The Lagrangian capacity of toric domains

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Definition 1.1

A **Liouville domain** is a pair (X, λ), where X is a compact, connected smooth manifold with boundary ∂X and $\lambda \in \Omega^1(X)$ is such that $\mathrm{d}\lambda \in \Omega^2(X)$ is symplectic, $\lambda|_{\partial X}$ is contact and the orientations on ∂X coming from $(X, \mathrm{d}\lambda)$ and coming from $\lambda|_{\partial X}$ are equal.

Definition 1.2

A **star-shaped domain** is a compact, connected $2n$-dimensional submanifold X of $\mathbb{C}^n$ with boundary ∂X such that (X, λ) is a Liouville domain, where

$$\lambda \coloneqq \frac{1}{2} \sum_{j=1}^{n} (x^j \mathrm{d}y^j - y^j \mathrm{d}x^j) \in \Omega^1(\mathbb{C}^n).$$

Equivalently, the **Liouville vector field** Z is outward pointing, where

$$\lambda = \iota_Z \mathrm{d}\lambda \Longrightarrow Z = \frac{1}{2} \sum_{j=1}^{n} \left(x^j \frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial y^j} \right).$$

Definition 1.3

The **moment map** is the map $\mu\colon \mathbb{C}^n \longrightarrow \mathbb{R}^n_{\geq 0}$ given by

$$\mu(z_1, \dots, z_n) \coloneqq \pi(|z_1|^2, \dots, |z_n|^2).$$

Define also

$$\begin{aligned} \Omega_X &\coloneqq \Omega(X) \coloneqq \mu(X) \subset \mathbb{R}^n_{\geq 0}, && \text{for every } X \subset \mathbb{C}^n, \\ X_\Omega &\coloneqq X(\Omega) \coloneqq \mu^{-1}(\Omega) \subset \mathbb{C}^n, && \text{for every } \Omega \subset \mathbb{R}^n_{\geq 0}, \\ \delta_\Omega &\coloneqq \delta(\Omega) \coloneqq \sup\{a \mid (a, \dots, a) \in \Omega\}, && \text{for every } \Omega \subset \mathbb{R}^n_{\geq 0}. \end{aligned}$$

We call δ_Ω the **diagonal** of Ω.

Definition 1.4

A **toric domain** is a star-shaped domain X such that $X = X(\Omega(X))$.
A toric domain $X = X_\Omega$ is

- **convex** if $\hat{\Omega} \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (|x_1|, \dots, |x_n|) \in \Omega\}$ is convex;
- **concave** if $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex.

Example 1.5

The following are toric domains:

$$E(a_1,\dots,a_n) := \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{\pi|z_j|^2}{a_j} \le 1 \right\} \qquad \textbf{(ellipsoid)}$$

$$B(a) := \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{\pi|z_j|^2}{a} \le 1 \right\} \qquad \textbf{(ball)}$$

$$Z(a) := \{ z \in \mathbb{C}^n \mid \pi|z_1|^2 \le a \} \qquad \textbf{(cylinder)}$$

$$P(a) := \{ z \in \mathbb{C}^n \mid \forall j = 1,\dots,n \colon \pi|z_j|^2 \le a \} \qquad \textbf{(cube)}$$

$$N(a) := \{ z \in \mathbb{C}^n \mid \exists j = 1,\dots,n \colon \pi|z_j|^2 \le a \}$$

(nondisjoint union of cylinders)

