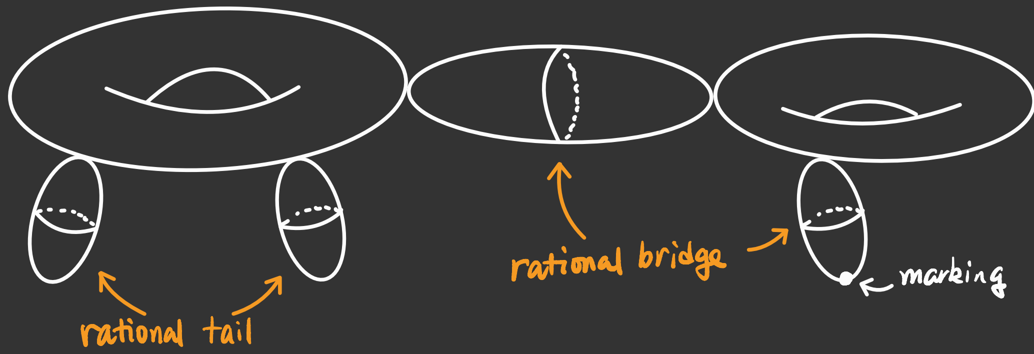
Quasimap Wallcrossing

Yang Zhou,
SCMS, Fudan

Prepared Pages

ε-stability

$C =$ 

Defn: (Ciocan-Fontanine–Kim–Maulik, '14)

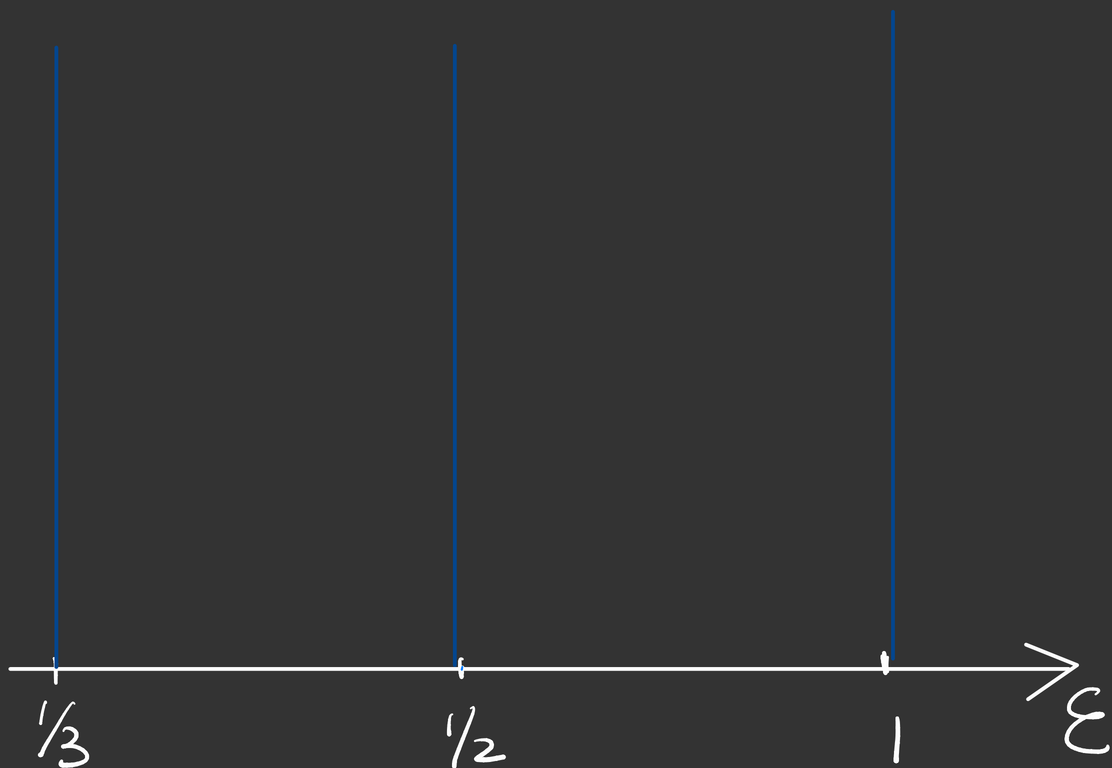
A quasi-map is ε-stable ($\varepsilon \in \mathbb{Q}_{>0}$) if

- Every rational bridge has degree > 0
- Every rational tail has degree $> \frac{1}{\varepsilon}$
- Every base point has length $\leq \frac{1}{\varepsilon}$

multiplicity of common zeros of s_i

$$= \min_{i=1,\dots,s} \text{vanishing order of } s_i \text{ at } x.$$

Wall-and-chamber structure:



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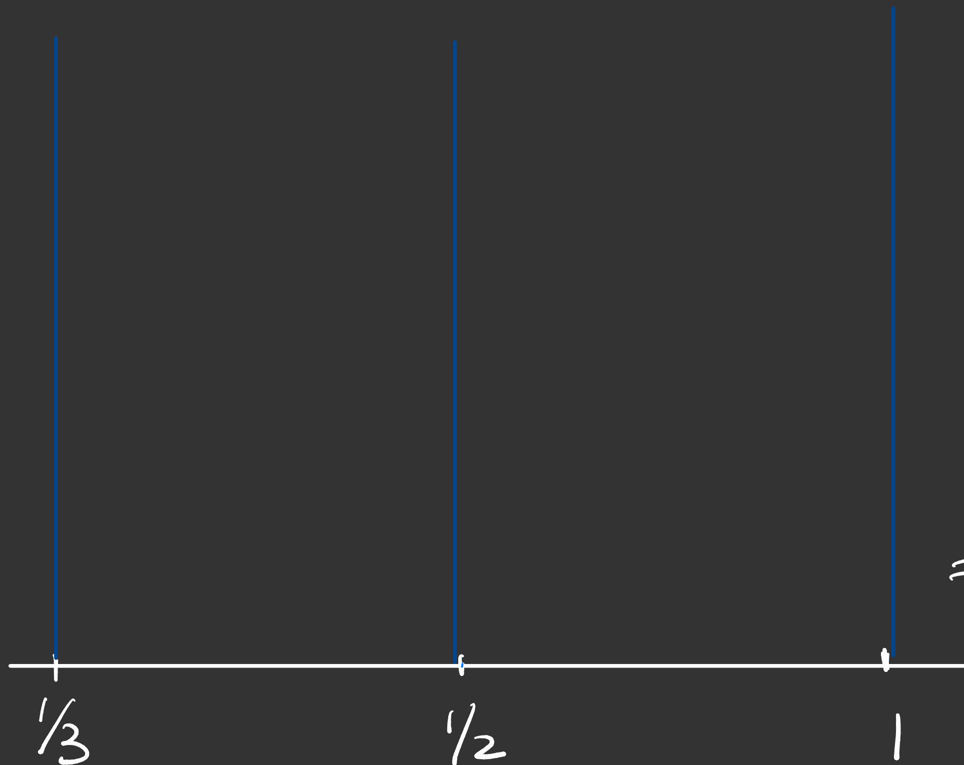
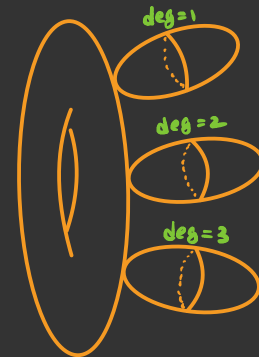


