On the Fukaya-Morse A_∞ category

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Plan

1. Fukaya-Morse A_∞ category ($+$"enhancement")
2. Picture 1: effective field theory (second quantization) ✗
3. Picture 2: HTQM (first quantization approach)

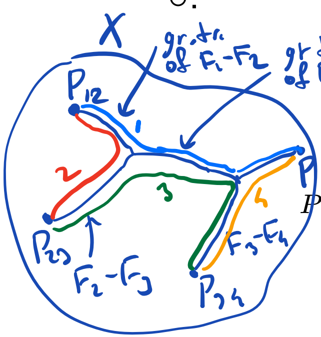
Fukaya A_∞ category and Morse degeneration

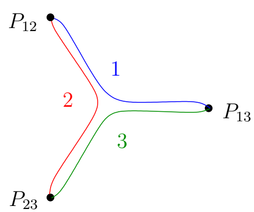
Fix X – a Riemannian manifold

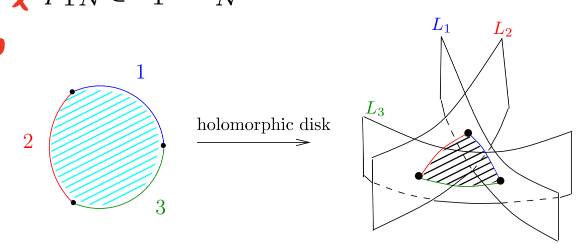
Fukaya '93
Fukaya-Oh '97
Kontsevich-Soibelman '00

	Fukaya-Morse A_∞ category on X	Fukaya A_∞ category on T^*X
Ob:	$F_1, \ldots, F_N$	Lagrangians $L_a = \text{graph}(\epsilon\, dF_a)$
Mor:	$\text{Mor}(F_a, F_b) = \text{Span}_{\mathbb{Z}}(\text{Crit}(F_a - F_b))$	$\text{Mor}(L_a, L_b) = \text{Span}_{\mathbb{Z}}$(intersection points of L_a, L_b) $d(F_a - F_b) = 0$
$\circ$:	$P_{i,i+1} \in \text{Crit}(F_i - F_{i+1})$ $m(P_{12}, P_{23}, \ldots, P_{N-1\,N}) = \sum_{P_{1N} \in \text{Crit}(F_1 - F_N)} \#\mathcal{M}^{\text{trees}}[P_{1N}]$	$p_{i,i+1} \in L_i \cap L_{i+1}$ $L_1 \to L_2 \to L_3 \quad L_{N-1} \to L_N$ $m(p_{12}, p_{23}, \ldots, p_{N-1\,N}) = \sum_{p_{1N} \in L_1 \cap L_N} \#\mathcal{M}^{\text{hol.disks}}[p_{1N}]$

$X \leftarrow T^*X$
$\epsilon \to 0$







Enhancement

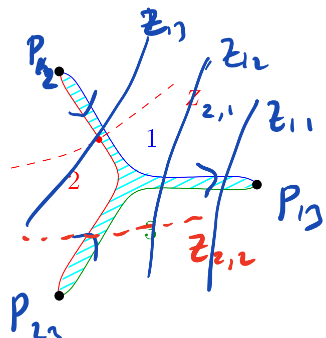
Enhancement: $\mathrm{Mor}(F_a, F_a) = C^{\text{sing}}_\bullet(X)$

Composition maps:

$F_1 \xrightarrow{Z_{1,1}} F_1 \xrightarrow{Z_{1,2}} \cdots F_1 \xrightarrow{P_{12}} F_2 \xrightarrow{Z_{21}} F_2 \xrightarrow{Z_{22}} F_2 \xrightarrow{P_{23}} F_3 \quad \cdots \quad \to F_{N-1} \xrightarrow{P_{N-1\,N}} F_N \to \cdots \to F_N$

$$m\colon \mathrm{Mor}(F_1, F_1)^{\otimes k_1} \otimes \mathrm{Mor}(F_1, F_2) \otimes \cdots \mathrm{Mor}(F_{N-1}, F_N) \otimes \mathrm{Mor}(F_N, F_N)^{\otimes k_N} \to \mathrm{Mor}(F_1, F_N)$$

$$m(\{Z_{1,\alpha}\}, P_{12}, \{Z_{2,\alpha}\}, \ldots, P_{N-1\,N}, \{Z_{N,\alpha}\}) = \sum_{P_{1N}} \#\mathcal{M} \cdot [P_{1N}]$$



- differentials: $m(Z) = \partial Z$, $m(P_{12}) = d_{\text{Morse}} P_{12}$
- $m(Z_1, Z_2) = Z_1 \cap Z_2$
- $m(Z_1, \ldots, Z_n) = 0$, $n \geq 3$

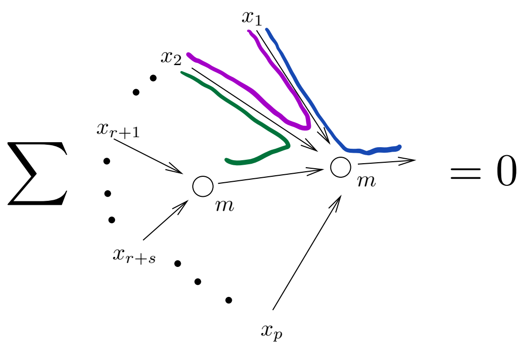
A_∞ relations

$$F_{a_1} \xrightarrow{x_1} F_{a_2} \xrightarrow{x_2} F_{a_3} \xrightarrow{x_3} \cdots \xrightarrow{x_p} F_{a_{p+1}}$$

Let $x_i \in \mathrm{Mor}(F_{a_i}, F_{b_i})$, $i = 1 \ldots p$, with $b_i = a_{i+1}$ - composable sequence of morphisms. Then:

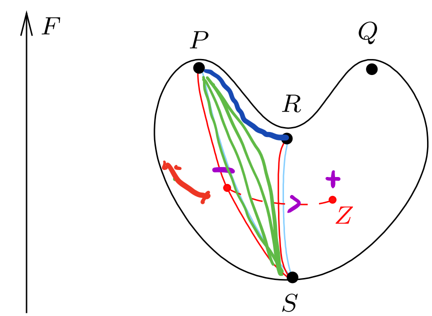
$$\sum_{r,s} m(x_1, \ldots, x_r, m(x_{r+1}, \ldots, x_{r+s}), x_{r+s+1}, \ldots, x_p) = 0$$

Or:



Example of an A_∞ relation: heart-shaped sphere

Example: $X = S^2$, $N = 2$ functions, $F = F_1 - F_2$.



$F_1 \xrightarrow{P} F_2 \xrightarrow{Z} F_2$

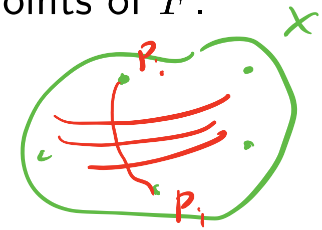
$$\overbrace{m(\underbrace{m(P)}_{R}, Z)}^{S} + \underbrace{m(P, m(Z))}_{-S} + m(\underbrace{m(P, Z)}_{0}) = 0$$

A_∞ relation: $x_1 = P \in \mathrm{Mor}(F_1, F_2)$, $x_2 = Z \in \mathrm{Mor}(F_2, F_2)$.

$$\boxed{m(d_{\mathrm{Morse}}(P), Z) + m(P, \partial Z) + d_{\mathrm{Morse}} m(P, Z) = 0}$$

Example: deformation of Morse differential by cycles

$N=2$, $F=F_1-F_2$. Fix $\{C_\alpha\}$ -cycles on X. $\{P_i\}$ – crit. points of F.



