Topological Quantum Field Theories for Character Stacks

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Representation varieties

Representation variety

G a complex algebraic group and M a compact manifold.

$$\mathfrak{X}_G(M) = \mathrm{Hom}\left(\pi_1(M), G\right)$$

Algebraic structure:
$\pi_1(M) = \langle \gamma_1, \ldots, \gamma_\ell \,|\, R_\alpha(\gamma_1, \ldots, \gamma_\ell) = 1 \rangle$. We have an identification

$$\begin{array}{cccc} \psi : & \mathrm{Hom}\left(\pi_1(M), G\right) & \longrightarrow & G^\ell \\ & \rho & \mapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_\ell)) \end{array}$$

with the algebraic set

$$\mathrm{Im}\ \psi = \left\{ (g_1, \ldots, g_\ell) \in G^\ell \,|\, R_\alpha(g_1, \ldots, g_\ell) = 1 \right\}.$$

Across the non-abelian Hodge theory

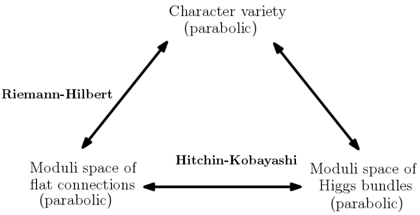
Character variety

With respect to the action of G by conjugation

$$\mathcal{R}_G(M) = \mathfrak{X}_G(M) /\!\!/ G,$$

where $X /\!\!/ G$ denotes the **GIT quotient.**

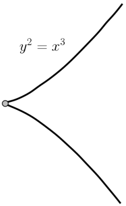
<u>Non-abelian Hodge theory</u>.



Extracting algebro-geometric data

$$\begin{array}{ccc} (\mathbf{Var}_{\mathbb{C}}, \sqcup, \times) \text{ semi-ring} & \rightsquigarrow & \mathrm{K}\mathbf{Var}_{\mathbb{C}} \text{ Grothendieck ring} \\ X \in \mathbf{Var}_{\mathbb{C}} & \rightsquigarrow & [X] \in \mathrm{K}\mathbf{Var}_{\mathbb{C}} \end{array}$$

$$[X_1 \sqcup X_2] = [X_1] + [X_2], \qquad [X_1 \times X_2] = [X_1] \cdot [X_2].$$



$$[X] = [X - \star] + [\star] = [\mathbb{C} - \star] + [\star] = [\mathbb{C}]$$

Notation: $[\mathbb{C}] = \mathbb{L} = q$.

Problem

Compute $[\mathfrak{X}_G(\Sigma)] \in \mathrm{K}\mathbf{Var}_{\mathbb{C}}$ (or even better $\in \mathbb{Z}[q]$).

Known results

- **Arithmetic method** *(Hausel, Rodríguez-Villegas, Letellier, Mereb, Florentino, Mellit, Schiffmann, Bozec...)*. $G = \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$.

Key idea: Katz's theorem on point counting

Let X be a $\mathbb{Z}$-scheme. Suppose that there exists a polynomial $\mathcal{P}(x) \in \mathbb{Z}[x]$ such that

$$|X(\mathbb{F}_{p^n})| = \mathcal{P}(p^n).$$

Then $[X] = \mathcal{P}(q) \in \mathrm{K}\mathbf{Var}_{\mathbb{C}}$, where $q = \mathbb{L}$.

'Con': The solution is not explicit

It is written in terms of the character tables of $\mathrm{GL}_n(\mathbb{F}_q), \mathrm{SL}_n(\mathbb{F}_q)$ (equivalently, on combinatorial data of partitions of n).

Known results

- **Geometric method** *(Logares, Muñoz, Newstead, Martínez, Baraglia, Hekmati...)*: Explicitly study the variety. $G = \mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_2(\mathbb{C}), \mathrm{SL}_3(\mathbb{C})$.

Key idea: Stratifications

Decompose a complex variety X into simpler subvarieties

$$X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n.$$

Compute the virtual classes $[X_i] \in \mathrm{K}\mathbf{Var}_{\mathbb{C}}$. Using the additivity in $\mathrm{K}\mathbf{Var}_{\mathbb{C}}$ obtain $[X] = [X_1] + [X_2] + \ldots + [X_n]$.

'Con': The method is case-specific

Only valid for small rank (and sometimes genus).

Topological Quantum Field Theories

- **Quantum method** *(GP, Logares, Muñoz)*.

