# The cobordism hypothesis in dimension 1

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Motivation: study closed smooth $n$-manifolds.

- Up to diffeomorphism?
- If we mod out we loose a lot of information, and we get just an unstructured set.
- $\Rightarrow$ consider the classifying space of closed smooth $n$-manifolds.

**Observation**

Complicated $n$-manifolds can often be better understood by presenting them as the result of gluing together simpler manifolds along boundaries.

$\Rightarrow$ Extend the scope to compact $n$-manifolds with boundary.

Invariants of manifolds which behave well with respect to gluing are also called *local*.

How to encode the operation of gluing manifolds along boundary components?

**Idea**

Consider compact $n$-manifolds as cobordisms between closed $(n-1)$-manifolds, with gluing given by composition.

## The cobordism ∞-category

For $n \geq 1$ we may organize the information of closed smooth $(n-1)$-manifolds and ($n$-dimensional) cobordisms between them into an ∞-category $\mathrm{Bord}_n^{n-1}$ whose

- Objects are closed smooth $(n-1)$-manifolds.
- The mapping space from $M$ to $N$ is the classifying space of cobordisms from $M$ to $N$, that is, smooth compact $n$-manifolds $W$ equipped with a diffeomorphism
$$\partial W \cong M \coprod N.$$
- Composition is given by gluing cobordisms together.

- Obtain local manifolds invariants from functors $Z\colon \mathrm{Bord}_n^{n-1} \to \mathcal{C}$, where $\mathcal{C}$ is a convenient target ∞-category.
- Closed $n$-manifolds are sent to endomorphisms of $Z(\varnothing)$.

What about locality on the level of $(n-1)$-manifolds?

$\Rightarrow$ Include also $(n-2)$-manifolds and all $(n-1)$-manifolds with boundary as cobordisms.

Need to pass to an $(\infty, 2)$-categorical object.

**The cobordism $(\infty, 2)$-category**

For $n \geq 1$ consider an $(\infty, 2)$-category $\mathrm{Bord}_n^{n-2}$ whose

- Objects are closed $(n-2)$-manifolds.
- $1$-Morphisms are cobordisms ($(n-1)$-manifolds with boundary).
- $2$-Morphisms are cobordisms between cobordisms ($n$-manifolds w. corners).

All compositions are given by gluing cobordisms together.

# Cobordism higher categories

## The cobordism $(\infty, k)$-category

For $n, k \geq 1$ consider an $(\infty, k)$-category $\mathrm{Bord}_n^{n-k}$ whose

- Objects are closed $(n-k)$-manifolds.
- $1$-Morphisms are cobordisms ($(n-k+1)$-manifolds with boundary).
- $2$-Morphisms are cobordisms between cobordisms ($(n-k+2)$-manifolds w. corners).
- For $i \geq 2$ the $i$-morphisms are cobordisms between cobordisms between cobordisms ... ($(n-k+i)$-manifolds with corners).

All compositions are given by gluing cobordisms together.

For $k = n$ we get an $(\infty, n)$-category $\mathrm{Bord}_n^0$ whose objects are $0$-manifolds, $1$-morphisms are $1$-dimensional cobordisms, $2$-morphisms are $2$-dimensional cobordisms, and so on.

What about locality on the level of $0$-manifolds?

A 0-manifold decomposes as a disjoint union of points.

**Idea**

Refine $\mathrm{Bord}_n^0$ to a *symmetric monoidal* $(\infty, n)$-category $\mathrm{Bord}_n$, with the monoidal structure given by disjoint union.

Completely local invariants (respect gluing in all dimensions) are given by symmetric monoidal functors

$$Z\colon \mathrm{Bord}_n \to \mathcal{C},$$

where $\mathcal{C}$ is a target symmetric monoidal $(\infty, n)$-category.

= unoriented topological field theories

**Variant**

Instead of just smooth manifolds we can consider manifolds endowed with a framing with respect to some fixed $n$-dimensional orthogonal vector bundle $E \to B$: the $i$-morphisms will now be $i$-dimensional cobordisms $W$ equipped with a continuous map $f\colon W \to B$ and an isomorphism $\tau\colon TW \oplus \mathbb{R}^{n-i} \cong f^*E$.

$\Rightarrow$ Obtain a symmetric monoidal $(\infty, n)$-category $\mathrm{Bord}_n^B$.

