Exceptional super Lie algebras in twisted M-theory

w/ Raghavendran and Saberi

- There is a unique eleven-dim^{l} theory of supergravity. It's supposed to be the low energy limit of M-theory.
- Exceptional Lie algebras of E_n type are known to be "hidden symmetries" of dim^{l} reductions of 11d SUGRA.
- The goal of the talk is to explain the appearances of exceptional simple super Lie alg's present in the twist of 11d SUGRA (before dim^{l} reduction).

- **Twisted SUSY**: Given a supersymmetric theory and a supercharge Q w/ $Q^2 = 0$ can consider the twist.

Observables in Q-twisted thy is the Q-cohomology of original observables.

At the level of the twist, symmetries can become *enhanced*.

Ex: The 4d $N = 1,2,4$ superconformal algebra is finite dim^l. After twisting, the $N=1$ superconformal algebra enhances to the alg. of holomorphic vector fields on $\mathbb{C}^2 = \mathbb{R}^4$. [W., Saberi]

Determined by the supercharge Q.

Similar enhancements for twisted SUGRA.

① Twisted supergravity w/ Costello-Li

Fields in ordinary gravity consist, in part, of a metric g.

BRST : introduce ghosts for reparametrization invariance. Ghosts are vector fields X.

$$g \longmapsto g + L_X g .$$

- In SOGRA, there are fermionic fields.

The "gravitino" $\psi \in \Omega^1(M, S)$

spinor bundle.

Have linear gauge symmetry

$$\psi \longmapsto \psi + d\,Q(x)$$

where

$$Q(x) \in \Gamma(M, S)$$

is a fermionic ghost

$$\mathbb{Z} \times \mathbb{Z}/2 \xrightarrow{\text{Tot}} \mathbb{Z}/2$$

$\mathbb{Z}/2$ parity \ ghost # $\mathbb{Z}$	0	-1
even	g $+$	X_- $-$
odd	ψ $-$	$Q(x)$ $+$

Costello–Li : Twisted SUGRA is a background where $Q(x)$ takes a nonzero value $Q \in \Gamma(M, S) \overset{\text{constant}}{?} S|_0$

Must satisfy EOM for SUGRA, like

Killing $\nabla_g Q = 0$

$\mathbb{R}^n$ Q = constant spinor.

and $Q^2 = 0$, like an ordinary twist.

- Perturbatively, gravity described by some dg Lie (or L_∞) algebra

$\wedge$ super

$$\left(\mathfrak{g}_{grav},\ d,\ [-,-] \right)$$

$MC(\mathfrak{g}_{grav}) \rightsquigarrow$ EOM for supergravity thy.

Twist concretely deforms this algebra:

$$d \rightsquigarrow d + [Q,-]$$

If L_∞ then:

$$\ell_2 \rightsquigarrow \ell_2 + \ell_3(Q,-,-)$$

$\vdots$

etc.

• In 11d SUSY, there are essentially two classes of twisting supercharges.

Super-translations (cplxified)

$$\mathbb{C}^{11} \oplus \Pi S \ni Q$$

$so(11) \circlearrowright$ ... 32-dim'l spin rep'n.

$$[Q_1, Q_2] = \Gamma(Q_1, Q_2).$$

clifford mult.

$[Q, Q] = 0$:

- "minimal" twist. satisfies

For this talk.

$$\dim \operatorname{Im}([Q,-]) = 6$$

Stabilized by $SU(5) \subset SO(11)$.

$\operatorname{Im}[Q,-] \subset \mathbb{C}^{11}$

- "non-minimal"

$$\dim(\operatorname{Im}[Q,-]) = 9$$

Stabilized by $\underline{G_2 \times SU(2)}$.

- One approach to computing twists based on the "pure spinor" formalism.

(complexified)

$$\mathfrak{siso}_{d,N} = \text{super Poincaré algebra.}$$

$$Y_{d,N} = \left\{ x \in \mathfrak{siso}^{\text{odd}}_{d,N} \;\middle|\; x^2 = 0 \right\}$$

$$\circlearrowright \mathfrak{siso}_{d,N}$$

- The "nilpotence" variety. [Eager, Saberi, Walcher]

$$\begin{tikzcd}
\left\{ \text{Sheaves on } Y_{d,N} \right\} \arrow[r, squiggly, "\text{PSF}", "{[\text{Berkovits, Cederwall, } \ldots]}"'] \arrow[d, "?"] & \left\{ \text{SUSY multiplets} \right\} \arrow[d, "\text{twist}"] \\
\left\{ \text{Sheaves on } \tilde{Y}_{d,N} \right\} \arrow[r, squiggly] & \left\{ \text{twisted multiplets} \right\}
\end{tikzcd}$$

