Introducing *homotopy.io*: a proof assistant for geometrical higher category theory

Jamie Vicary
Department of Computer Science, University of Cambridge

With Nathan Corbyn, Lukas Heidemann, Nick Hu, David Reutter and Calin Tataru

Topological Quantum Field Theory Club
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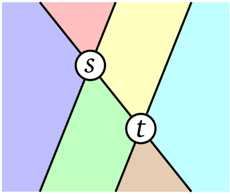
This is the mathematics of the 21st century.
We are only starting to glimpse what is possible.

Graphical calculus

It is conjectured that n-categories have an n-dimensional graphical calculus, which is the dual of the ordinary 'commutative diagrams'.



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 & B \arrow[dr] & & \\
A \arrow[ur, bend left] \arrow[dr, bend right] & & C \arrow[dr, bend left] & \\
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 & & f \arrow[uu, Rightarrow, "t"] \arrow[ur, bend right] &
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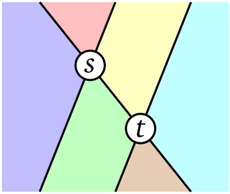

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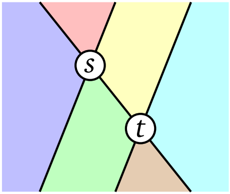

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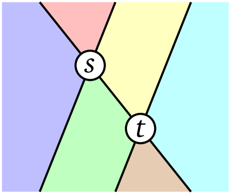

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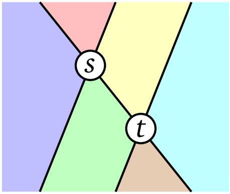

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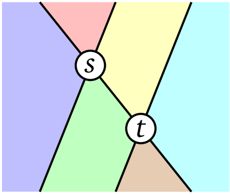

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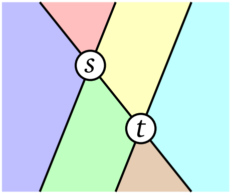

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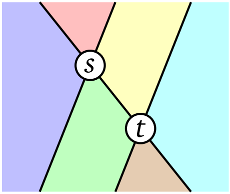

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- In higher dimensions, no formal theory has been developed.
- Nonetheless, regularly used as an informal language.

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A model of higher categories is *semistrict* if it is as strict as possible, while still allowing arbitrary homotopy. However, yields long proofs!

For proof construction, better to be *semiweak*: as weak as possible, except strictly associative and unital. Yields unique composites.

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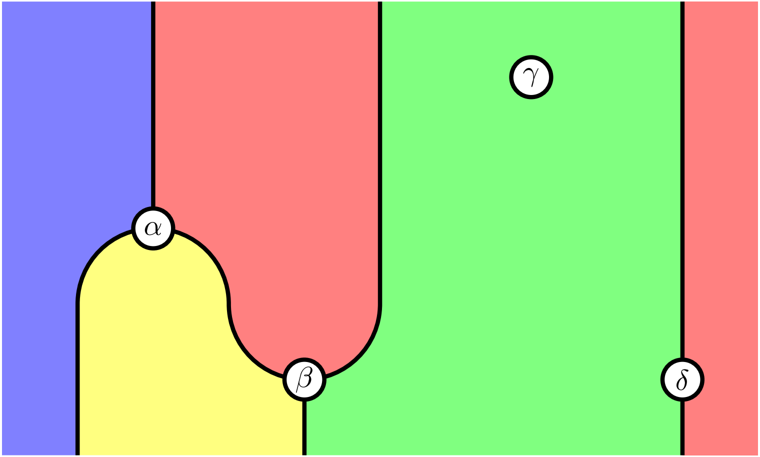
- High-level methods assist construction of complex homotopies.

Monotone functions

Consider the following string diagram in a bicategory.