Generating function for compositions

$$m\colon \mathrm{Mor}(F_1,F_2)\otimes \mathrm{Mor}(F_2,F_2)^{\otimes k}\to \mathrm{Mor}(F_1,F_2),$$

$F_1 \xrightarrow[P_i]{} F_2 \xrightarrow[C]{} F_2 \to$ $\qquad C^{F_2}$

$$m_i^j(T)=\sum_{k\geq 1}\sum_{\alpha_1,\ldots,\alpha_k}\underbrace{\#\mathcal{M}(P_i,C_{\alpha_1},\ldots,C_{\alpha_k},P_j)}_{\#\text{grad traj } P_i\to P_j \text{ passing through cycles}}\quad T_{\alpha_1}\cdots T_{\alpha_k}$$

T_α - generating parameters, $|T_\alpha|=1-\mathrm{codim}C_\alpha$.

A_∞ relations $\Rightarrow$ $\boxed{(d_{\mathrm{Morse}}+m(T))^2=0}$.

$d_{Morse}+m(T) \circlearrowright MC(F)[[T]]$

Explanation from HPT:

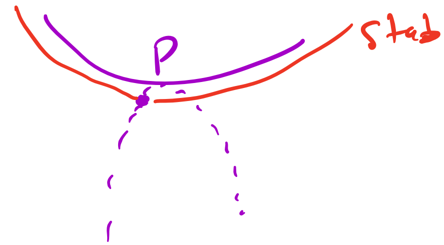
$G = L_v + \varepsilon d^*$

$H = L_v + \varepsilon \Delta$

$$\begin{array}{c} K \circlearrowright \quad \Omega^\bullet(X),\ \overset{d_1}{d} + \overset{d_{pert} = d_2}{\sum_\alpha T_\alpha \delta_{C_\alpha}\wedge} \\ p\big\downarrow \quad \big\uparrow i \\ MC(X,F)\ ,\ d_{\mathrm{Morse}} + m(T) \end{array}$$

$p d_2 i - p d_2 K d_2 i + p d_2 K d_2 K d_2 i - \cdots -$

Morse contraction



- $i\colon \underset{\text{crit. point}}{P} \mapsto \delta_{\mathrm{Unstab}_P}$
- $p\colon \omega \mapsto \sum_P \left(\int_X \omega \wedge \delta_{\mathrm{Stab}_P} \right) \cdot [P]$
- $K = \int_0^\infty dt\, \iota_v\, e^{-t\mathcal{L}_v} : \quad \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$.

  v – gradient vector field.

  Integral kernel: $\delta_Y \in \Omega_{\mathrm{distr}}(X \times X)$;

  $Y = \{(x,y) \mid x = \mathrm{Flow}_t(v) \circ y \text{ for some } t > 0\}$

Picture 1a: homotopy transfer

$$\underline{K} \circlearrowright \quad V = \Omega^\bullet(X) \otimes \mathrm{Mat}_{N\times N} = \bigoplus_{a,b=1}^{N} \Omega^\bullet_{ab}(X) \quad \text{– dg algebra}$$

MC(F_a-F_b)

Ω_{aa}

$$\underline{p}\downarrow \qquad \uparrow\underline{i}$$

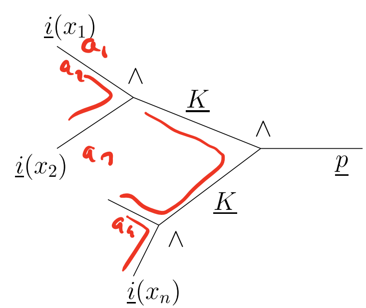
$$\mathbb{M} = \bigoplus_{a,b} \mathbb{M}_{ab}, \qquad \mathbb{M}_{ab} = \begin{cases} MC(F_a - F_b), & a \neq b \\ \Omega^\bullet_{aa}, & a = b \end{cases}$$

$$\underline{i}, \underline{p}, \underline{K} = \begin{cases} \text{Morse contraction for } F_a - F_b, & a \neq b \\ \text{trivial } (\underline{i} = \underline{p} = \mathrm{id}, \underline{K} = 0), & a = b \end{cases}$$