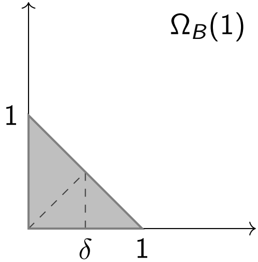

Figure: Ball and cube

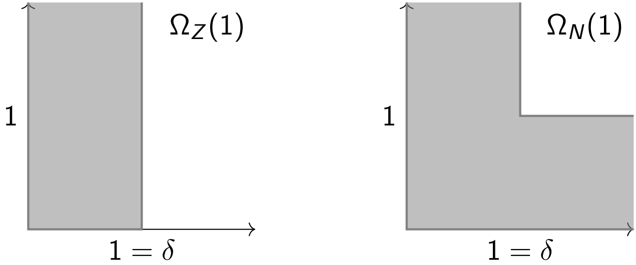

Figure: Cylinder and nondisjoint union of cylinders

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Definition 2.1

A **symplectic capacity** is a map c which to every symplectic manifold (possibly in a restricted subclass) assings an element of $[0, +\infty]$, such that

- (Monotonicity) If $(X, \omega_X) \longrightarrow (Y, \omega_Y)$ is a symplectic embedding of codimension 0 (possibly in a restricted subclass) then $c(X, \omega_X) \leq c(Y, \omega_Y)$;
- (Conformality) If $\alpha > 0$ then $c(X, \alpha\omega) = \alpha c(X, \omega)$.

Example 2.2

Let (X, ω) be a symplectic manifold. We define the **cube capacity** of X by

$$c_P(X, \omega) \coloneqq \sup\{a \mid \exists \text{ symplectic embedding } P(a) \longrightarrow X\}.$$

Definition 2.3 ([CM18, Section 1.2])

Let (X,ω) be a symplectic manifold. If L is a Lagrangian submanifold of X, then we define the **minimal symplectic area of** L by

$$A_{\min}(L) := \inf\{\omega(\sigma) \mid \sigma \in \pi_2(X,L),\, \omega(\sigma) > 0\}.$$

Definition 2.4 ([CM18, Section 1.2])

The **Lagrangian capacity** of (X,ω) is

$$c_L(X) := \sup\{A_{\min}(L) \mid L \subset X \text{ is an embedded Lagrangian torus}\}.$$

Proposition 2.5 ([CM18, Section 1.2])

The Lagrangian capacity c_L satisfies:

- *(Monotonicity) If $(X,\omega) \longrightarrow (X',\omega')$ is a symplectic embedding with $\pi_2(X',\iota(X)) = 0$, then $c_L(X,\omega) \leq c_L(X',\omega')$.*
- *(Conformality) If $\alpha \neq 0$, then $c_L(X,\alpha\omega) = |\alpha|\, c_L(X,\omega)$.*

Lemma 2.6

If X is a star-shaped domain, then $c_L(X) \geq c_P(X)$.

Proof.

Let $\iota\colon P(a) \longrightarrow X$ be a symplectic embedding, for some $a > 0$. We want to show that $c_L(X) \geq a$. Define

$$T \coloneqq \{z \in \mathbb{C}^n \mid |z_1|^2 = a/\pi, \ldots, |z_n|^2 = a/\pi\} \subset \partial P(a),$$
$$L \coloneqq \iota(T) \subset X.$$

Then,

$$\begin{aligned} c_L(X) &\geq A_{\min}(L) && \text{[by definition of } c_L\text{]} \\ &= A_{\min}(T) && \text{[since } \pi_2(X, \iota(P(a))) = 0\text{]} \\ &= a && \text{[by Stokes' theorem].} \end{aligned}$$

□

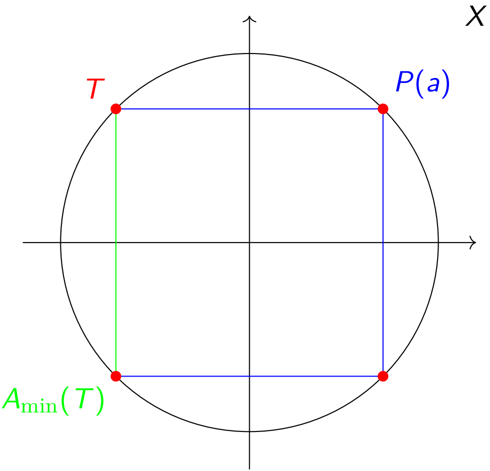


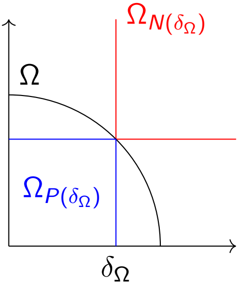
Figure: Proof of $c_L(X) \geq c_P(X)$ for $X = B(r) \subset \mathbb{C}^2$

Lemma 2.7

If X_Ω is a convex or concave toric domain, then $c_P(X_\Omega) \geq \delta_\Omega$.