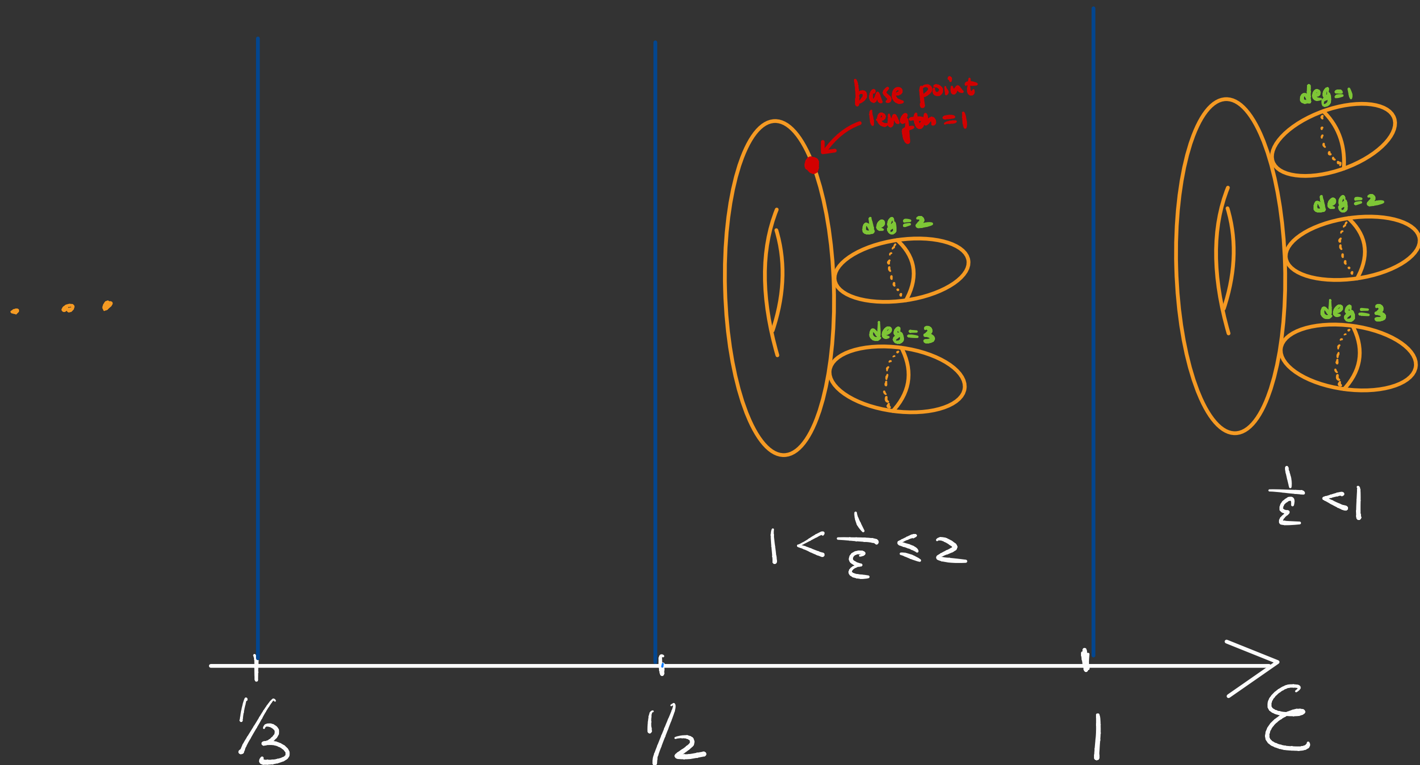


$\frac{1}{\varepsilon} < 1$

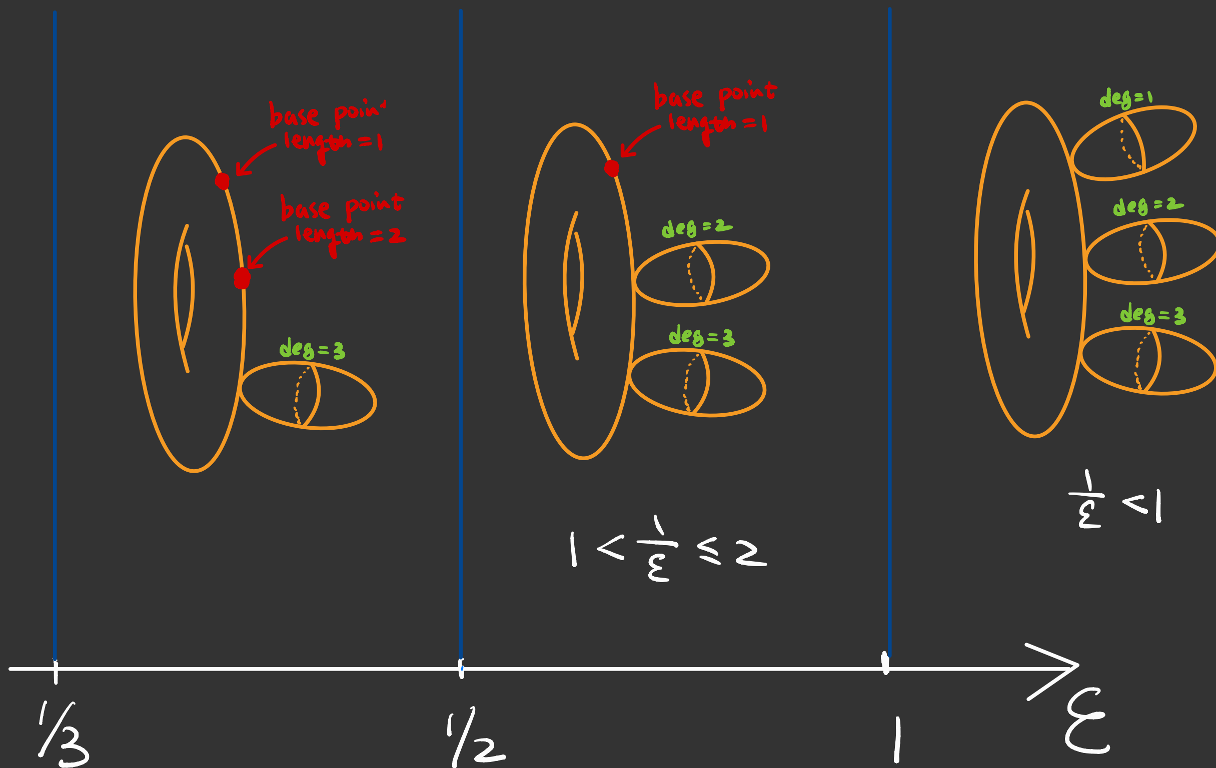
$\Rightarrow$ stable maps

$\Rightarrow$ GW theory

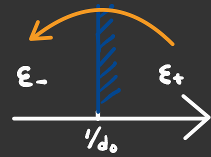
Wall-and-chamber structure:



Wall-and-chamber structure:



Wall-crossing formula.



Conjecture: (Ciocan-Fontanine — Kim)

$$[Q^{\varepsilon_-}_{g,n}(X,\beta)]^{vir} - [Q^{\varepsilon_+}_{g,n}(X,\beta)]^{vir}$$

$$"=" \sum_{k\geq 1}\sum_{\vec{\beta}}\frac{1}{k!}\prod_{i=1}^{k} ev_{n+i}^{*}\,\mu_{\beta_i}(z)\Big|_{z=-\psi_{n+i}} \cap [Q^{\varepsilon_+}_{g,n+k}(X,\beta')]^{vir}$$

where $\vec{\beta}=(\beta',\beta_1,\cdots,\beta_k)$. $\beta=\beta'+\beta_1+\cdots+\beta_k$. $\deg(\beta_i)=d_0$

$$\mu_{\beta_i} \longleftarrow \left\{ \mathbb{P}^1 \overset{\beta_i}{\dashrightarrow} X \right\}$$

base point
length $= d_0$

Numerical wall-crossing formula.

$$F_g^{\varepsilon}(\mathbf{t}(z)) := \sum_{n=0}^{\infty} \sum_{\beta \geq 0} \frac{q^{\beta}}{n!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{g,n,\beta}^{X,\varepsilon}$$

$$\mathbf{t}(z) \text{ "}\in\text{" } H^*_{CR}(X, \mathbb{Q})[z]$$

$$\mu^{\geq \varepsilon}(q,z) := \sum_{\varepsilon \leq \frac{1}{\deg(\beta)} < \infty} \mu_{\beta}^{\varepsilon}(z)\, q^{\beta}$$

