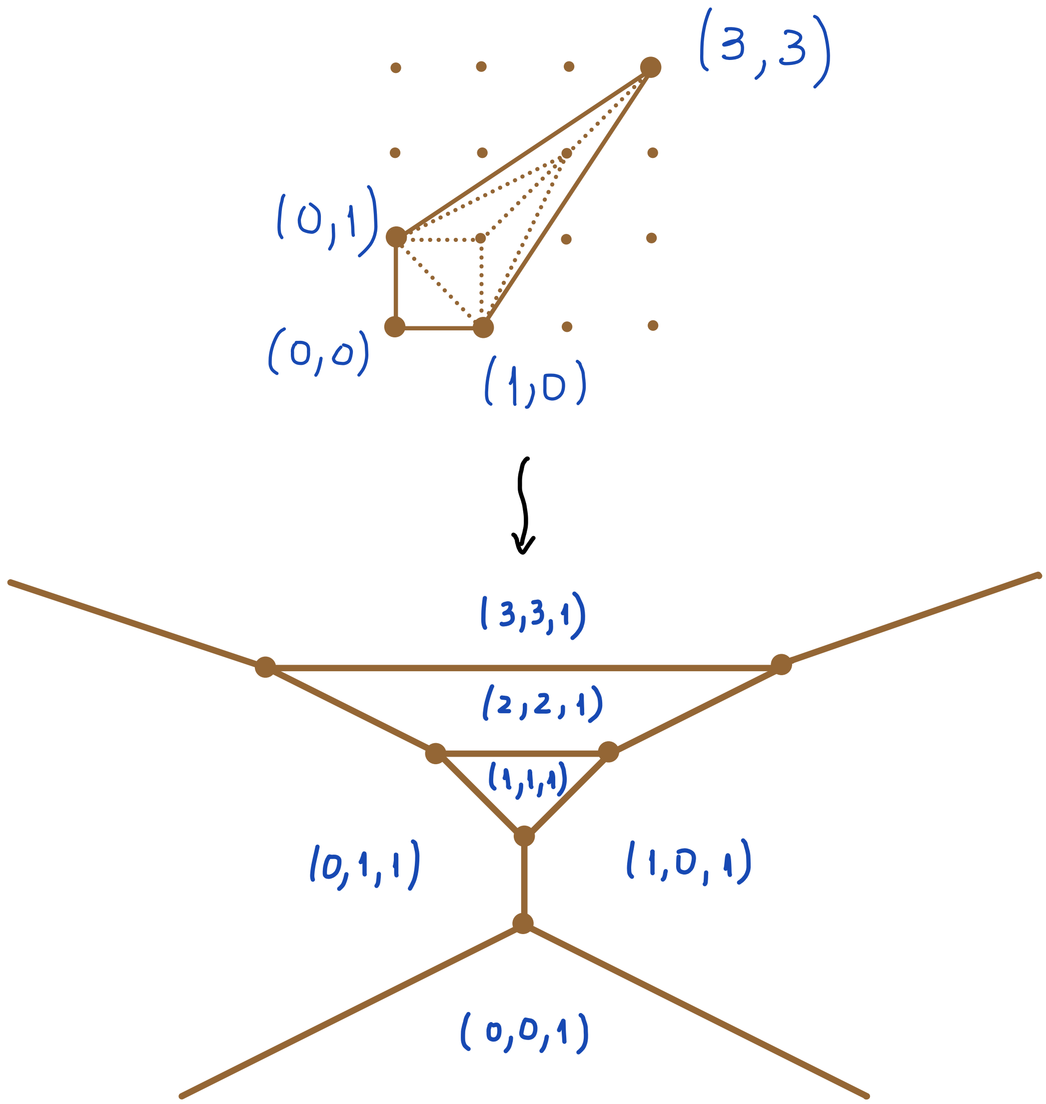
- Smooth example



$\rightsquigarrow$



$$(M_D, \xi_D) = (S^2 \times S^3, \xi_3), \qquad \boxed{m = 1}$$



- **Thm.** [Batyrev-Dais '96, Stapledon '08, AMM]

$$\dim H^*_{orb}(X_{\mathcal{I}};\mathbb{Q}) = \begin{cases} \delta_{mj}(D), & *=2j,\ j\in \frac{1}{m}\mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

$* \in \mathbb{Q}$

$$\boxed{H^*_{orb} = \text{Chen-Ruan cohomology}}$$

Cor.

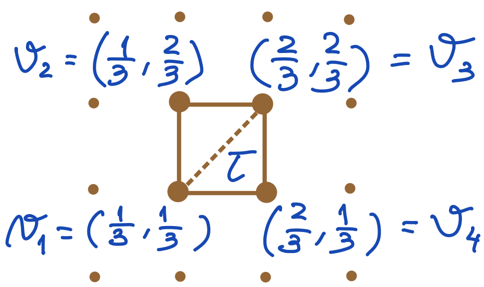
$$cb_{2j}(D) = \sum_{k\geq 0} \dim H^{2(n-j+k)}_{orb}(X_{\mathcal{I}};\mathbb{Q}),\ \forall j\in\mathbb{Q}.$$

- **Remark:** McLean-Ritter have similar result for isolated finite quotient singularities, which overlaps with this corollary when M_D is a Lens space, i.e D is a simplex.

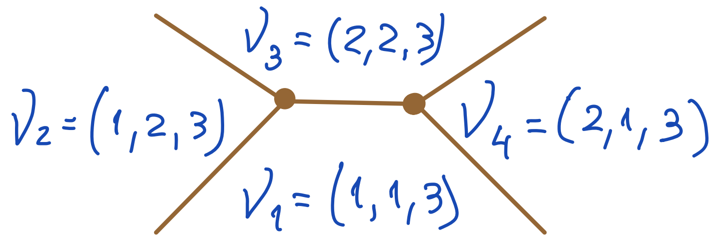
- **Example**

$$(M_D, \xi_D) = (S^2 \times S^3, \xi_1)/\mathbb{Z}_3 \cong \text{preq. of } (S^2 \times S^2, 3\sigma \times 3\sigma)$$

$$\boxed{m=3}$$



$\rightsquigarrow$



X_τ

$$\delta_4 = \delta_7 = \delta_2 = \delta_5 = \delta_0 = \delta_3 = 1$$

$$\delta_1 = \delta_8 = \delta_6 = 0$$

$$\Rightarrow H^*_{orb}(X_\tau ; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & * = 0, 4/3, 2, 8/3, 10/3, 14/3 \\ 0, & \text{otherwise} \end{cases}$$