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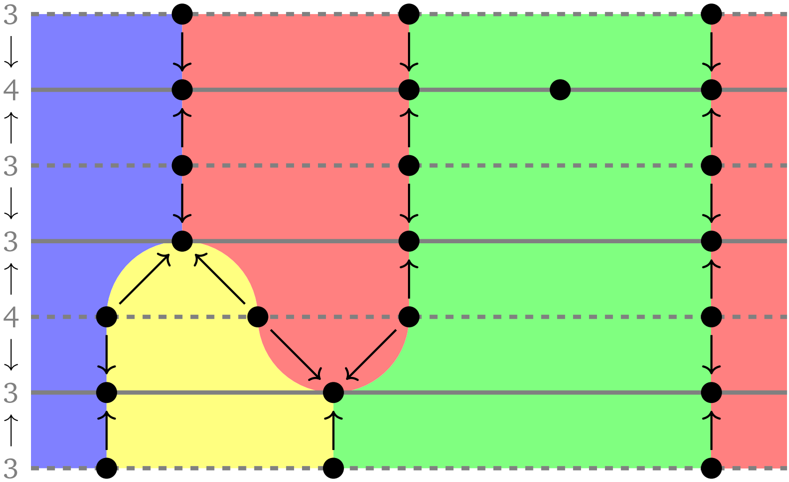
Consider the following string diagram in a bicategory.

It gives rise to an alternating sequence of monotone functions.



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Thus we obtain $S : Z_{\mathcal{C}} \to \Delta_+$ and $R : Z_{\mathcal{C}}^{\mathrm{op}} \to \Delta_=$.

Diagrams from zigzags

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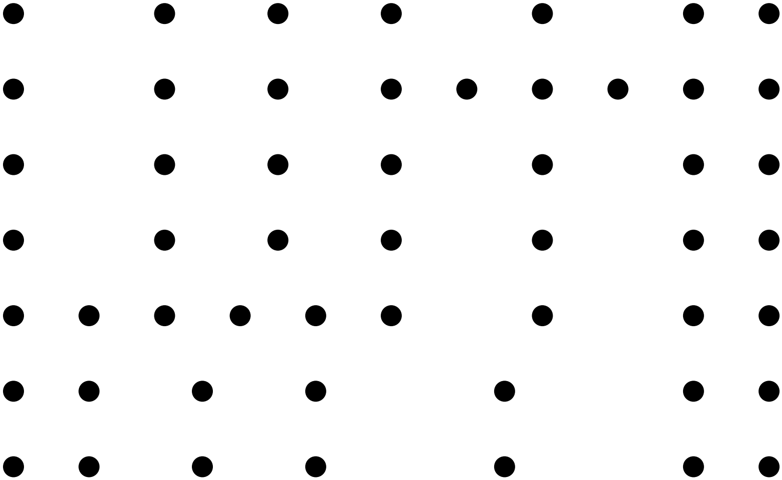
Here are 53 examples of 0-diagrams, all objects of 1:



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Here are 7 examples of 1-diagrams, all objects of $Z_1 = \Delta_+$:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \leftarrow \bullet$$

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet$$

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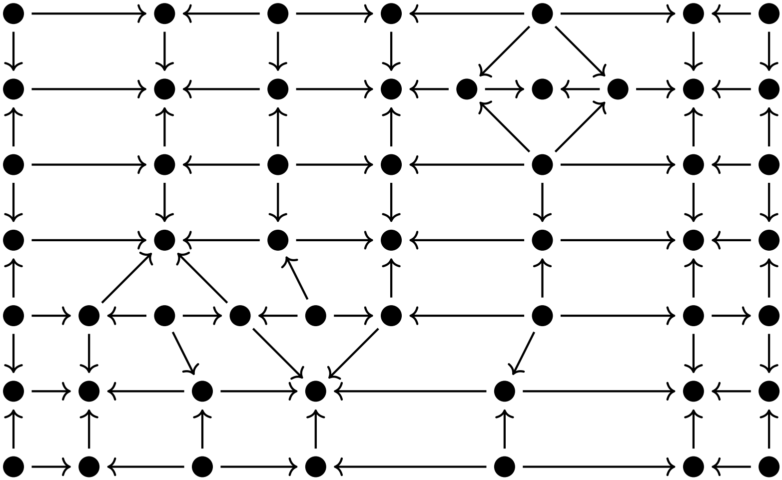
Here is 1 example of a 2-diagram, an object of Z_{Δ_+}:



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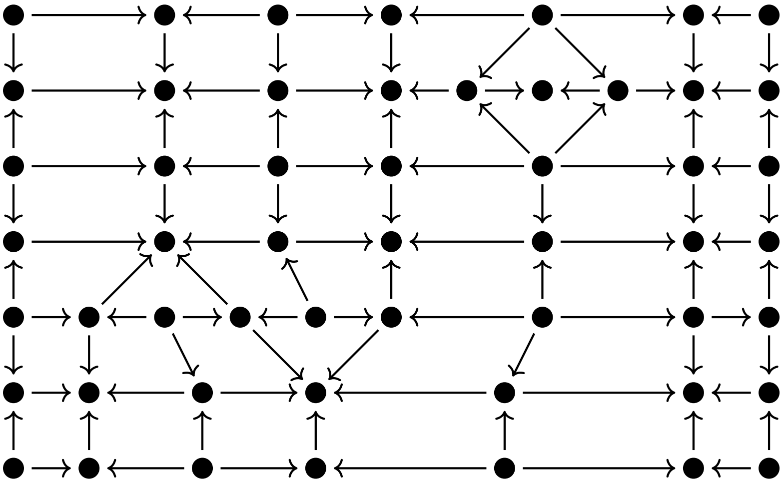

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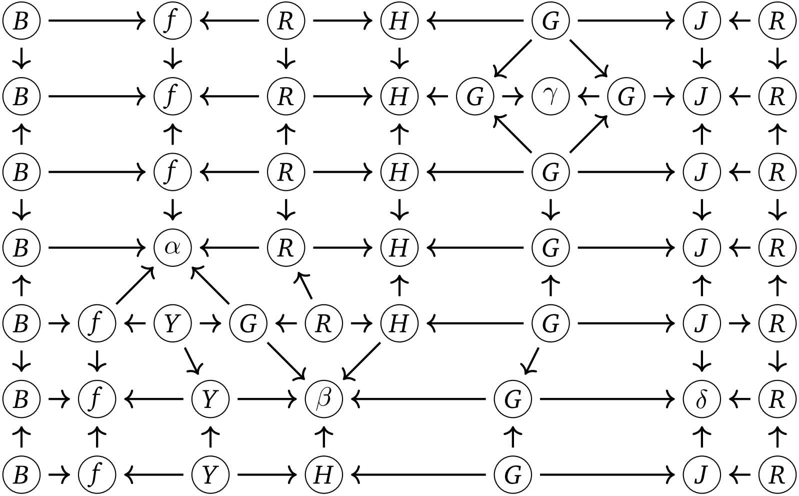

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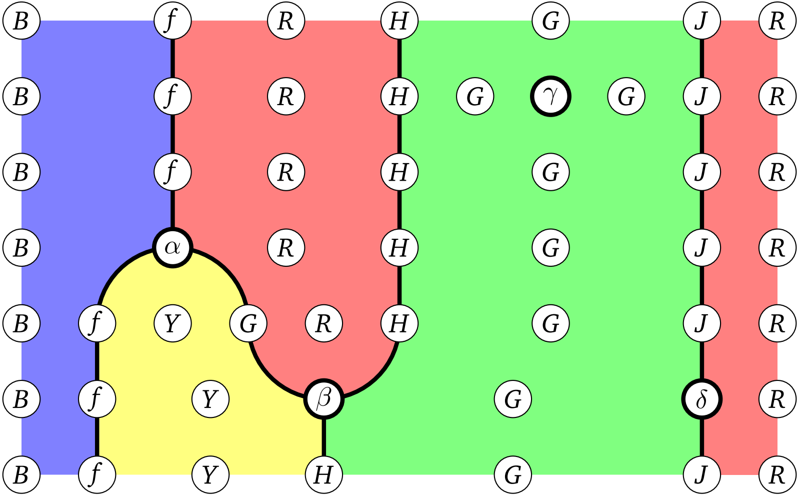