PSF roughly $\exists\ \mathcal{Y}$ sheaf on Y_d

$$C^\infty(\mathbb{R}^d) \otimes \Gamma(Y_d, \mathcal{Y})$$

$\circlearrowleft$ $\mathrm{Siso}_{d,N}$ $\circlearrowleft$ D_{univ}

Given $Q \in Y_{d,N}$, simpler algebra :

$$\mathrm{Siso}_d^Q = H^\bullet(\mathrm{siso}_{d,N}, [Q, -])$$

$$\tilde{Y}_d^Q = \{ x \in \mathrm{Siso}_{d,N}^Q \mid x^2 = 0 \}$$

Ex : $d = 6$, $N = (2, 0)$

Q = minimal supercharge .

$$\tilde{Y}^{Q}_{6} = \left\{ \begin{matrix} \text{rank } 1 \\ 2\times 3 \text{ matrices} \end{matrix} \right\} \subset \mathbb{C}^{2\times 3} = \mathbb{C}^{6}.$$

PS machine applied to

$$\tilde{\mathcal{F}} = \mathcal{O}_{Y^{Q}_{6,(2,0)}}$$

Gives the holomorphic twist of the exotic $N = (2,0)$ tensor multiplet.

(No Lagrangian description). [Saberi, W.]

<u>Ex</u>: $d = 11$, $N = 1$. Q = minimal.

$$\tilde{Y}^{Q}_{11,1} = Gr(2,5).$$

$$\mathcal{F} = \mathcal{O}_{\tilde{Y}}$$

hol. twist of 11d SUGRA. [Saberi, W.]

• The **minimal** twist of 11d SUGRA is defined

on $X \times S$

X: CY5, $X = \mathbb{C}^5$ for now; S: smooth 1-mfld, $S = \mathbb{R}$ for now.

The **linear BV** complex of fields is linear EOM, linear gauge symmetries.

odd / even

$$\mathrm{PV}^{1,\bullet}(\mathbb{C}^5) \otimes \Omega^\bullet(\mathbb{R}) \xrightarrow{\partial_\Omega} \mathrm{PV}^{0,\bullet}(\mathbb{C}^5) \otimes \Omega^\bullet(\mathbb{R})$$

μ ; ν

$\mathrm{PV}^{1,\bullet}(\mathbb{C}^5)$: hol. vf's ; $\mathrm{PV}^{0,\bullet}(\mathbb{C}^5) = \Omega^{0,\bullet}(\mathbb{C}^5)$

$$\Omega^{0,\bullet}(\mathbb{C}^5) \otimes \Omega^\bullet(\mathbb{R}) \xrightarrow{\partial} \Omega^{1,\bullet}(\mathbb{C}^5) \otimes \Omega^\bullet(\mathbb{R}).$$

β ; γ

$\Omega^{0,\bullet}$: Dolbeault cplx ; $\bar\partial + d_{\mathbb{R}}$.

Some comments:

• $\mu \in \mathrm{PV}^{1,1} \otimes \Omega^0$ is a component of the metric.

$\mu(z,\bar z)\frac{\partial}{\partial z_i} d\bar z_j$

- $\beta \in \Omega^{0,3} \otimes \Omega^{0}, \Omega^{0,2} \otimes \Omega^{1}$

  $\gamma \in \Omega^{1,2} \otimes \Omega^{0}, \Omega^{1,1} \otimes \Omega^{1}$

  are components of the supergravity/higher CS field $C \in \Omega^{3}(\mathbb{R}^{11})$.

Linearized gauge transformations + EOM. encoded in the complex above.

- Propose the following interacting BV theory. [Raghavendran, Saberi, W.]

$$\underbrace{S_{free}(\beta, \gamma ; \mu, \nu)}_{\text{cplx above}} + I(\gamma ; \mu, \nu)$$

$$I = \frac{1}{2}\int_{\mathbb{C}^5\times\mathbb{R}} \frac{1}{1-\nu}\mu^2 \partial \gamma \qquad \gamma \in \Omega^{1,*}_{\mathbb{C}^5} \otimes \Omega^*_{\mathbb{R}}.$$

$$+ \frac{1}{6}\int_{\mathbb{C}^5\times\mathbb{R}} \underline{\gamma \partial \gamma \partial \gamma}.$$

First line understood as fml series

$$\frac{1}{1-\nu}\mu^2 \partial\gamma = \mu^2\partial\gamma + \nu\mu^2\partial\gamma + \cdots$$

<u>Thm</u>: $S_{free} + I$ satisfies the <u>CME</u>

$$\{S_{free} + I, S_{free} + I\} = 0.$$

- <u>Note</u>: Last line is precisely the holomorphic analogue of the CS action

$$\int \underline{c\, dc\, dc}$$

c = connection in ghost grade.