Then.

$$F_g^{\varepsilon}(\mathbf{t}(z)) = F_g^{\infty}(\mathbf{t}(z) + \mu^{\geq \varepsilon}(q,-z)) \quad (g \geq 1.)$$

In $g=0$,

$$J^{\varepsilon}(\mathbf{t}(z), q, z) = 1 + \frac{t}{z} + \sum_{0<\deg(\beta)\le\frac{1}{\varepsilon}} I_\beta(z) q^\beta$$

$$+ \sum_{\beta\geq 0,\, k\geq 0} \frac{q^\beta}{k!} \sum_p T_p \left\langle \frac{T^p}{z(z-\psi)}, \mathbf{t}(\psi), \cdots, \mathbf{t}(\psi) \right\rangle^{X,\varepsilon}_{0,1+k,\beta}.$$

Then

$$J^{+\infty}(\mathbf{t}(z) + \mu^{\geq\varepsilon}(q,-z), q, z) = J^{\varepsilon}(\mathbf{t}(z), q, z).$$

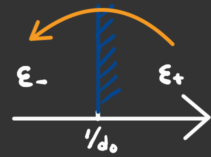
Setting $\mathbf{t}(z)=0$, $\varepsilon\to 0^+$

$$J^{+\infty}(\mu^{\geq 0}(q,-z), q, z) = I(q,z)$$

If X is simply connected CY3, invertible.

$$\mu^{\geq 0}(q,z) = I_1(q) + I_0(q)\cdot z. \qquad I_1(q) \in H^2(X;\mathbb{Q})$$

Wall-crossing formula.



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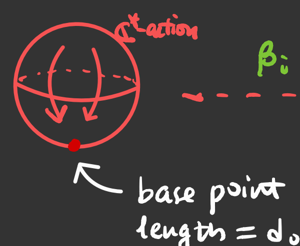
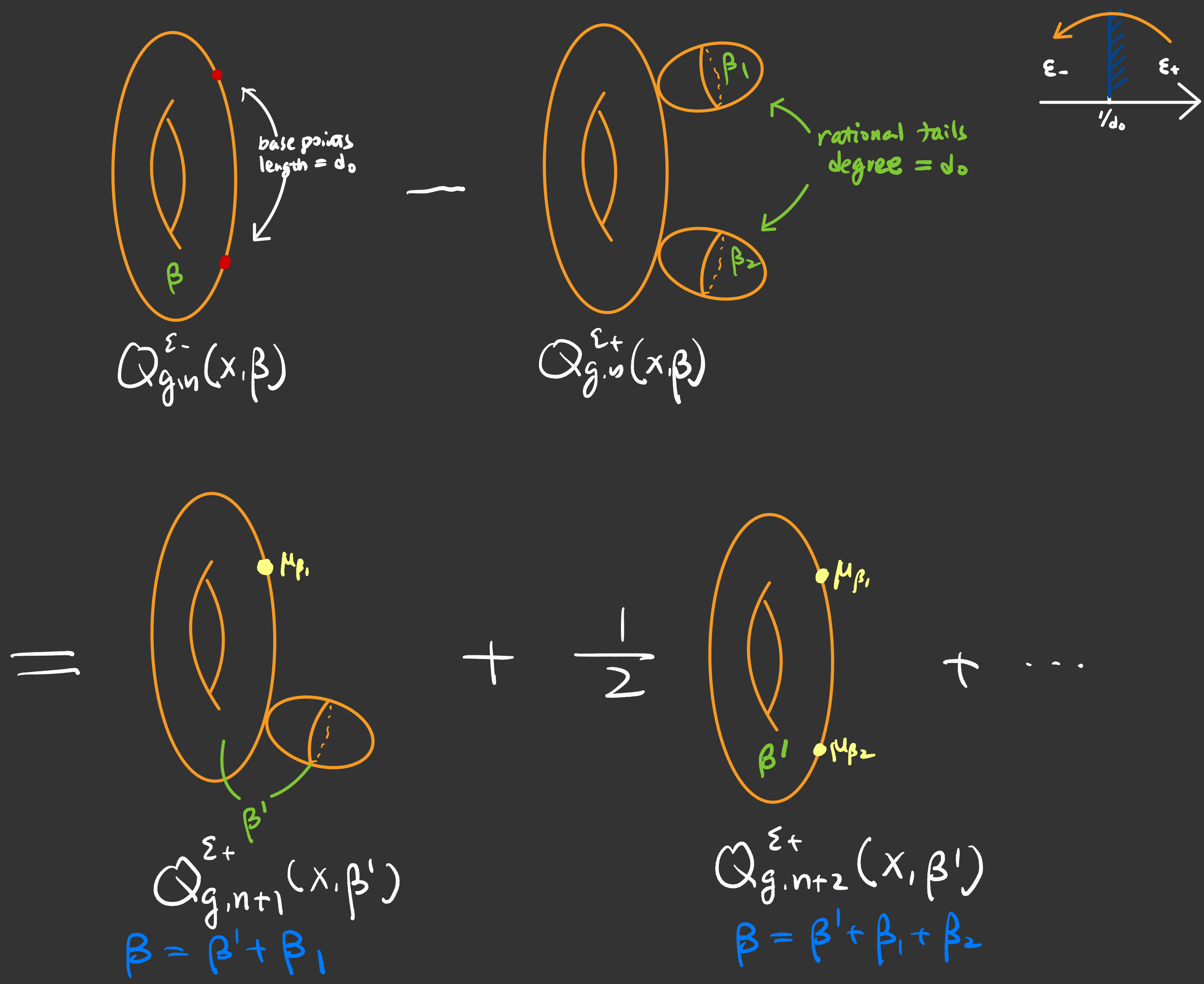
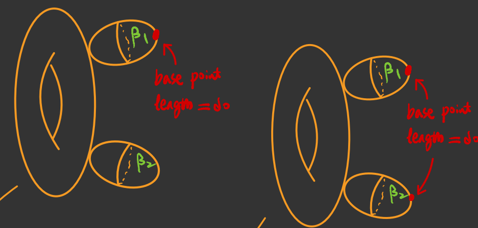
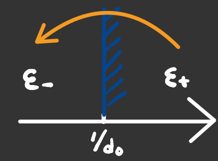