# Examples

## Example

When $E \to B$ is the universal orthogonal vector bundle $E_n \to \mathrm{B}O(n)$ the associated tangential structure is trivial, and we get again the unoriented cobordism $(\infty, n)$-category $\mathrm{Bord}_n$.

## Example

When $B = *$ we obtain for each $i$-dimensional cobordism $W$ a framing on $TW \oplus \mathbb{R}^{n-i}$. This yields the framed cobordism $(\infty, n)$-category $\mathrm{Bord}_n^{\mathrm{fr}}$.

## Example (of a topological field theory)

Let $\mathrm{Alg}_n$ be the symmetric monoidal $(\infty, n)$-category whose

- Objects are $\mathbb{E}_n$-ring spectra.
- $1$-Morphisms from $A$ to $B$ are $\mathbb{E}_{n-1}$-algebras in $(A, B)$-bimodules.
- Given $M, N$ two $\mathbb{E}_{n-1}$-algebras in $(A, B)$-bimodules, $2$-morphisms from $M$ to $N$ are $\mathbb{E}_{n-2}$-algebras in $(M, N)$-bimodules in $(A, B)$-bimodules.
- And so on: $i$-morphisms are $\mathbb{E}_{n-i}$-algebras in some iterated bimodule $\infty$-category.

Given an $\mathbb{E}_n$-algebra $A$ we may construct a topological field theory $\mathrm{Bord}_n^{\mathrm{fr}} \to \mathrm{Alg}_n$ via factorization homology; it sends an $i$-dimensional cobordism $W$ to the $\mathbb{E}_{n-i}$-algebra $\int_{W \times \mathbb{R}^{n-i}} A$.

Idea: the symmetric monoidal $(\infty, n)$-category $\mathrm{Bord}_n^B$ has a universal property, which permits to *classify* $B$-framed topological field theories.

For this, one needs the categorical notion of *adjoints*.

**Definition**

Let $\mathcal{C}$ be an $(\infty, n)$-category. For $1 \leq i \leq n-1$ let $f\colon x \to y$ an $i$-morphism between two parallel $(i-1)$-morphisms $x, y$. An $i$-morphism $g\colon y \to x$ is said to be *right adjoint* to $f$ if there exist $(i+1)$-morphisms $u\colon \mathrm{id}_x \to g \circ f$ (unit) and $v\colon f \circ g \to \mathrm{id}_y$ (counit) such that the composite $(i+1)$-morphisms

$$f \xrightarrow{\mathrm{id} \circ u} f \circ g \circ f \xrightarrow{v \circ \mathrm{id}} f \quad \text{and} \quad g \xrightarrow{u \circ \mathrm{id}} g \circ f \circ g \xrightarrow{\mathrm{id} \circ v} g$$

are homotopic to the respective identities. In that case we also say that $f$ is left adjoint to $g$.

If $\mathcal{C}$ carries a monoidal structure then we can also extend the above definition to $0$-morphisms, or objects, by using the monoidal product $\otimes$ in place of composition. In this case one often talks of *duals* instead of adjoints. If $\mathcal{C}$ is furthermore symmetric monoidal then the notions of left and right duals coincide.

We will say that a symmetric monoidal $(\infty, n)$-category *has duals* if every $i$-morphism has both a left and a right adjoint for every $0 \leq i \leq n-1$.

# The cobordism hypothesis

## $\mathrm{Bord}_n^B$ has duals

For $0 \leq i \leq n-1$, both the left and right adjoint of a $B$-framed $i$-cobordism $(W, f\colon W \to B, \eta\colon TW \oplus \mathbb{R}^{n-i} \to f^*E)$ is $(W, f, \overline{\eta})$, considered as going in the opposite direction, and where $\overline{\eta}$ is obtained from $\eta$ by reversing the first coordinate of the $\mathbb{R}^{n-i}$ component; the difference between the left and the right adjoint lie in the identification of the framing on the incoming and outgoing boundaries.

In 95' Baez and Dolan suggested the following:

## Conjecture (The cobordism hypothesis, framed version)

$\mathrm{Bord}_n^{\mathrm{fr}}$ *is the free symmetric monoidal* $(\infty, n)$*-category with duals generated by the canonically framed point* $* \in \mathrm{Bord}_n^{\mathrm{fr}}$*. In particular, for every symmetric monoidal* $(\infty, n)$*-category with duals* $\mathcal{C}$*, evaluation at* $*$ *induces an equivalence*

$$\iota\,\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \iota\mathcal{C},$$

*where* $\iota$ *means taking core* $\infty$*-groupoids.*

In fact, one can show that the $(\infty, n)$-category of symmetric monoidal functors between any two symmetric monoidal $(\infty, n)$-categories with duals is already an $\infty$-groupoid, and so in the above formulation of the cobordism hypothesis one can also remove the $\iota$ symbol on the left hand side.