Proof.

Since X_Ω is convex or concave, we have $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$.



The result follows since $c_P(X_\Omega) := \sup\{a \mid \exists\, P(a) \hookrightarrow X_\Omega\}$. □

Theorem 2.8 ([GH18, Theorem 1.18])

If X_Ω is a convex or concave toric domain, then $c_P(X_\Omega) = \delta_\Omega$.

We now consider the results by Cieliebak–Mohnke for the Lagrangian capacity of the ball and the cylinder.

Proposition 2.9 ([CM18, Corollary 1.3])

The Lagrangian capacity of the ball is

$$c_L(B^{2n}(1)) = \frac{1}{n} = \delta_{\Omega(B^{2n}(1))}.$$

Proposition 2.10 ([CM18, p. 215-216])

The Lagrangian capacity of the cylinder is

$$c_L(Z^{2n}(1)) = 1 = \delta_{\Omega(Z^{2n}(1))}.$$

- By Lemmas 2.6 and 2.7, if X_Ω is a convex or concave toric domain then $c_L(X_\Omega) \geq \delta_\Omega$.
- But as we have seen in Propositions 2.9 and 2.10, if X_Ω is the ball or the cylinder then $c_L(X_\Omega) = \delta_\Omega$.

Conjecture 2.11 ([CM18, Conjecture 1.5])

The Lagrangian capacity of the ellipsoid is

$$c_L(E(a_1, \ldots, a_n)) = \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1} = \delta_{\Omega(E(a_1,\ldots,a_n))}.$$

Conjecture 2.12 ([Per22, Conjecture 6.24])

If X_Ω is a convex or concave toric domain then

$$c_L(X_\Omega) = \delta_\Omega.$$

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To prove our results about the Conjecture 2.12, we will need to use the following symplectic capacities.

McDuff–Siegel capacities	$\tilde{\mathfrak{g}}_k^{\leq \ell}$	[MS22]
Higher symplectic capacities	$\mathfrak{g}_k^{\leq \ell}$	[Sie20]
Gutt–Hutchings capacities	c_k^{GH}	[GH18]

for $k, \ell \in \mathbb{Z}_{\geq 1}$. We will only need to consider these capacities for $\ell = 1$, i.e. $\tilde{\mathfrak{g}}_k^{\leq 1}, \mathfrak{g}_k^{\leq 1}$.

Theorem 3.1 ([Per22, Theorem 6.41])

If X_Ω is a 4-dimensional convex toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

For every $k \in \mathbb{Z}_{\geq 1}$,

$$
\begin{aligned}
\delta_\Omega &\leq c_P(X_\Omega) && \text{[by Lemma 2.7]} \\
&\leq c_L(X_\Omega) && \text{[by Lemma 2.6]} \\
&\leq \tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega)/k && \text{[by [Per22, Theorem 6.40]]} \\
&= c_k^{\mathrm{GH}}(X_\Omega)/k && \text{[dim 4 and [MS22, Proposition 5.6.1]]} \\
&\leq c_k^{\mathrm{GH}}(N(\delta_\Omega))/k && \text{[X_Ω is convex, hence $X_\Omega \subset N(\delta_\Omega)$]} \\
&= \delta_\Omega (k+1)/k && \text{[by [GH18, Lemma 1.19]].}
\end{aligned}
$$

□

Theorem 3.2 ([Per22, Theorem 7.65])

Assume that a suitable virtual perturbation scheme exists. If X_Ω is a convex or concave toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