Theorem (GP, GP-Logares-Muñoz)

For any complex algebraic group G and any $n \geq 1$, there exists a TQFT

$$Z : \mathbf{Bordp}_n \to \mathrm{K}\mathbf{Var}_{\mathbb{C}}\text{-}\mathbf{Mod},$$

computing virtual classes of G-representation varieties.

"Computing virtual classes"

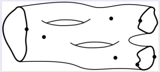
$$W : \emptyset \longrightarrow \emptyset \quad \rightsquigarrow \quad \begin{array}{rccc} Z(W) : & \mathrm{K}\mathbf{Var}_{\mathbb{C}} & \longrightarrow & \mathrm{K}\mathbf{Var}_{\mathbb{C}} \\ & 1 & \mapsto & [\mathfrak{X}_G(W)] \end{array}$$

Pros

Arbitrary group G, dimension. 2-category structure for deformation theory.

Cons?

- **Bordisms with basepoints**: **Bordp**$_n$.



$\Rightarrow$ No classification in terms of Frobenius algebras.

- **Lax monoidality**: No longer an isomorphism

$$\Delta_{M_1,M_2} : Z(M_1) \otimes_R Z(M_2) \longrightarrow Z(M_1 \sqcup M_2).$$

$\Rightarrow$ $Z(M)$ may not be dualizable (i.e. infinitely generated).

- **Not addressing the GIT quotient**: Computes $[\mathfrak{X}_G(\Sigma)]$ but not $[\mathfrak{X}_G(\Sigma) /\!\!/ G]$.

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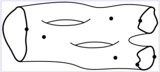

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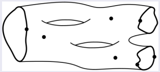

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Sketch of construction

Honouring the physics

$$\begin{tikzcd} \mathbf{Bordp}_n \arrow[r, "\text{Field theory}"] \arrow[rr, dashed, bend right, "Z"'] & \{\text{Fields}\} \arrow[r, "\text{Quantization}"] & R\text{-}\mathbf{Mod} \end{tikzcd}$$

$$\begin{array}{ccc} \mathbf{Bordp}_n & \xrightarrow{\text{Field theory}} & \mathrm{Span}(\mathbf{Var}_{\mathbb{C}}) \\ (M, A) & \mapsto & \mathfrak{X}_G(\Pi(M, A)) \\ (S^1, \star) & \mapsto & \mathfrak{X}_G(S^1) = G \\ \emptyset & \mapsto & \mathfrak{X}_G(\emptyset) = \star \end{array}$$

$$W : M_1 \to M_2 \quad \mapsto \quad \begin{tikzcd} & \mathfrak{X}_G(W) \arrow[dl, "i"'] \arrow[dr, "j"] & \\ \mathfrak{X}_G(M_1) & & \mathfrak{X}_G(M_2) \end{tikzcd}$$

Functoriality

Seifert-van Kampen theorem (fundamental groupoids)

Sketch of construction

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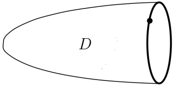
$$\begin{array}{ccc} \mathrm{Span}(\mathbf{Var}_{\mathbb{C}}) & \xrightarrow{\text{Quantization}} & \mathrm{K}\mathbf{Var}_{\mathbb{C}}\text{-}\mathbf{Mod} \\ & & \\ X & \mapsto & \mathrm{K}\mathbf{Var}/X \\ \mathfrak{X}_G(\Pi(M, A)) & \mapsto & \mathrm{K}\mathbf{Var}/\mathfrak{X}_G(\Pi(M, A)) \\ \mathfrak{X}_G(\emptyset) = \star & \mapsto & \mathrm{K}\mathbf{Var}/\star = \mathrm{K}\mathbf{Var}_{\mathbb{C}} \end{array}$$

$$\begin{tikzcd}[column sep=small] & \mathfrak{X}_G(W) \arrow[dl, "i"'] \arrow[dr, "j"] & \\ \mathfrak{X}_G(M_1) & & \mathfrak{X}_G(M_2) \end{tikzcd} \quad \mapsto \quad \underbrace{\begin{tikzcd}[column sep=small] & \mathrm{K}\mathbf{Var}/\mathfrak{X}_G(W) \arrow[dr, "j_!"] & \\ \mathrm{K}\mathbf{Var}/\mathfrak{X}_G(M_1) \arrow[ur, "i^*"] & & \mathrm{K}\mathbf{Var}/\mathfrak{X}_G(M_2) \end{tikzcd}}_{\text{‘Motivic Fourier-Mukai transf.’}}$$