# Orthogonal group action

The group $O(n)$ acts on $\mathrm{Bord}_n^{\mathrm{fr}}$ by modifying the framing.
$\Rightarrow$
Any $(\infty, n)$-category with duals $\mathcal{C}$ admits a canonical action of $O(n)$ on its core $\infty$-groupoid $\iota\mathcal{C} \simeq \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C})$.

This observation allows one to formulate the cobordism hypothesis for a general $n$-dimensional orthogonal vector bundle $E \to B$.

Let $\mathrm{Fr}(E) \to B$ be the associated frame bundle. Note that every $x \in \mathrm{Fr}(E)$ determines a $B$-framing on the $0$-dimensional manifold $*$. In particular, we have a natural map

$$\mathrm{Fr}(E) \to \iota\,\mathrm{Bord}_n^B,$$

which is furthermore $O(n)$-equivariant.

**Conjecture (The cobordism hypothesis, $B$-framed version)**

*For every symmetric monoidal $(\infty, n)$-category with duals $\mathcal{C}$ restriction along the above map induces an equivalence*

$$\iota\,\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^B, \mathcal{C}) \simeq \mathrm{Map}_{O(n)}(\mathrm{Fr}(E), \iota\mathcal{C}).$$

# In dimension 0

When $n = 0$ the group $O(n) = O(0)$ is trivial and $E \to B$ is a $0$-dimensional vector space, so we can forget about it.

**The cobordism hypothesis in dimension 0**

Let $B$ be a space. Then the symmetric monoidal $\infty$-groupoid $\mathrm{Bord}_0^B$ is freely generated from the space $B$.

This statement is not difficult to show: $\mathrm{Bord}_0^B$ can be identified with the symmetric monoidal $\infty$-groupoid $\iota\,\mathrm{Fin}_{/B}$ whose objects are finite sets $I$ equipped with a map $I \to B$ and whose morphisms are isomorphisms over $B$. It corresponds to the space $\coprod_n B^n_{\mathrm{h}\Sigma_n}$, which is indeed the free $\mathbb{E}_\infty$-monoid generated from $B$.

# Inductive formulation

Given an orthogonal vector bundle $E \to B$, let us denote by $S(E) \to B$ the corresponding sphere bundle, and by $E^{\perp} \to S(E)$ the sub-vector bundle of $E|_{S(E)}$ whose fiber over $(x, v) \in S(E)$ is the orthogonal complement of $v$ in $E_x$. Then for $0 \leq i \leq n-1$, the data of a $B$-framing of the form

$$f\colon W \to B, \eta\colon TW \oplus \mathbb{R}^{n-i} \cong f^* E$$

is equivalent to the data of an $S(E)$-framing of the form

$$f'\colon W \to S(E), \eta'\colon TW \oplus \mathbb{R}^{n-i-1} \cong (f')^* E^{\perp}.$$

We thus obtain a canonical functor

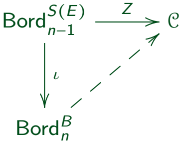
$$\iota\colon \mathrm{Bord}_{n-1}^{S(E)} \to \mathrm{Bord}_n^B .$$

## Definition

For $k \geq 0$ we say that a functor of $(\infty, n)$-categories $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is $k$-connective if it is essentially surjective, and if for every $i = 0, \ldots, k-1$ and every two parallel $i$-morphisms $f, g$ of $\mathcal{C}$ then induced functor of $(\infty, n-i-1)$-categories $\mathrm{Map}(f, g) \to \mathrm{Map}(\mathcal{F}(f), \mathcal{F}(g))$ is essentially surjective.

By construction, the functor $\iota$ above is $(n-1)$-connective. It might fail however to be an equivalence on underlying $(\infty, n-1)$-categories due to the existence of non-trivial h-cobordisms.

- **Inductive formulation**: reduce to showing that dotted extensions




  are classified by $B$-indexed families of unit $n$-cells $1_{\mathcal{C}} \to Z(S(E)_b)$ for $b \in B$, exhibiting the images under $Z$ of the two hemispheres as adjoint to each other.