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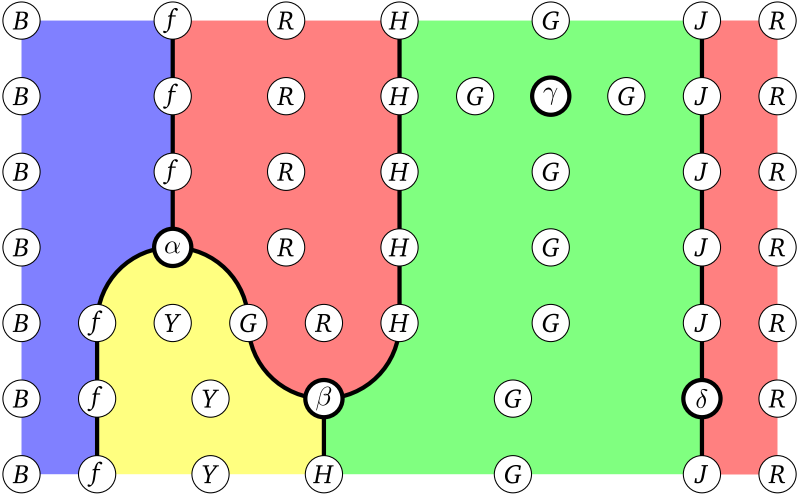

Type checking verifies that label neighbourhoods match definition.

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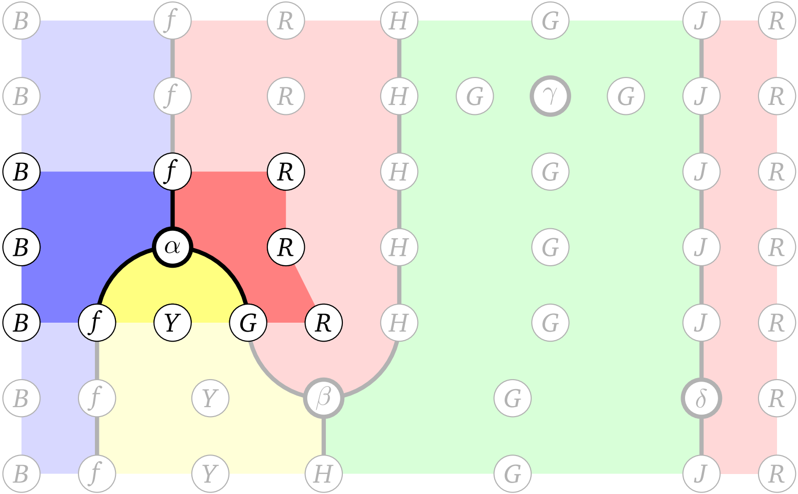

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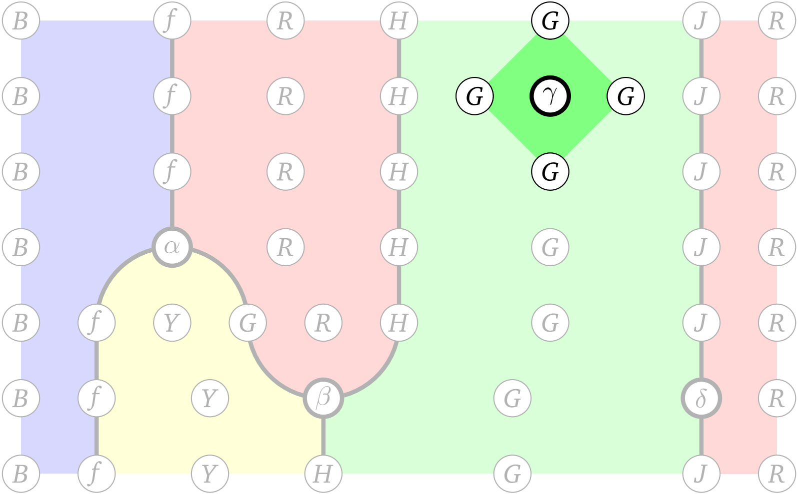

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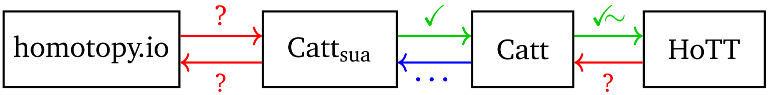

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Thanks for listening!