- First term is of "BF" type.

It endows

$$\text{Divergence-free w/ v.f's} \simeq \underline{\overset{0}{\mathrm{PV}^{1,\bullet}}} \xrightarrow{\partial_\Omega} \underline{\overset{1}{\mathrm{PV}^{0,\bullet}}}$$

divergence

w/ an L_∞ structure. This L_∞ structure is homotopy equivalent to the standard one.

- Numerous checks:
  - Dim^l reduction to 10d agrees w/ Costello-Li's description of twist of type IIA SUGRA using top^l strings and Kodaira-Spencer gravity.

- Compactification along CY3 is twist of 5d SUGRA coupled to appropriate gauge theory.

- Character of local op's $\Leftarrow\Rightarrow$ fact. homology over $S^{10} \times S^1$ returns M-theory index.

- Holographic checks ...

Thm: Thy admits a 1-loop quantization by work w/ Rabinovich and Gwilliam.

Higher loop anomalies ??

② Exceptional super Lie algebras.

Kac classified simple ∞-dim$^{\text{e}}$ super Lie algebras.

Of which, there are a few exceptional series. Examples of interest:

- $E(5,10) =$

div free

<u>even</u>	<u>odd</u>
$\mathrm{Vect}_0(\mathbb{C}^5)$	$\Omega^{2,cl}(\mathbb{C}^5)$
μ	α

$$[\mu,\mu'] = L_\mu \mu' \ , \ [\mu,\alpha] = L_\mu \alpha .$$

$$[\alpha,\alpha'] = \Omega^{-1} \vee (\underline{\alpha} \wedge \underline{\alpha'}) \in \mathrm{Vect}.$$

• $E(3,6) =$

<u>even</u> $\quad \mathrm{Vect}(\mathbb{C}^3) \oplus \mathcal{O}(\mathbb{C}^3) \otimes \mathfrak{sl}(2)$

$\quad\quad \mu \quad\quad\quad\quad f, e, h$

<u>odd</u> $\quad \Omega^{-\frac{1}{2}}(\mathbb{C}^3) \otimes R$

$\quad\quad\quad \omega$

where:

• $\Omega^{-\frac{1}{2}} = \Gamma\left(\mathbb{C}^3, K_{\mathbb{C}^3}^{-\frac{1}{2}} \otimes T^*_{\mathbb{C}^3}\right).$

• $R \cong \mathbb{C}^2 \circlearrowleft \mathfrak{sl}(2).$

Interesting brackets:

$$\left(\Omega^{-\frac{1}{2}} \otimes R\right)^{\otimes 2} \twoheadrightarrow \Omega^{-1} \otimes \wedge^2 R \cong T$$

$$\left(\Omega^{-\frac{1}{2}} \otimes R\right)^{\otimes 2} \twoheadrightarrow \Omega^{-\frac{1}{2}} \otimes \Omega^{\frac{1}{2}} \otimes S^2 R$$

$\mathcal{O} \otimes sl(2)$.

{ completely encoded by the action fnl.

<u>Thm</u>: The L_∞ algebra of twisted 11d SUGRA on $\mathbb{C}^5 \times \mathbb{R}$ is L_∞ equiv to a L_∞ central extension

$$\mathbb{C} \longrightarrow \widehat{E(5,10)} \longrightarrow E(5,10)$$

even   Lie 2-algebra

Defined by cocycle

$$\phi(\underset{\text{vf's}}{\bar\mu}, \underset{}{\bar\mu'}, \underset{\Omega^{2,cl}}{\underline{\alpha}}) = \langle \underline{\mu \wedge \mu'}, \underline{\alpha} \rangle \Big|_{z=0} \in \mathbb{C}.$$

<u>Idea</u>: Relationship of μ fields obvious.

The closed two-form $\alpha = \partial \gamma$

$$\Big(\underset{\beta}{\overset{\text{even}}{\Omega^0}} \xrightarrow{\partial} \underset{\gamma}{\overset{\text{odd}}{\Omega^1}} \Big) \xrightarrow{\partial} \underset{\alpha}{\Omega^{2,cl}}$$

$$\uparrow 2$$

$$\mathbb{C} \oplus \Omega^1 / d \simeq \Omega^{2,cl} .$$

Rmk: This extension is compatible with the M2 brane extension of the 11-dim'l super Poincaré alg of Baez, Huerta, Sati, Schreiber, ...

$$\mathbb{C}[2] \longrightarrow \text{m2 brane} \longrightarrow \mathfrak{siso}_{11d}$$

} Q-twist

Q-cohomology

$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, equal] & \text{M2 brane}^{Q} \arrow[r] \arrow[d] & \text{SiSo}^{Q}_{11d} \arrow[d] \\
\mathbb{C} \arrow[r] & \widehat{E(5,10)} \arrow[r] & E(5,10)
\end{tikzcd}
$$

- Why does this simple algebra appear?