- **Reduction to the unoriented case**: for a fixed $n$, the inductive formulation for the universal orthogonal vector bundle $E_n \to \mathrm{B}O(n)$ implies the inductive statement in general. We can hence forget about framing and just consider unoriented manifolds.
- **Remove unnecessary information**: one may assume without loss of generality that $Z$ is $(n-1)$-connective (since $\iota$ is so).
- **Reduction of the categorical level**: for $n \geq 2$, pairs $(\mathcal{C}, Z)$ with $Z$ an $(n-2)$-connective functor are classified by lax symmetric monoidal functors $\mathrm{Bord}_{n-1}^{n-2} \to \mathrm{Cat}_\infty$ via the association $Z \mapsto M_Z$, where $M_Z(X) = \mathrm{Map}_{\mathcal{C}}(1_{\mathcal{C}}, Z(X))$.
- **Unstraightening**: lax symmetric monoidal functors $\mathrm{Bord}_{n-1}^{n-2} \to \mathrm{Cat}_\infty$ can equivalently be encoded via symmetric monoidal cocartesian fibrations $\mathcal{X} \to \mathrm{Bord}_{n-1}^{n-2}$ via the Grothendieck construction.

# Lurie's strategy

- **Reduction to the inductive step**.
- **Reduction of the inductive step to its unoriented instance**.
- **Removing unnecessary information**.
- **Reduction of the categorical level ($n \geq 2$)**.
- **Unstraightening**.

These parts can all be set up rigorously with modern technology.

## Final step (generators and relations)

Let $\mathcal{X} \to \mathrm{Bord}_{n-1}^{n-2}$ and $\mathcal{Y} \to \mathrm{Bord}_{n-1}^{n-2}$ be the symmetric monoidal cocartesian fibrations corresponding to the $(n-2)$-connective symmetric monoidal functors $\mathrm{id}\colon \mathrm{Bord}_{n-1} \to \mathrm{Bord}_{n-1}$ and $\mathrm{Bord}_{n-1} \to \mathrm{Bord}_n$. Then $\mathcal{Y}$ is generated from $\mathcal{X}$ by generators and relations corresponding to handles of various indices and the cancellation of handles of successive indices.

This part is done using Morse theory and Igusa's theory of framed functions. In particular, it uses the contractibility of the space of framed functions, which, though was not yet proven in 2009, has been proven since by Eliashberg and Mishachev.

## But...

Even when using framed functions to break the passage from $\mathcal{X}$ to $\mathcal{Y}$ into steps using an index filtration, the bottom line generators and relations statement required at each step lacks a proof, which has not been completed since.

Also, the second half of Lurie's strategy does not cover the case $n = 1$.

# Current status of the proof

- In the aftermath of Lurie's 2009 paper, Schommer-Pries announced knowing how to complete the last part using a paper of Ayala–Francis–Rozenblyum. Unfortunately, a mistake was discovered in that paper, and a proof by Schommer-Pries never appeared.
- A new proof by Schommer-Pries could be coming soon, in a forthcoming paper called "the relative tangle hypothesis".
- Ayala and Francis themselves describe a conditional proof of the cobordism hypothesis based on a certain conjecture, though the validity of that conjecture is unclear due to the previously mentioned mistake.
- Grady and Pavlov have recently put online a manuscript claiming to prove not just the cobordism hypothesis, but a vast generalization thereof. Their paper has not yet been refereed and for the moment its status is not known.

# The cobordism hypothesis in dimension 1

Let $E \to B$ be a 1-dimensional orthogonal vector bundle, so that $\pi\colon S(E) \to B$ is a degree 2 covering space. In particular, for each $b \in B$ the fiber $S(E)_b$ is a set of size 2.

## Inductive formulation $0 \to 1$

For a symmetric monoidal $\infty$-category with duals $\mathcal{C}$, extensions

$$\begin{array}{ccc} \mathrm{Bord}_0^{S(E)} & \xrightarrow{Z} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathrm{Bord}_1^{B} & & \end{array}$$

are classified by $B$-indexed families of unit morphisms

$$1_{\mathcal{C}} \to Z(S(E)_b) = \bigotimes_{x \in S(E)_b} Z(x)$$

exhibiting the two objects of $\{Z(x)\}_{x \in S(E)_b}$ as dual to each other.