Type checking

Definition. Given an untyped n-diagram D, its *volume* $|D|$ is defined to be 1 if $n = 0$, or else we define $|D| = \sum_i |S_i|$, the sum of the volumes of the singular levels.

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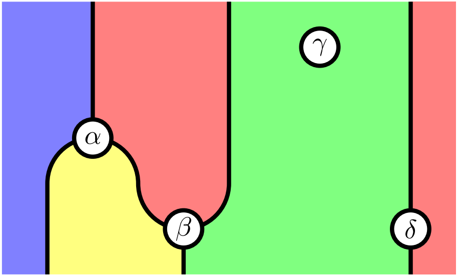
This 2-diagram has $|D| = 10$, meaning it has 10 "pieces":



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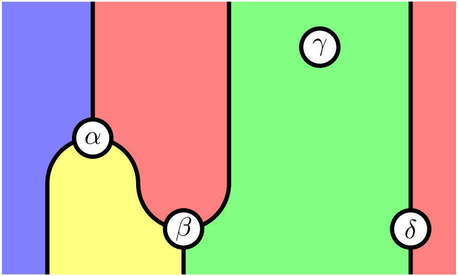

Essentially, type checking verifies that these "pieces" are either identities, or elements of the signature.

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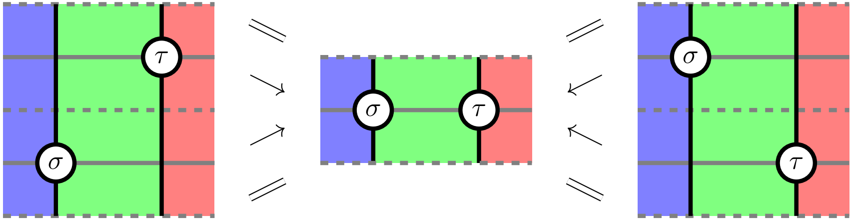

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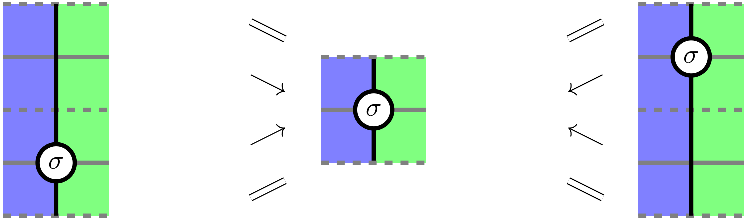

Here is the first piece. Note it has volume 1 as required.

It contains "unnecessary" identities on the left and right. As a result, it is not itself an identity.

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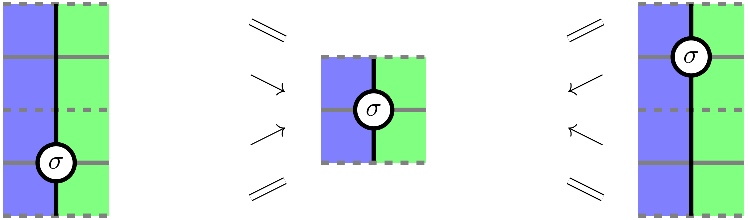

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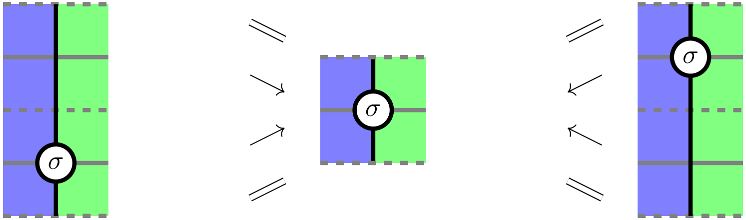

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In this way, homotopies come "built in".