Induced A_∞ algebra structure on $\mathbb{M}$:

Kadeishvili
-Kontsevich-Soibelman

$$m_n(x_1, \ldots, x_n) = \sum$$



inputs: $x_i \in \underset{\text{with } b_i = a_{i+1}}{\mathbb{M}_{a_i b_i}} = \begin{cases} \text{Morse chain}, & a_i \neq b_i \\ \text{form/sing. chain } \delta_Z, & a_i = b_i \end{cases}$

Picture 1b: effective action

dgL

BF theory: $S = \int_X \langle B \overset{\wedge}{,} dA + \frac{1}{2}[A,A]\rangle$

Fields:

- $A \quad \in \quad \Omega^\bullet(X) \otimes \underbrace{(\mathrm{Mat}_{N\times N} \otimes \mathbb{A})}_{\mathfrak{g}}[1] \quad = \quad \bigoplus_{a,b} \Omega_{ab} \otimes \mathbb{A}[1]$

  where $\mathbb{A}$ = upper-triangular $\widetilde{N} \times \widetilde{N}$ matrices.
- $B \quad \in \quad \Omega^\bullet(X) \otimes \mathfrak{g}^*[d-2]$

Next: integrate out off-diagonal components A_{ab}, $a \neq b$ subject to gauge-fixing

$$\boxed{\iota_{v_{ab}} A_{ab} = 0}$$

– axial gauge but in different directions for different components! We induce the effective action on diagonal fields + remnants of off-diagonal fields.

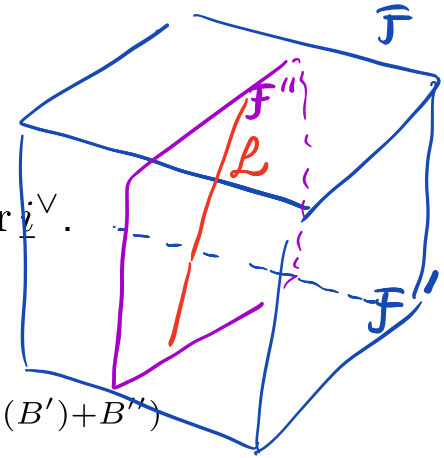
Picture 1b: effective action cont'd

More explicitly:

BV pushforward: $\underset{\text{fields}}{\mathcal{F}} \to \underset{\text{IR fields}}{\mathcal{F}'} = T^*[-1](\mathbb{M} \otimes \mathbb{A}[1])$

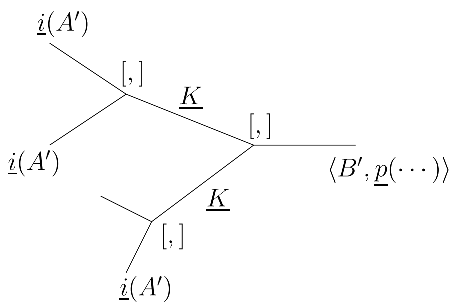
Splitting of fields: $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ with $\mathcal{F}'' = \ker \underline{p} \oplus \ker \underline{i}^\vee$.

Gauge-fixing Lagrangian: $\mathcal{L} = \mathrm{im}(K) \oplus \mathrm{im}(K^\vee) \subset \mathcal{F}''$



$$e^{\frac{i}{\hbar} S_{\mathrm{eff}}(A',B')} = \int_{\mathcal{L}} \mathcal{D}A''\, \mathcal{D}B''\, e^{\frac{i}{\hbar} S(\underline{i}(A')+A'',\underline{p}^\vee(B')+B'')}$$

$S_{\mathrm{eff}} = \sum_{\text{trees}}$



$= \sum_{n \geq 1} \frac{1}{n!} \langle B', l_n(A', \ldots, A')\rangle$

$\{l_n\}$ – L_∞ algebra operations on $\mathbb{M} \otimes \mathbb{A}$.