The first step in Lurie's strategy applies here:

## Proposition

*The $[0 \to 1]$-cobordism hypothesis implies the cobordism hypothesis for $n = 1$.*

# The $[0 \to 1]$-inductive formulation

**Proposition**

*The $[0 \to 1]$-cobordism hypothesis implies the cobordism hypothesis for $n = 1$.*

**Proof.**

By the cobordism hypothesis in dimension $0$, the data of a symmetric monoidal functor $Z\colon \mathrm{Bord}_0^{S(E)} \to \mathcal{C}$ corresponds to an $\infty$-groupoid map $f\colon S(E) \to \iota\mathcal{C}$. There is a $\mathcal{C}_2$-action on both sides, on $S(E)$ by its structure as a frame bundle and on $\iota\mathcal{C}$ since it is a symmetric monoidal $\infty$-category with duals. We then have an induced $C_2$-action on the $\infty$-groupoid $\mathcal{X} := \mathrm{Map}(S(E), \iota\mathcal{C})$, such that $\mathcal{X}^{\mathrm{hC}_2} = \mathrm{Map}_{C_2}(S(E), \iota\mathcal{C})$. Given $f \in \mathcal{X}$, we also have a $C_2$-action on the space of natural equivalences $\mathrm{Equiv}(\overline{f}, f)$, whose $C_2$-homotopy fixed points correspond to $C_2$-homotopy fixed point structures on $f$. We then compute

$$\begin{aligned}
\mathcal{X}^{\mathrm{hC}_2} \times_{\mathcal{X}} \{f\} = \mathrm{Equiv}(\overline{f}, f)^{\mathrm{hC}_2} &= \Big[ \lim_{x \in S(E)} \mathrm{Map}_{\iota\mathcal{C}}(\overline{f(\overline{x})}, f(x)) \Big]^{\mathrm{hC}_2} \\
&= \Big[ \lim_{x \in S(E)} \mathrm{Map}_{\mathcal{C}}^{\mathrm{unit}}(1_{\mathcal{C}}, f(\overline{x}) \otimes f(x)) \Big]^{\mathrm{hC}_2} \\
&= \lim_{b \in B} \mathrm{Map}_{\mathcal{C}}^{\mathrm{unit}}(1_{\mathcal{C}}, Z(S(E)_b)).
\end{aligned}$$

Assuming the inductive formulation we have that extensions of $Z$ to $\mathrm{Bord}_1^B$ are classified by the data on the right hand side, and hence by that on the left. □

# Proof of the $[0 \to 1]$-cobordism hypothesis

We now discuss how to prove the $[0 \to 1]$-inductive step of the cobordism hypothesis.

## II

The second step in Lurie's strategy is the reduction to unoriented case. Lurie's argument works in the $[0 \to 1]$-case, but we will skip it as our argument will not be very sensitive to the tangential structure.

## III

The third step in Lurie's strategy is a reduction to the case where $Z\colon \mathrm{Bord}_0^{S(E)} \to \mathcal{C}$ is essentially surjective. Indeed, we can always replace $\mathcal{C}$ with the essential image of $Z$.

## IV

The fourth step, if we try to adapt it creatively to $n = 1$, is not very useful: it would essentially say that the data of a symmetric monoidal $\infty$-category under $\mathrm{Bord}_0^{S(E)}$ corresponds to a symmetric monoidal $\infty$-category with an action of $\mathrm{Bord}_0^{S(E)}$, but this doesn't seem to peel away any of the complexity.

This is roughly because we are already on categorical level 1, and so if we want to simplify we would need to peel further to categorical level 0, that is, to $\infty$-groupoids.

We proceed to enlarge $\mathrm{Bord}_0^{S(E)}$ by freely adding to it a $B$-indexed family of morphisms $\mathrm{ev}_b\colon S(E)_b \to \varnothing$. This results in a familiar object: it is simply the subcategory

$$\mathrm{Bord}_1^{B,\supset} \subseteq \mathrm{Bord}_1^{B}$$

containing all objects and just the morphisms whose underlying cobordism is a disjoint union of identity segments and $\supset$-shaped cobordisms (from two points to the empty set). We note that

$$\iota\,\mathrm{Bord}_1^{B,\supset} = \iota\,\mathrm{Bord}_1^{B} = \mathrm{Bord}_0^{S(E)} = \coprod_{n=0}^{\infty} S(E)^n_{\Sigma_n}$$