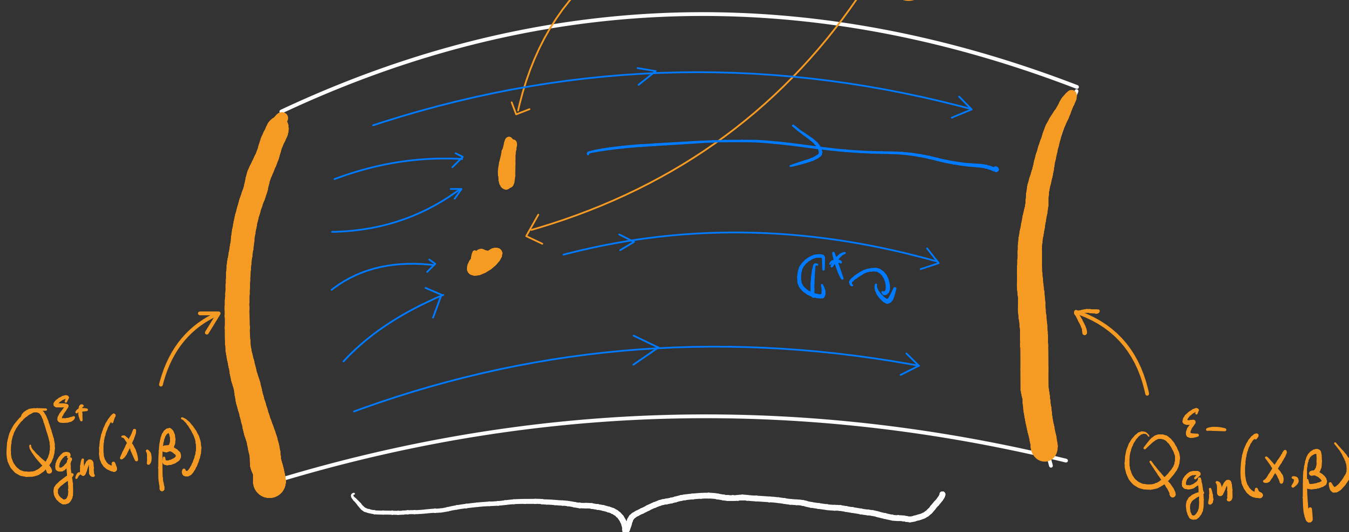


ε_-
ε_+
$1/d_0$
base points
length = d_0
β
$Q^{\varepsilon_-}_{g,n}(X,\beta)$
$-$
β_1
β_2
rational tails
degree = d_0
$Q^{\varepsilon_+}_{g,n}(X,\beta)$
$=$
μ_{β_1}
β'
$Q^{\varepsilon_+}_{g,n+1}(X,\beta')$
$\beta = \beta' + \beta_1$
$+ \frac{1}{2}$
μ_{β_1}
β'
μ_{β_2}
$Q^{\varepsilon_+}_{g,n+2}(X,\beta')$
$\beta = \beta' + \beta_1 + \beta_2$
$+ \cdots$

Idea of proof:



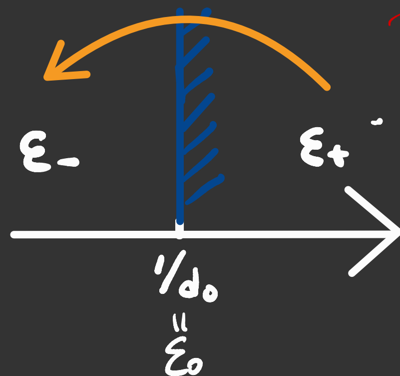




$MQ^{\varepsilon_0}_{g,n}(X,\beta)$

master space.