For every $k \in \mathbb{Z}_{\geq 1}$,

$$
\begin{aligned}
\delta_\Omega &\leq c_P(X_\Omega) && \text{[by Lemma 2.7]} \\
&\leq c_L(X_\Omega) && \text{[by Lemma 2.6]} \\
&\leq \tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega)/k && \text{[by [Per22, Theorem 6.40]]} \\
&\leq \mathfrak{g}_k^{\leq 1}(X_\Omega)/k && \text{[by [MS22, Section 3.4]]} \\
&= c_k^{\mathrm{GH}}(X_\Omega)/k && \text{[by [Per22, Theorem 7.64]]} \\
&\leq c_k^{\mathrm{GH}}(N(\delta_\Omega))/k && \text{[X_Ω is convex, hence $X_\Omega \subset N(\delta_\Omega)$]} \\
&= \delta_\Omega(k+n-1)/k && \text{[by [GH18, Lemma 1.19]].}
\end{aligned}
$$

□

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Let (X, λ) be a nondegenerate Liouville domain.

- Choose a point $x \in \operatorname{int} X$ and a **symplectic divisor** (germ of a symplectic submanifold of codimension 2) $D \subset X$ through x.
- The boundary $(\partial X, \lambda|_{\partial X})$ is a **contact manifold** and therefore has a **Reeb vector field**. Let γ be a Reeb orbit.
- The **completion** of (X, λ) is the exact symplectic manifold

$$(\hat{X}, \hat{\lambda}) \coloneqq (X, \lambda) \cup_{\partial X} (\mathbb{R}_{\geq 0} \times \partial X, e^r \lambda|_{\partial X}).$$

- Let $\mathcal{M}_X^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle$ denote the moduli space of J-holomorphic curves in $\hat{X}$ which are positively asymptotic to the Reeb orbit γ and which have contact order k to D at x.

Definition 4.1 ([MS22, Definition 3.3.1])

For $k \in \mathbb{Z}_{\geq 1}$ the **McDuff–Siegel capacities** of X are given by

$$\tilde{\mathfrak{g}}_k^{\leq 1}(X) \coloneqq \sup_{J \in \mathcal{J}(X,D)} \inf_{\gamma} \mathcal{A}(\gamma),$$

where $\mathcal{A}(\gamma) \coloneqq \int_{S^1} \gamma^* \lambda|_{\partial X}$ and the infimum is over Reeb orbits γ such that $\mathcal{M}_X^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle \neq \varnothing$.

Theorem 4.2 ([Per22, Theorem 6.40])

If (X, λ) is a Liouville domain then

$$c_L(X) \leq \inf_k \frac{\tilde{\mathfrak{g}}_k^{\leq 1}(X)}{k}.$$

Proof (1/5).

- Let $k \in \mathbb{Z}_{\geq 1}$ and $L \subset \operatorname{int} X$ be an embedded Lagrangian torus. Denote $a := \tilde{\mathfrak{g}}_k^{\leq 1}(X)$. We wish to show that there exists $\sigma \in \pi_2(X, L)$ such that $0 < \omega(\sigma) \leq a/k$.
- Choose a suitable Riemannian metric on L, such that there exists a symplectic embedding $\phi \colon D^*L \longrightarrow \operatorname{int} X$ with $\phi|_L = \mathrm{id}_L$. Choose a point $x \in \operatorname{int} D^*L$, a symplectic divisor D through x, and a sequence $(J_t)_t$ of almost complex structures on $\hat{X}$ realizing SFT neck stretching along S^*L.
- By definition of $\tilde{\mathfrak{g}}_k^{\leq 1}(X) =: a$, there exists a Reeb orbit γ together with a sequence $(u_t)_t$ of J_t-holomorphic curves $u_t \in \mathcal{M}_X^{J_t}(\gamma)\langle \mathcal{T}^{(k)} x \rangle$ with energy $E(u_t) \leq a$.