and in particular carries a $C_2$-action compatible with the $C_2$-action on $S(E)$. Since $\mathrm{Bord}_0^{S(E)}$ is freely generated from $S(E)$ the $S(E)$-family of arrows

$$\mathrm{ev}_{\pi(x)}\colon S(E)_{\pi(x)} = x \coprod \overline{x} \to \varnothing$$

extends to symmetric monoidal functor

$$\mathrm{ev}\colon \mathrm{Bord}_0^{S(E)} \to (\mathrm{Bord}_1^{B,\supset})_{/\varnothing}$$

sending an object $X \in \mathrm{Bord}_0^{S(E)}$ to a morphism of the form $\mathrm{ev}_X\colon X \coprod \overline{X} \to \varnothing$.

**Definition**

Let us say that a symmetric monoidal functor $Z\colon \mathrm{Bord}_1^{B,\supset} \to \mathcal{C}$ is *non-degenerate* if for every $b \in B$ the morphism $Z(\mathrm{ev}_b)\colon Z(S(E)_b) \to 1_{\mathcal{C}}$ is a counit exhibiting the two objects $\{Z(x)\}_{x \in S(E)_b}$ as dual to each other. We then write

$$\mathrm{Fun}^{\mathrm{nd}}(\mathrm{Bord}_1^{B,\supset}, \mathcal{C}) \subseteq \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{B,\supset}, \mathcal{C})$$

the full subcategory spanned by the non-degenerate functors.

By the dual argument used in the previous proposition we have the following:

**Proposition**

*We have a natural equivalence*

$$\iota\,\mathrm{Fun}^{\mathrm{nd}}(\mathrm{Bord}_1^{B,\supset}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Map}_{C_2}(S(E), \iota\mathcal{C}).$$

To a non-degenerate functor $Z\colon \mathrm{Bord}_1^{B,\supset} \to \mathcal{C}$ we could associate the lax symmetric monoidal functor $M_Z\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}_\infty$ given by

$$M_Z(x) = \mathrm{Map}_{\mathcal{C}}(1_{\mathcal{C}}, Z(x)).$$

## Lurie step IV - $[0 \to 1]$-version

Suppose given a lax symmetric monoidal functor $M\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}_\infty$. We may try to define an $\infty$-category $\mathcal{B}(M)$ as follows:

- The objects of $\mathcal{B}(M)$ are the objects of $\mathrm{Bord}_1^{B,\supset}$ (or $\mathrm{Bord}_0^{S(E)}$).
- For $X, Y \in \mathrm{Bord}_1^{B,\supset}$ the mapping space $\mathrm{Map}_{\mathcal{B}(M)}(X, Y)$ is given by $M(\overline{X} \otimes Y)$.
- For $X, Y, Z \in \mathrm{Bord}_1^{B,\supset}$ the composition law

$$\mathrm{Map}_{\mathcal{B}(M)}(X, Y) \times \mathrm{Map}_{\mathcal{B}(M)}(Y, Z) \to \mathrm{Map}_{\mathcal{B}(M)}(X, Z)$$

  is given by the composite

$$M(\overline{X} \otimes Y) \times M(\overline{Y} \otimes Z) \to M(\overline{X} \otimes Y \otimes \overline{Y} \otimes Z) \to M(\overline{X} \otimes Z),$$

  where the first map is induced by the lax monoidal structure of $M$ and the second by post-composition with $Z(\mathrm{ev}_Y)$.

But we have no units!

$\Rightarrow$

The object $\mathcal{B}(M)$ is a *non-unital* $\infty$-category. In fact, it's a symmetric monoidal one.

**Definition**

In a non-unital $\infty$-category $\mathcal{E}$, a *quasi-unit* is a morphism $f: x \to x$ which has the property that the operations of pre-composition and post-composition with $f$ are equivalences. We say that $\mathcal{E}$ is *quasi-unital* if every object admits a quasi-unit.

The collection of quasi-unital $\infty$-categories can be organized into an $\infty$-category $\mathrm{Cat}_\infty^{\mathrm{qu}}$, where the morphisms are functors which send quasi-units to quasi-units.

**Definition**

We will say that a lax symmetric monoidal functor $M: \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}$ is *non-degenerate* if $\mathcal{B}(M)$ is quasi-unital, and will say that a natural transformation $M \Rightarrow M'$ between non-degenerate functors is non-degenerate if the associated functor $\mathcal{B}(M) \to \mathcal{B}(M')$ preserves quasi-units.