- Allows base points of length d_0
- ~~Disallows rat'l tails of degree d_0~~

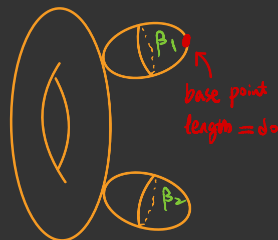


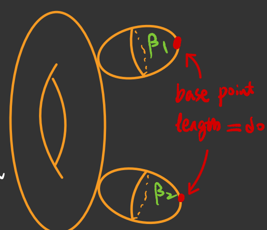
- ~~Disallows base points of length d_0~~
- Allows rational tails of degree d_0

Key observation:

If we allow both degree $= d_0$ rational tails and length $= d_0$ base points

Then

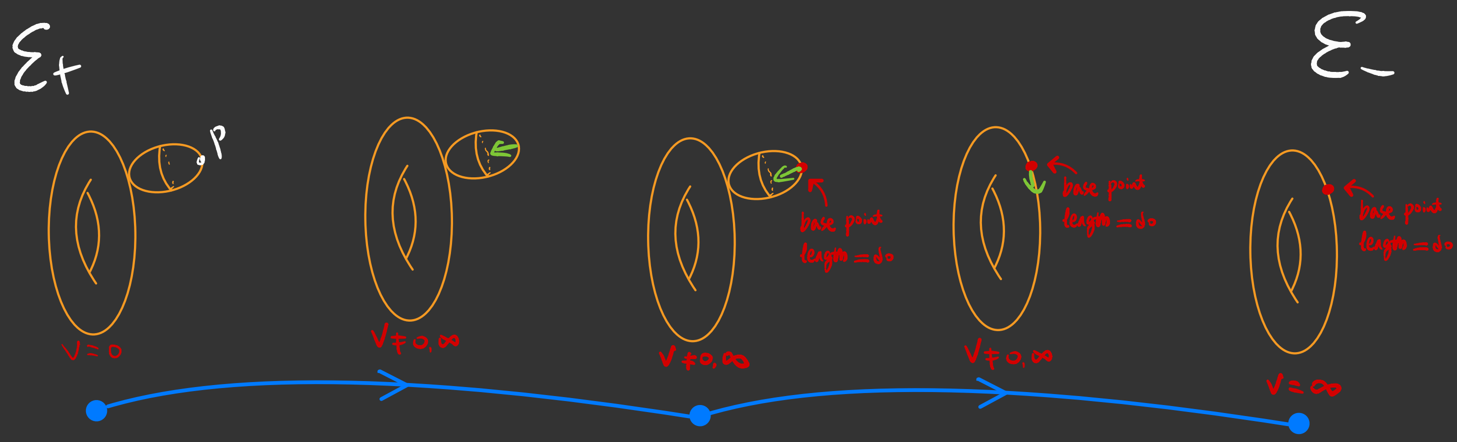
Aut() $= \infty$,

Aut() $= \infty$,

$\vdots$

$v \in T_pC \cup \{\infty\}$ s.t. when $v = 0$ Σ_+-stable

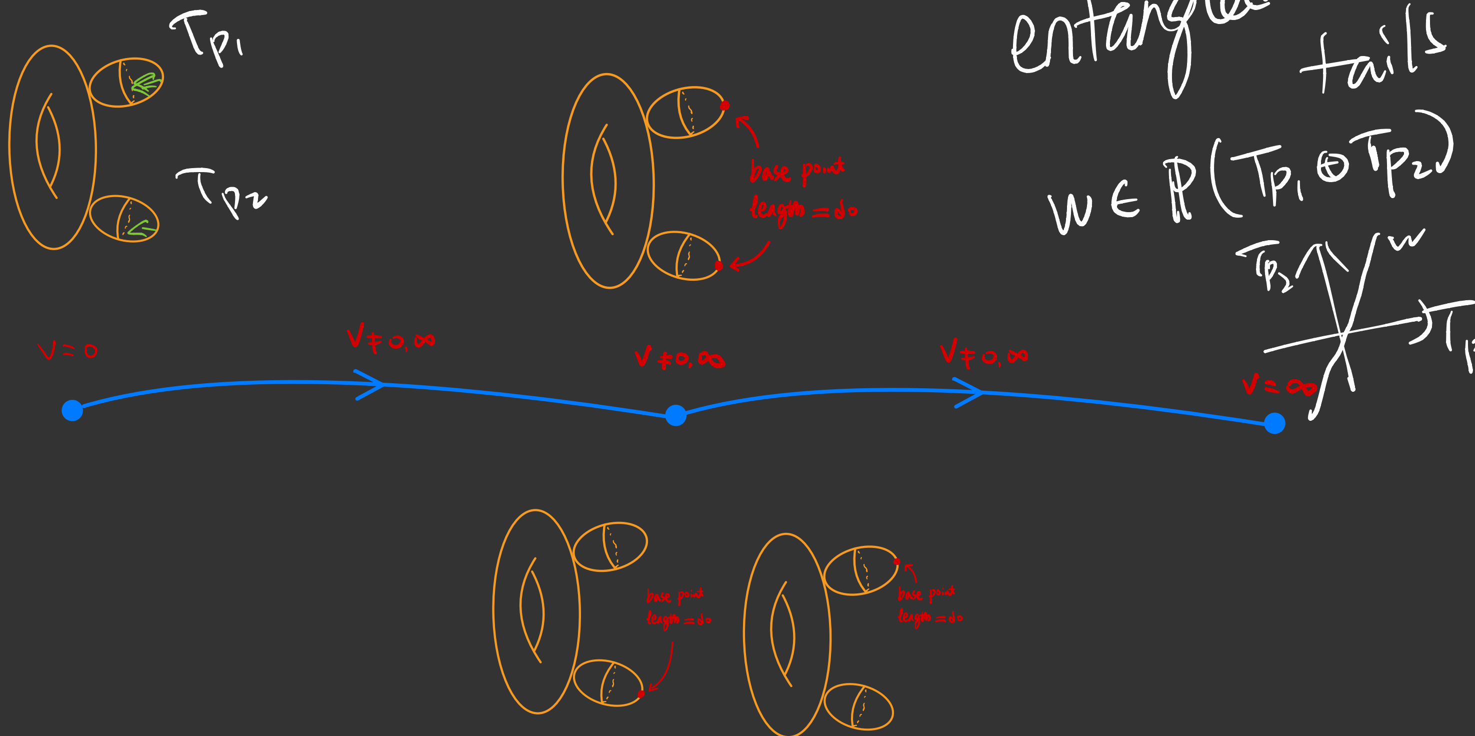
when $v = \infty$ Σ_--stable

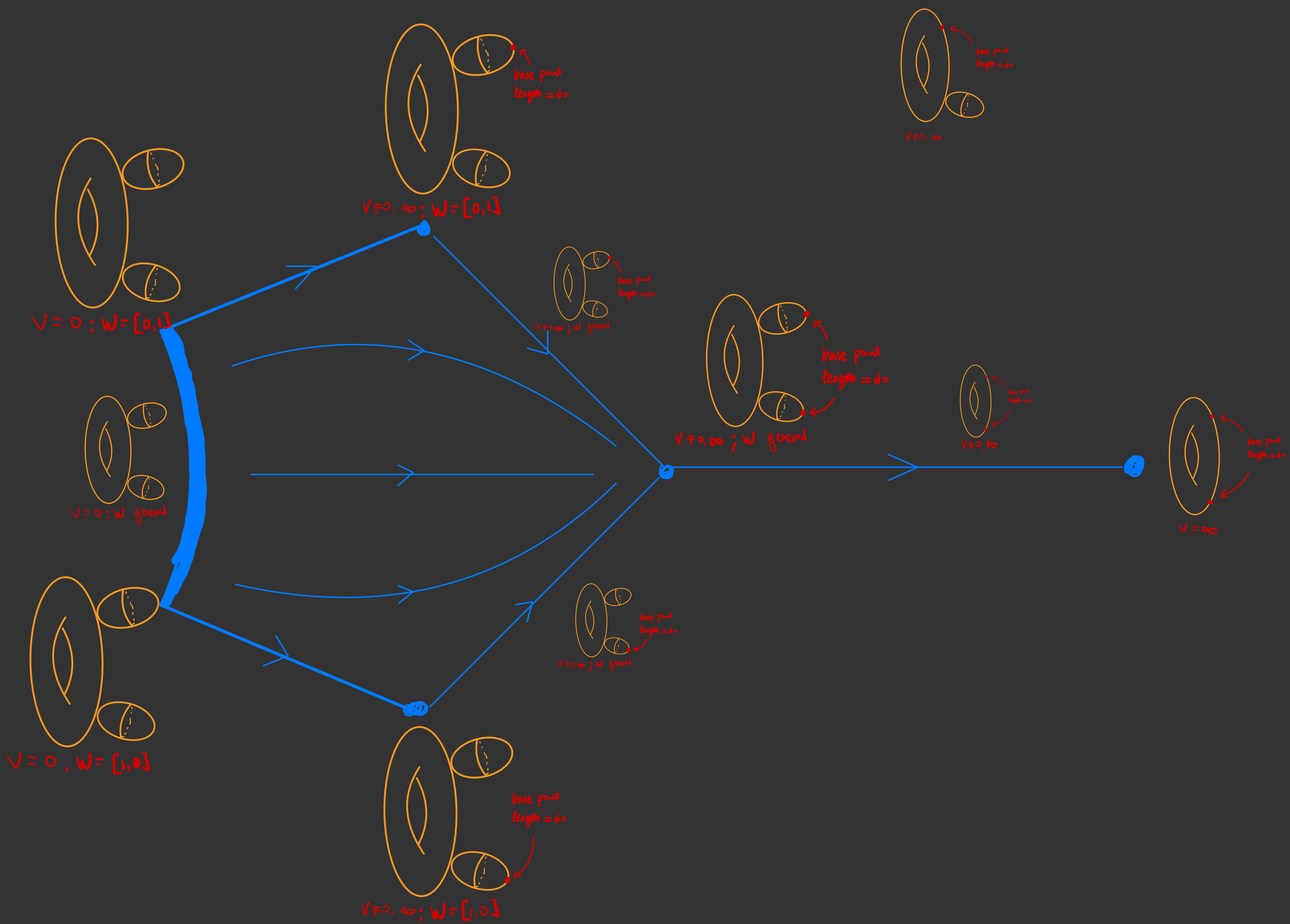


correction term

entangled rational tails

$w \in \mathbb{P}(T_{p_1} \oplus T_{p_2})$





V = 0 ; W = [0,1]
V = 0 ; W general
V = 0 , W = [1,0]
V ≠ 0, ∞ ; W = [0,1]
base point
length = d₀
V ≠ 0, ∞ ; W = [1,0]
V ≠ 0, ∞ ; W general
V ≠ 0, ∞
V = ∞

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X = smooth projective variety $/\mathbb{C}$

Counting curves in X.

$$\overline{M}_{g,n}(X,\beta) = \Big\{ C = $$

$$\xrightarrow{f} X \Big\}$$

RS of genus g.

target manifold

Two difficulties:

① $M_{g,n}(X,\beta)$ not compact. → $\overline{M}_{g,n}(X,\beta)$ stable map compactification.