- Other simple super Lie alg's appear in M-thry.

$E(3,6)$ = ∞-dim'l enhancement of 6d superconformal algebra

single M5 brane (irreducible!)

single M5 brane have description of the minimal twist

③ Higher Virasoro symmetry.

The famous 6d thy is superconformal.

But superconformal algebras are all finite dim^{l} in higher dimensions (≥ 3).

Main idea: There are infinite-dim^{l} enhancements of derived superconformal algebras. I'll focus on the twisted case.

One perspective: M-thy on

$$\mathrm{AdS}_7 \times S^4$$

is holographically dual to the 6d superconformal thy.

Matching symmetries, this means that the SUCO algebra should act on the 11d thy.

At the twisted level

$$\mathrm{SUCA}^{Q}_{6d} \simeq osp(6|1).$$

whose even part is

$$sl(4) \times sl(2)$$

Recall, part of fields of 11d thy are divergence free vf's on $\mathbb{C}^5$ which are constant along $\mathbb{R}$. The twisted version of AdS geometry comes from coupling M5 branes

$$\mathbb{C}^3_z \times 0 \times 0 \subset \mathbb{C}^3_z \times \mathbb{C}^2_w \times \mathbb{R}.$$

- $sl(4)$ acts by conformal symmetries* of $\mathbb{C}^3_z$.

  (These aren't quite div. free. Eg

$$\sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i} - \frac{3}{2} \sum_{a=1}^{2} w_a \frac{\partial}{\partial w_a})$$

- $sl(2)$ rotates $\mathbb{C}^2_w$.

This defines embedding

$$osp(6|1) \hookrightarrow E(5,10).$$

__Claim__: This embedding factors

$$
\begin{tikzcd}
\mathfrak{osp}(6|1) \arrow[rr, hook] \arrow[dr, hook] & & E(5,10) \\
& E(3,6) \arrow[ur, hook] &
\end{tikzcd}
$$

__even__ $\quad \mathrm{Vect}(\mathbb{C}^3) \oplus \mathcal{O}_{\mathbb{C}^3} \otimes \mathfrak{sl}(2) \hookrightarrow E(3,6)$.

__Upshot__: $E(3,6)$ is an ∞-dim'l enhancement of the twisted 6d SUCO algebra.

__Thm__: The twisted $\mathcal{N} = (2,0)$ theory defines an irreducible module for $E(3,6)$.

Remarks

- This thm admits a lift to factorization algebras on $\mathbb{C}^3$. Exists on any 3-fold X, in fact.

- MC eqn on X:

$$\omega \wedge \partial \omega = 0 \quad , \omega \in \Omega^{1,0}(X)$$

1) If $X = \Sigma \times M^4$, $\omega \in \Omega^{1,0}(\Sigma)$

then ω-twist is

$$\simeq \underset{\text{on } \Sigma}{\text{Virasoro}} \otimes H^{\cdot}_{dR}(M^4).$$

2) If $X = \Sigma \times \mathbb{C}^2$ and

$\omega \in \Omega^{1,0}(\Sigma)$, $\partial\omega \neq 0$

then ω-twist is $\simeq$ to Virasoro on Σ, (and is trivial away from $\Sigma \times \{0\}$).

Soln's to some EOM w/ gauge symmetries

|| perturbatively

formal moduli spaces.

|| deformation

$B\mathfrak{g}$

In BV $B\mathfrak{g}$ is equipped w/ some shifted symplectic structure.

$$\mathfrak{g} \simeq \widehat{E(5,10)}$$

on $\mathbb{C}^5 \times \mathbb{R}$.

gauge symmetries in twisted SUGRA 4.

$$\Omega^{0,\bullet}(\mathbb{C}^5) \otimes \Omega^\bullet(\mathbb{R})$$

$$\simeq \Omega^{0,\bullet}(\mathbb{C}^5) \simeq \mathcal{O}^{hol}(\mathbb{C}^5).$$

Poincare

M5 brane worldvolume thy

$\wr$

6d superconformal thy.

Not a Lagrangian thy.

w/ Ingmar we described the

twist of a single M5 brane

= "abelian" 6d (2,0) thy.

$\simeq$ X cplx 3-fold

Intermediate Jacobian of X

+ fermionic stuff ....

$$\circlearrowleft$$

$E(3,6)$.

$X = \mathbb{C}^3 \rightsquigarrow$ module $\underline{\underline{E(3,6)}}$.

$$\parallel$$

$\Sigma \times M^4$

opers on Σ.