The operation $M \mapsto \mathcal{B}(M)$ can then be organized into a functor

$$\mathrm{Fun}^{\mathrm{nd}}(\mathrm{Bord}_1^{B,\supset}, \mathrm{Grp}) \to (\mathrm{Cat}_\infty^{\mathrm{qu},\mathrm{SM}})_{\mathrm{Bord}_0^{S(E)}/}$$

from the $\infty$-category of non-degenerate lax symmetric monoidal functors $\mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}$ and non-degenerate natural transformations between them to the $\infty$-category of symmetric monoidal quasi-unital $\infty$-categories equipped with a symmetric monoidal functor from $\mathrm{Bord}_0^{S(E)}$.

# From quasi-unital to unital

If $Z\colon \mathrm{Bord}_1^{B,\supset} \to \mathcal{C}$ is an essentially surjective symmetric monoidal functor and $\mathcal{C}$ has duals then $B(M_Z)$ is naturally equivalent to the underlying (symmetric monoidal) non-unital $\infty$-category of $\mathcal{C}$, which is quasi-unital.

**Proposition (H.)**

*The forgetful functor*

$$\mathrm{Cat}_\infty \to \mathrm{Cat}_\infty^{\mathrm{qu}}$$

*is an equivalence.*

**Corollary**

*If $Z\colon \mathrm{Bord}_1^{B,\supset} \to \mathcal{C}$ is an essentially surjective symmetric monoidal functor and $\mathcal{C}$ has duals then the lax symmetric monoidal functor $M_Z\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}$ captures enough information to reconstruct $\mathcal{C}$ (as a symmetric monoidal $\infty$-category under $\mathrm{Bord}_0^{S(E)}$).*

Arguing along these lines one reduces the $[0 \to 1]$-cobordism hypothesis to the following statement:

**Proposition (The cobordism hypothesis in dimension 1, bottom line)**

*The lax symmetric monoidal functor $M_\iota\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}$ associated to the inclusion $\iota\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Bord}_1^{B}$ is initial in $\mathrm{Fun}^{\mathrm{nd}}(\mathrm{Bord}_1^{B,\supset}, \mathrm{Grp})$.*

# Proof of bottom line

**Proposition (The cobordism hypothesis in dimension 1, bottom line)**

*The lax symmetric monoidal functor $M_\iota$ is initial in* $\mathrm{Fun}^{\mathrm{nd}}(\mathrm{Bord}_1^{B,\supset}, \mathrm{Grp})$.

Using a suitable Grothendieck construction we can pass from lax symmetric monoidal functors $M\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}$ to symmetric monoidal left fibrations $\mathcal{T} \to \mathrm{Bord}_1^{B,\supset}$. Unwinding the definitions, in the case of $\iota\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Bord}_1^{B}$ the symmetric monoidal $\infty$-category $\widetilde{M}$ has

- Objects $B$-framed $1$-manifolds with boundary.
- Morphisms are given by framed open embeddings which are surjective on components (and which are not required to respect the boundary).

We then let $\mathcal{T}_0 \subseteq \mathcal{T}$ be the full symmetric monoidal subcategory spanned by those $X \in \mathcal{T}$ whose underlying $1$-manifold is a disjoint union of segments.

## Proposition

*Let* $\widetilde{M} \to \mathrm{Bord}_1^{B,\supset}$ *be a symmetric monoidal left fibration. Restriction along* $\mathcal{T}_0 \subseteq \mathcal{T}$ *induces an equivalence*

$$\mathrm{Fun}^{\otimes}_{/\mathrm{Bord}_1^{B,\supset}}(\mathcal{T}, \widetilde{M}) \xrightarrow{\simeq} \mathrm{Fun}^{\otimes}_{/\mathrm{Bord}_1^{B,\supset}}(\mathcal{T}_0, \widetilde{M}).$$

## Proof.

An inverse to this map is given by relative left Kan extension over $\mathrm{Bord}_1^{B,\supset}$. The fact that this left Kan extension exists, is symmetric moonidal and covers every symmetric monoidal $\mathcal{T} \to \widetilde{M}$ results from combining the following two crucial points:

- $\widetilde{M} \to \mathrm{Bord}_1^{B,\supset}$ is a left fibration (in particular, its fibers are $\infty$-groupoids).
- For every $X \in \mathcal{T}$ the comma category $(\mathcal{T}_0)_{/X} := \mathcal{T}_0 \times_{\mathcal{T}} \mathcal{T}_{/X}$ is weakly contractible.