Functoriality

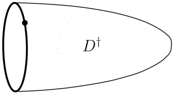
Base change (a.k.a. Beck-Chevalley property)

Explicit maps for surfaces



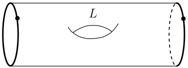
$$\emptyset \longrightarrow (S^1, \star)$$

$$Z(D) : \mathrm{K}\textbf{Var}_{\mathbb{C}} \to \mathrm{K}\textbf{Var}/G$$



$$(S^1, \star) \longrightarrow \emptyset$$

$$Z(D^\dagger) : \mathrm{K}\textbf{Var}/G \to \mathrm{K}\textbf{Var}_{\mathbb{C}}$$



$$(S^1, \star) \longrightarrow (S^1, \star)$$

$$Z(L) : \mathrm{K}\textbf{Var}/G \to \mathrm{K}\textbf{Var}/G$$

Key point

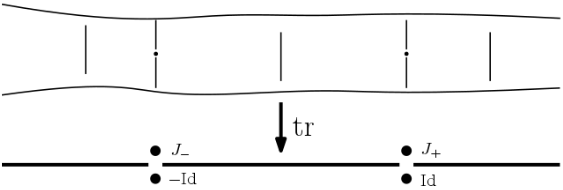
It is enough to understand three linear maps
(and two of them are trivial).

Case $G = \mathrm{SL}_2(\mathbb{C})$

Conjugacy classes in $\mathrm{SL}_2(\mathbb{C})$

$$\pm\mathrm{Id} \quad J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in \mathbb{C} - \{0, \pm 1\}$.



Case $G = \mathrm{SL}_2(\mathbb{C})$

Finiteness miracle

$$\langle \mathbb{1}_\star \rangle \overset{Z(D)}{\rightsquigarrow} \langle \mathbb{1}_1 \rangle \overset{Z(L)}{\rightsquigarrow} \langle \mathbb{1}_1, \mathbb{1}_{-1}, \mathbb{2}_1, \mathbb{2}_{-1} \rangle \overset{Z(L)}{\rightsquigarrow} \langle \mathbb{1}_1, \mathbb{1}_{-1}, \mathbb{2}_1, \mathbb{2}_{-1} \rangle$$

$$\Rightarrow \langle \mathbb{1}_1, \mathbb{1}_{-1}, \mathbb{2}_1, \mathbb{2}_{-1} \rangle \subseteq \mathrm{K}\mathbf{Var}/\mathrm{SL}_2(\mathbb{C}) \text{ is enough}$$

Theorem (Martínez-Muñoz) & (GP)

$$\begin{aligned}[\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(\Sigma_g)] = {} & \left(q^2-1\right)^{2g-1} q^{2g-1} + \frac{1}{2}(q-1)^{2g-1}q^{2g-1}(q+1)\left(2^{2g}+q-3\right) \\ & + \frac{1}{2}(q+1)^{2g+r-1}q^{2g-1}(q-1)\left(2^{2g}+q-1\right) + q(q^2-1)^{2g-1}.\end{aligned}$$

Recall: $q = [\mathbb{C}]$.

Problem 1: TQFT doesn't work for quotients

The GIT quotient is old fashioned

$$\mathfrak{M}_G(M) = [\mathfrak{X}_G(M)/G] \qquad \text{(stacky quotient)}$$

Recall: A stack is a pseudo-functor (of 2-categories)

$$\mathfrak{M} : \mathbf{Sch}/S \to \mathbf{Grpd}$$

which is a sheaf for the fppf topology.
Example: Sheaf of points for an S-scheme X

$$\underline{X} = \mathrm{Hom}_{\mathbf{Sch}/S}(-, X).$$

Informal idea

$\mathfrak{M}(U)$ captures the U-families of a moduli problem.

The quotient stack $[X/G]$

$$[X/G]\,(U) = \left\{ \begin{array}{ccc} P & \xrightarrow{\text{Equivariant}} & X \\ {\scriptstyle \text{Principal } G\text{-bdl}}\big\downarrow & & \\ U & & \end{array} \right\}$$

Particular case: $BG := [\star/G] = \{\text{Principal } G\text{-bundles}\}$.

$$\mathbf{Stack}/BG \cong \{\text{Algebraic spaces equipped with } G\text{-action}\}$$

$$\Rightarrow (\mathfrak{M}_G(M) \to BG) = \mathfrak{X}_G(M) + \text{Adj. } G\text{-action}$$

Goal

Compute the motive $[\mathfrak{M}_G(M)] \in \mathrm{K}\,(\mathbf{Stack}/BG)$.