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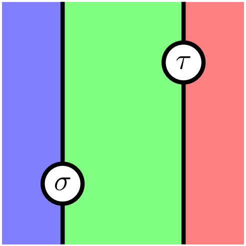

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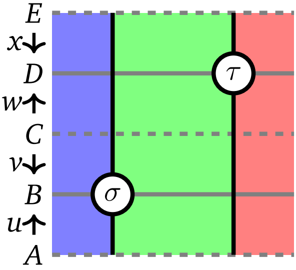

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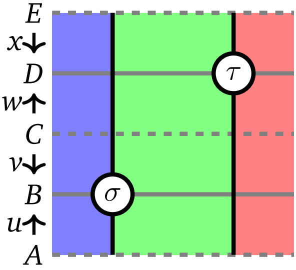

We build the contraction as a pushout of cospans in Z_1^n, and the homotopy itself as an associated zigzag map.

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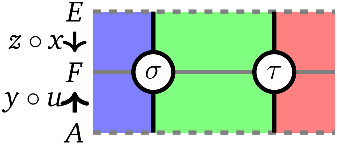

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Colimit algorithm

(With David Reutter.)

Theorem. For a category **C**, the following correctly constructs colimits in $Z_{\mathbf{C}}$, or correctly fails:

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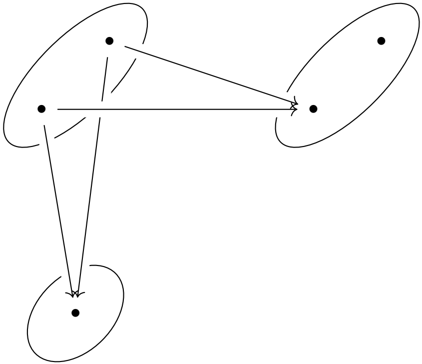

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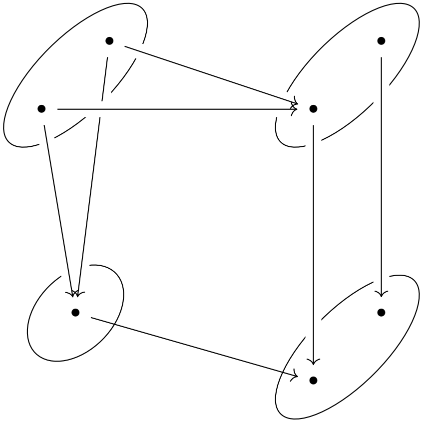

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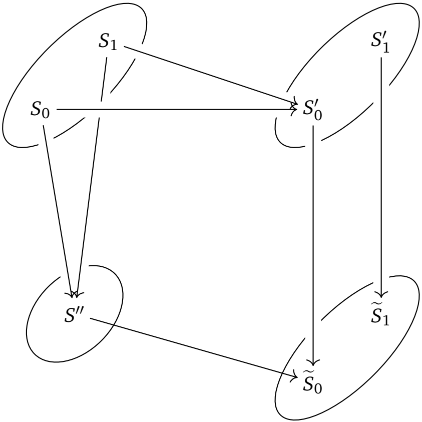

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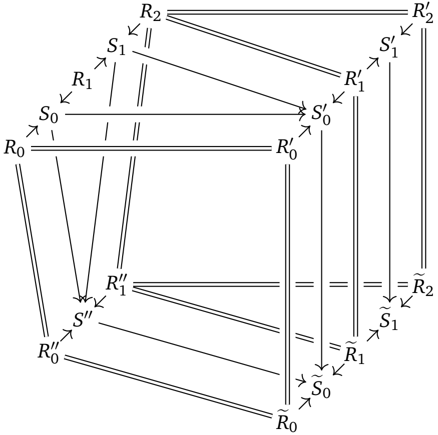

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