For the second statement it is enough to check for $X$ whose underlying $1$-manifold is the circle. The $\infty$-category $(\mathcal{T}_0)_{/X}$ then has objects are non-empty disjoint unions of segments embedded in the circle, with morphisms being component surjective embeddings over the circle. This $\infty$-category is a familiar one which arises naturally in the context of factorization homology, and its contractibility is not hard to verify. □

We recognize the symmetric monoidal $\infty$-category $\mathcal{T}_0$ as being the monoidal envelope of the $\infty$-operad $\mathbb{E}^B_{\mathrm{nu}}$ whose colors are $B$-framed 1-discs and whose spaces of multi-operations $(X_1, ..., X_n) \to Y$ are empty if $n = 0$ and are given by spaces of $B$-framed embeddings $X_1 \coprod ... \coprod X_n \to Y$ if $n \geq 1$.

By the universal property of symmetric monoidal envelopes, symmetric monoidal functors $\mathcal{T}_0 \to \widetilde{M}$ over $\mathrm{Bord}_1^{B,\supset}$ correspond to $\mathbb{E}^B_{\mathrm{nu}}$-algebra objects in the $\mathbb{E}^B_{\mathrm{nu}}$-monoidal $\infty$-groupoid encoded by the restricted left fibration

$$\widetilde{M}' := \widetilde{M} \times_{\mathrm{Bord}_1^{B,\supset}} (\mathbb{E}^B_{\mathrm{nu}})^{\otimes} \to (\mathbb{E}^B_{\mathrm{nu}})^{\otimes}.$$

The data of an $\mathbb{E}^B_{\mathrm{nu}}$-monoidal $\infty$-groupoid is the same as that of a $B$-indexed family $\{\mathcal{X}_b\}_{b \in B}$ of non-unital monoidal $\infty$-groupoids, with $\mathbb{E}^B_{\mathrm{nu}}$-algebra objects corresponding to $B$-families $\{A_b \in \mathrm{Alg}_{\mathbb{E}^1_{\mathrm{nu}}}(X_b)\}$ of non-unital algebra objects.

# End of proof

In the case above, the $B$-family corresponding to $\widetilde{M}'$ has $X_b = \widetilde{M}_{S(E)_b}$, that is the fiber of $\widetilde{M} \to \mathrm{Bord}_1^{B,\supset}$ over the object $S(E)_b \in \mathrm{Bord}_1^{B,\supset}$ (corresponding to a disjoint union of two $B$-framed points with inverse framing).

In particular, each symmetric monoidal functor $\mathcal{F}\colon \mathcal{T} \to \widetilde{M}$ over $\mathrm{Bord}_1^{B,\supset}$ gives, when transported along the equivalences

$$\mathrm{Fun}^{\otimes}_{/\mathrm{Bord}_1^{B,\supset}}(\mathcal{T}, \widetilde{M}) \simeq \mathrm{Fun}^{\otimes}_{/\mathrm{Bord}_1^{B,\supset}}(\mathcal{T}_0, \widetilde{M}) \to \mathrm{Alg}_{\mathbb{E}^B_{\mathrm{nu}}}(\widetilde{M}'),$$

a family $\{A_b\}_{b\in B}$ of non-unital algebra objects in the family of non-unital monoidal $\infty$-groupoids $\{\widetilde{M}_{S(E)_b}\}$.

If $\widetilde{M} \to \mathrm{Bord}_1^{B,\supset}$ comes from a non-degenerate lax monoidal functor $M\colon \mathrm{Bord}_1^{B,\supset} \to \mathrm{Grp}$, then $\widetilde{M}_{S(E)_b}$ is quasi-unital, in which case $\mathcal{F}$ corresponds to a non-degenerate natural transformation $M_\iota \Rightarrow M$ if and only if the underlying object of each $A_b$ is a quasi-unit in $\widetilde{M}_{S(E)_b}$ (that is, the operations $A_b \otimes (-)$ and $(-) \otimes A_b$ are equivalences).

In any quasi-unital monoidal $\infty$-groupoid, the space of non-unital algebra objects whose underlying object is a quasi-unit is contractible.

$\Rightarrow$

$M_\iota$ is initial in $\mathrm{Fun}^{\mathrm{nd}}(\mathrm{Bord}_1^{B,\supset}, \mathrm{Grp})$ (bottom line statement).

Thank You!