Winter sales!

Theorem (GP-Hablicsek-Vogel)

There exists a lax monoidal TQFT

$$Z : \mathbf{Bordp}_n \to \mathrm{K}\left(\mathbf{Stack}/BG\right)\text{-}\mathbf{Mod},$$

computing virtual classes of G-character stacks

$$Z(W)(1) = [\mathfrak{M}_G(M) \to BG] \in \mathrm{K}\left(\mathbf{Stack}/BG\right).$$

Sketch of sketch of proof: Repeat the construction of the non-stacky case.

- Field theory: Works because ‘taking stacky quotients’ preserves pullbacks.
- Quantization: Same argument + technical work.

$\mathrm{AGL}_1(\mathbb{C})$-character stack

TQFT: The "core submodule" is generated by $\langle \mathbb{1}_1, \mathbb{1}_J \rangle$ but it is no longer a $\mathbb{Z}[q]$-module.

Z ()

$$= \begin{pmatrix} 1 + q(q-2)[\mathbb{G}_a/G] + (q+1)[\mathbb{G}_m/G] & q(q-2)[\mathrm{AGL}_1(k)/G] \\ q(q-2)[\mathbb{G}_a/G] & q^2 + q(q-1)(q-2)[\mathbb{G}_a/G] \end{pmatrix}.$$

Theorem (GP-Hablicsek-Vogel)

The virtual class of the $\mathrm{AGL}_1(\mathbb{C})$-character stack is

$$\mathfrak{M}_{\mathrm{AGL}_1(\mathbb{C})}(\Sigma_g) = BG + ((q-1)^{2g} - 1)[\mathbb{G}_a/G] + \frac{q^{2g}-1}{q-1}[\mathbb{G}_m/G] + \frac{(q^{2g-2}-1)\left((q-1)^{2g}-1\right)}{q-1}[\mathrm{AGL}_1(k)/G].$$

Problem 2: Monoidality

The monoidality problem

Does there exist a **monoidal** TQFT

$$\mathcal{Z} : \mathbf{Bordp}_n \to \mathrm{K}\left(\mathbf{Stack}/BG\right)\text{-}\mathbf{Mod}$$

computing the virtual classes of G-representation stacks?

Remark: This is a monoidal Kan extension problem

$$\begin{tikzcd} \mathbf{Bord}_n \arrow[r, dashed, "\mathcal{Z}"] & \mathrm{K}\left(\mathbf{Stack}/BG\right)\text{-}\mathbf{Mod} \\ \mathbf{Tub}_n \arrow[u, hook] \arrow[ur, "Z"'] & \end{tikzcd}$$

Theorem (GP)

$$\mathrm{Fun}^{\otimes}(\mathbf{Tub}_n, R\text{-}\mathbf{Mod}) = \text{lax monoidal TQFTs}$$

The monoidality problem

Theorem No-Go (GP)

Let G be an algebraic group.

- If $\dim G \geq 1$, then there does **not exist** a **monoidal** TQFT computing virtual classes of character stacks.
- If $\dim G = 0$, then there **exists** a **monoidal** TQFT computing the point count of character stacks.

Indeed: For $\dim G = 0$, the monoidal TQFT is a modification of the previous construction (other quantization).

Moral

Lax monoidal TQFTs are mandatory for applications to algebraic topology.

Future fun

- Classification of lax monoidal TQFTs.
- Quiver representation varieties.

$$\begin{tikzcd} \bullet \arrow[drr] & \bullet \arrow[l] \arrow[r] \arrow[dr] & \bullet \\ & & \bullet \end{tikzcd}$$

- Derived geometry $D^b(X)$ + Fourier-Mukai.
- TQFT across the non-abelian Hodge correspondence.
- Mirror symmetry conjectures for character varieties.

$$D^b(X) \overset{?}{\cong} \mathrm{Fuk}(\widehat{X}), \qquad Z_G \overset{?}{\longleftrightarrow} Z_{G^\vee}$$

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Thank you very much for your attention!