$\#\mathrm{Aut}(C,x,f) < \infty$ = $\boxed{\text{stable}}$

<u>Thm</u>: $\overline{M}_{g,n}(X,\beta)$ is a proper DM stack. (cpt + Hausdorff). i.e.

not contracted by f.

② $M_{g,n}(X,\beta)$ is <u>almost</u> always highly singular.

$\exists$ perfect obstruction theory on $\overline{M}_{g,n}(X,\beta)$ $\Longrightarrow$ virtual fundamental class

$\in H_*(\overline{M}_{g,n}(X,\beta))$

$\rightsquigarrow$ intersection numbers $\rightsquigarrow$ Gromov-Witten theory.

Alternative compactifications of $M_{g,n}(X,\beta)$.

Quasimaps. for G.I.T. quotients.

$$X = (x_1^5 + \cdots + x_5^5 = 0) \subseteq \mathbb{P}^4$$

$$\left\{ C \xrightarrow{f} X \right\} \xleftrightarrow[\mathcal{L} = f^*\mathcal{O}(1)]{} \left\{ \begin{array}{l} \begin{array}{c} \mathcal{L} \\ \downarrow \\ C \end{array} + \varphi_1, \cdots, \varphi_5 \in \Gamma(C, \mathcal{L}), \\ \text{s.t. } \ ① \ (\vec{\varphi} = 0) = \phi \\ \qquad ② \ \sum \varphi_i^5 = 0 \end{array} \right\}$$

For $\overline{M}_{g,n}(X,\beta)$.

For quasimaps,

Just let the zeros collide.

Do not bubble.

Def: A quasimap $C \dashrightarrow X$ = quintic

is a line bundle $\mathcal{L} \to C$, and

$\varphi_1, \dots, \varphi_5 \in \Gamma(C, \mathcal{L})$ s.t.

$(\vec{\varphi} = 0)$ is finite, disjoint from the

nodes and markings.

call base points

$$X = W /\!\!/_{\theta} G$$

affine variety with mild sing.

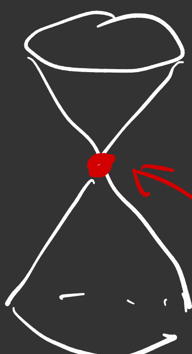
reductive group

$$\{C \to X\} \longleftrightarrow \left\{ \begin{array}{c} G\text{-}P \\ \downarrow \\ C \end{array}, \quad \begin{array}{c} P \times_G W \\ \downarrow \uparrow \sigma \\ C \end{array} \;\; \text{s.t. } \sigma(C) \subset P \times W^s(\theta) \right\}$$

$W^{ss}(\theta) = W^s(\theta)$.

$G \curvearrowright W^s(\theta)$ freely.

$\Longrightarrow X = [W^s(\theta)/G]$

$W =$  $= (x_1^5 + \cdots + x_5^5 = 0) \subseteq \mathbb{C}^5$, $\quad G = \mathbb{C}^*$,

unstable point.

Thm: (Ciocan-Fontaine—Kim—Maulik)

For any $\varepsilon \in \mathbb{Q}_{>0}$.

$Q^{\varepsilon}_{g,n}(X,d)$ is proper DM stack

(moduli of ε-stable quasimaps), with p.o.t.

$\Longrightarrow$ $[Q^{\varepsilon}_{g,n}(X,d)]^{vir}$ virtual fundamental class

ε-stable quasimap invariants.

Proved in

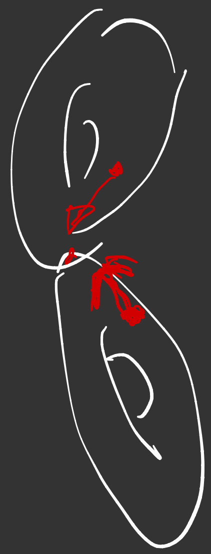
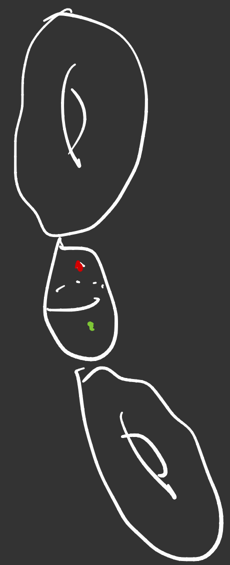
1) $g=0$ or X with strong torus action or C.I. in products of projective spaces by Ciocan-Fontanine-Kim

2) C.I. in projective space Clader-Janda-Ruan

3) $g=0$ for toric c.i. Jun Wang.

Thm (2-):

Wall-crossing conjecture is true in full generality.

As $g \to \infty$, difficultity gets closer.

$g=0$ $\quad X \subset \mathbb{P}^4$ $\quad$ Convex $\to$ QHP

quintic

$g=1$ $\quad$ stable map

quasimap