Proof (2/5).

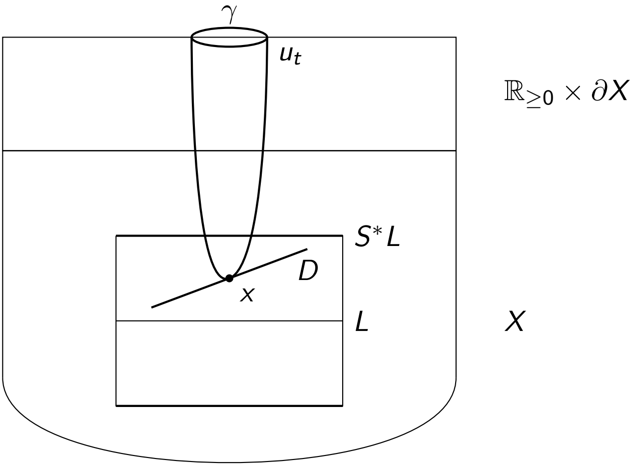

Figure: The proof so far

Proof (3/5).

- By the SFT-compactness theorem, the sequence $(u_t)_t$ converges to a broken holomorphic curve $F = (F^1, \dots, F^N)$, where each F^ν is a holomorphic curve. Denote by C the component of $F^1 \subset T^*L$ which carries the tangency constraint.

Proof (4/5).

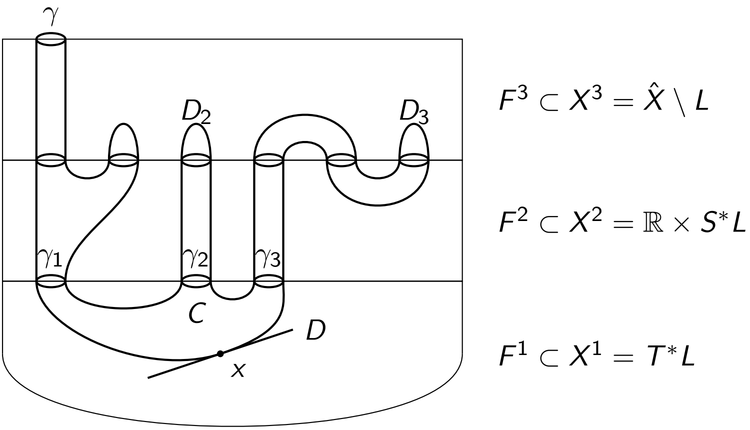

Figure: The broken holomorphic curve F in the case $N = 3$

Proof (5/5).

- The choices of almost complex structures J_t can be done in such a way that the simple curve corresponding to C is regular, i.e. it is an element of a moduli space which is a manifold.
- Using the dimension formula for this moduli space, it is possible to conclude that C must have at least $k+1$ punctures.
- This implies that C gives rise to at least $k > 0$ disks $D_1, \ldots, D_k$ in X with boundary on L. The total energy of the disks is less or equal to a. Therefore, one of the disks must have energy less or equal to a/k. □

References

[CM18] Kai Cieliebak and Klaus Mohnke. Punctured holomorphic curves and Lagrangian embeddings. *Inventiones mathematicae*, 212(1):213–295, April 2018.

[GH18] Jean Gutt and Michael Hutchings. Symplectic capacities from positive S^1-equivariant symplectic homology. *Algebraic & Geometric Topology*, 18(6):3537–3600, October 2018.

[MS22] Dusa McDuff and Kyler Siegel. Symplectic capacities, unperturbed curves, and convex toric domains. *arXiv:2111.00515 [math]*, February 2022.

[Per22] Miguel Pereira. *Equivariant symplectic homology, linearized contact homology and the Lagrangian capacity*. PhD thesis, Universität Augsburg, March 2022.

[Sie20] Kyler Siegel. Higher symplectic capacities. *arXiv:1902.01490 [math-ph]*, February 2020.

Thank you for listening!