L_∞ relations $\Leftrightarrow$ BV master equation $\{S_{\mathrm{eff}}, S_{\mathrm{eff}}\} = 0$

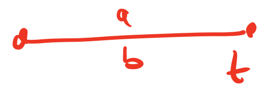
From L_∞ back to A_∞

Recovering A_∞ products on $\mathbb{M}$:

$$l_n(x_1 \otimes t_{12}, \dots, x_n \otimes t_{n\,n+1}) = m_n(x_1, \dots, x_n) \otimes t_{1\,n+1}$$

$t_{ij} \in \mathbb{A}$ matrix with (i,j)-entry 1 and all other entries 0.

Picture 2: HTQM



Topological quantum mechanics:

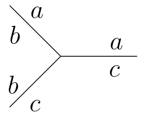
- Space of states: $\mathcal{H}_{ab} = \Omega^\bullet(X)$
  (for a particle of (a, b)-type, $a \neq b$).
- BRST operator $Q = d$.
- Hamiltonian $H = \mathcal{L}_{v_{ab}} = [Q, G]_+$.
- $G = \iota_{v_{ab}}$.
- Evolution operator (superpropagator):
  $U(t, dt) = e^{-t\,H - dt\,G} \quad \in \quad \Omega^\bullet(\mathbb{R}_+) \otimes \mathrm{End}(\mathcal{H}_{ab})$

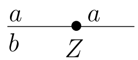
$N=2$ SUSY QM

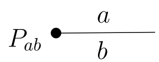
$(d_t + ad_Q)\, U = 0$

HTQM on metric trees

HTQM on metric trees:

- 3-valent vertex 

$\sim$ $\mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \xrightarrow{\wedge} \mathcal{H}_{ac}$
- 2-valent vertex 

$\sim$ operator $\mathcal{H}_{ab} \xrightarrow{\wedge \delta_Z} \mathcal{H}_{ab}$
- 1-valent vertex 

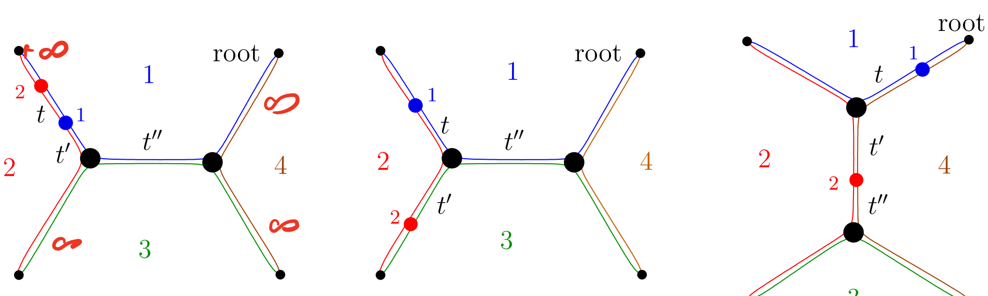
$\sim$ state $\delta_{\mathrm{Unstab}_{P_{ab}}}$
- (a,b)-edge of length t $\sim$ $U_{ab}(t, dt)$

Out of these building blocks, we build a form on the space of metric trees:

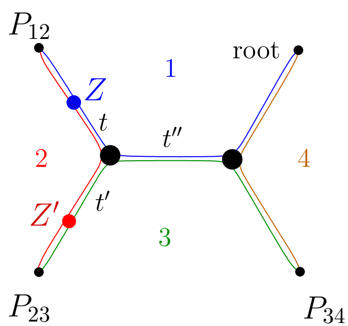
$$I \in \Omega^\bullet(MT_{N;k_1,\ldots,k_N}) \otimes \mathrm{Hom}(\mathrm{Mor}_{1,1}^{\otimes k_1} \otimes \mathrm{Mor}_{1,2} \otimes \cdots \otimes \mathrm{Mor}_{N-1,N} \otimes \mathrm{Mor}_{N,N}^{\otimes k_N}, \mathrm{Mor}_{1,N})$$

where $\mathrm{Mor}_{a,b} := \mathrm{Mor}(F_a, F_b)$.

Example: three top-cells in $MT_{4;1,1,0,0}$



Example

$I\Bigg($ 

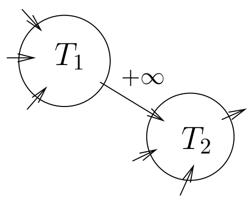
$\Bigg) = \sum_{P_{14}\in \mathrm{Crit}(F_1 - F_4)} \overline{I} \cdot [P_{14}]$

$$\overline{I} = \int_X \delta_{\mathrm{Stab}_{P_{14}}} \wedge U_{13}(t'', dt'')\Big(U_{12}(t, dt)(\delta_Z \wedge \delta_{\mathrm{Unstab}_{P_{12}}})\wedge$$

$$\wedge\, U_{23}(t', dt')(\delta_{Z'} \wedge \delta_{\mathrm{Unstab}_{P_{23}}})\Big) \wedge \delta_{\mathrm{Unstab}_{P_{34}}}$$

Properties of I

1. $(d_{MT} + Q)I = 0$
2. Factorization on IR boundary of MT:

$I\Bigg($ 

$\Bigg) = \Big\langle I(T_2) \wedge I(T_1) \Big\rangle$

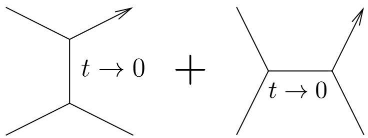
3. Period $\displaystyle\int_{MT} I = m$ – the composition map in Fukaya-Morse category.

A_∞ relations from IR factorization of HTQM

$$(d_{MT}+Q)I = 0 \qquad \Rightarrow \qquad \int_{\partial MT} I = -Q \underbrace{\int_{MT} I}_{m}$$

$m(\cdots m(\underline{\underline{\bullet}})\cdots)$

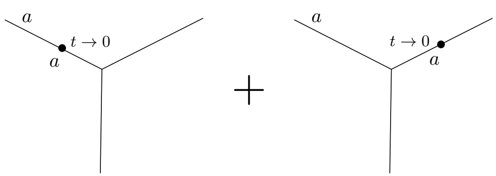
L.h.s. (contributions of boundary strata of MT):

- a 

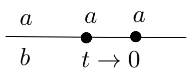
$= 0$

- b 

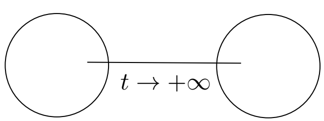
$= 0$

- c 

$= 0$

- d 

$\to$ terms $m(\cdots m(Z,Z')\cdots)$ in A_∞ relation.

- e 

$\to$ terms $m(\cdots m(\underbrace{\cdots}_{\geq 2 \text{ colors}})\cdots)$ in A_∞ relation.

References

- K. Fukaya, “Morse homotopy, A^∞-category and Floer homologies.” In Proceeding of Garc Workshop on Geometry and Topology. Seoul National Univ, 1993
- K. Fukaya, Y. G. Oh, “Zero-loop open strings in the cotangent bundle and Morse homotopy,” Asian Journal of Mathematics 1, no. 1 (1997): 96-180.
- M. Kontsevich, Y. Soibelman, “Homological mirror symmetry and torus fibrations.” arXiv preprint math/0011041 (2000).
- O. Chekeres, A. Losev, P. Mnev, D. Youmans, “Two field-theoretic viewpoints on the Fukaya-Morse A_∞ category,” arXiv:2112.12756.