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On very stable bundles

Notation: X = smooth projective curve$/\mathbb{C}$

$g(X)$ = genus $= g \geq 2$. K = canonical bundle of X

$N_X(n,d)$ = coarse moduli space of semi-stable vector bundles of rank n and degree d

$\longrightarrow$ projective variety

$\longrightarrow$ $\dim N_X(n,d) = n^2(g-1)+1$.

Definition (Drinfeld)
(letter to P. Deligne '81)

E is very stable if E has no non-zero nilpotent Higgs field $\varphi \in H^0(X, End(E)\otimes K)$

$\varphi : E \longrightarrow E \otimes K$.

$i \geq 1$: $\varphi^i : E \xrightarrow{\varphi} E\otimes K \xrightarrow{\varphi} E\otimes K^2 \longrightarrow \cdots \longrightarrow E\otimes K^i$

φ is nilpotent if $\varphi^i = 0$ for some $i>0$
the smallest such i = index of nilpotency of φ
If φ nilpotent, index of nilpotency $\leq n = rk(E)$

Properties of very stable bundles

(1) Drinfeld/Laumon very stable $\Longrightarrow$ stable.

Proof: we will show: not stable $\Rightarrow$ not very stable

E not stable $\Rightarrow \exists F \subsetneq E$ with $\mu(F) \geq \mu(E) \geq \mu(E/F)$

so we can construct a nilpotent Higgs field (of index 2)

$$0 \to F \xrightarrow{i} E \xrightarrow{\pi} E/F \to 0.$$

$$\swarrow \mu$$

$$0 \to FK \xrightarrow{i} EK \to E/F\,K \to 0$$

by taking $\varphi = i \circ u \circ \pi$ if $\mathrm{Hom}(E/F, FK) \neq 0$

But $\mu(\mathrm{Hom}(E/F, FK)) = \underbrace{-\mu(E/F) + \mu(F)}_{\geq 0} + 2g-2$

$\geq 2g-2 > g-1$

so by Riemann-Roch $\mathrm{Hom}(E/F, FK) \neq 0$.

$\square$

(2) very stable $\Longrightarrow$ regular for maximal subbundles

Consider subsheaves F of E

$$\begin{array}{ccc} F & \hookrightarrow & E \\ rk = n' & & rk = n \\ \deg = d' & & \deg = d. \end{array}$$

$$\mathrm{Quot}^{n',d'}(E) = \{ \text{subsheaves } F \hookrightarrow E,\ rk(F) = n',\ \deg(F) = d' \}$$

Assume that $\exp\dim \mathrm{Quot}^{n',d'}(E) = \chi(\mathrm{Hom}(F, E/F)) = 0$

$$\Longleftrightarrow \mu(\mathrm{Hom}(F, E/F)) = \mu(E/F) - \mu(F) = g-1.$$

In that case a general bundle E has a finite number of subbundles $F \hookrightarrow E$ of rank n' and degree d'.

Proposition: If E is very stable, then $\mathrm{Quot}^{n',d'}(E)$ is a smooth (= étale) 0-dimensional scheme.

Proof: It is enough to show that the Zariski-tangent space at $[F] \in \mathrm{Quot}^{n',d'}(E)$ is ZERO

$$T_{[F]}\mathrm{Quot}^{n',d'}(E) = \mathrm{Hom}(F, E/F) = 0$$

But Serre duality + Riemann Roch $\Rightarrow$

$$\mathrm{Hom}(E/F, F \otimes K) = 0 \Rightarrow \mathrm{Hom}(F, E/F) = 0.$$

(seen before!)

(3) Drinfeld - Laumon '88.

There exists a non-empty open subset of very stable bundles

$$\Omega \subset N_X(n,d).$$

(a general vector bundle is very stable).

Sketch of proof: look at the Hitchin system

- $Higgs_X(n,d)$ = coarse moduli space of semi-stable Higgs bundles (E,φ) with $\varphi \in H^0(X, End(E)\otimes K)$ over X of rank n and degree d

  $\longrightarrow$ quasi-projective variety.

  $\longrightarrow$ $\dim Higgs_X(n,d) = 2 \cdot \dim N_X(n,d)$.

- $$h : Higgs_X(n,d) \longrightarrow \bigoplus_{i=1}^{n} H^0(X, K_X^i)$$
  $$(E,\varphi) \longmapsto (h(E,\varphi))_{i=1..n}$$
  $$h(E,\varphi)_i = tr(\varphi^i) \text{ or } tr(\Lambda^i \varphi)$$

  Hitchin map. (proper, flat)

Facts: φ nilpotent $\iff$ $(E,\varphi) \in h^{-1}(0)$

$h^{-1}(0)$ = nilpotent cone $\subset Higgs_X(n,d)$

- There is a natural symplectic form on the smooth locus of $Higgs_X(n,d)$ extending the standard symplectic form on $T^*N_X(n,d) \subset Higgs_X(n,d)$.

Laumon '88 the nilpotent cone $h^{-1}(0)$ is a Lagrangian subvariety of $\mathrm{Higgs}_X(m,d)$

thus: $h^{-1}(0)$ is equidimensional, $\dim h^{-1}(0) = \dim N_X(m,d)$

Choose an irreducible component $\mathcal{C} \subset h^{-1}(0)$ satisfying.

(*) A general Higgs bundle (E,φ) in $\mathcal{C}$ has underlying semi-stable vector bundle E.

Remark: $m=2,3$: Any $\mathcal{C}$ satisfies (*)
$m>3$: not known whether any $\mathcal{C}$ satisfies (*)

rational forgetful map:

$$\begin{array}{ccc} \mathrm{Higgs}_X(m,d) & \overset{\pi}{\dashrightarrow} & N_X(n,d) \\ \cup & & \cup \\ \mathcal{C} & \dashrightarrow & W = \text{non very stable locus} \\ & & \overset{\shortparallel}{\pi(\mathcal{C})} \end{array}$$

$\mathbb{C}^*$-action on $\mathcal{C}$: fibers on $\pi_{|\mathcal{C}}$ have $\dim \geq 1$.
so $\dim W < \dim \mathcal{C}$
thus $\Omega = N_X(n,d) \setminus W$ is non-empty and open

Donagi-Pantev: non very stable = wobbly

Drinfeld's conjecture (letter to P. Deligne '81).
the wobbly locus W in $N_X(n,d)$ is pure of codimension 1.

Rem: $\{\text{strictly semi-stable}\} \subset W$

Some classical geometry

Description of Wobbly divisors

Examples: $g=2$, $n=2$, take fixed determinant moduli-spaces $SN_X(2,d)$

- $d=0$ $\quad SN_X(2,0) = \mathbb{P}^3$

$$W = \underset{\{\text{str. semi-stable bundles}\}}{\overset{}{\text{Kummer surface}}} \cup \underset{\{\text{tropes}\}}{16 \text{ hyperplanes}}$$

Kummer surface = {str. semi-stable bundles}; 16 hyperplanes = {tropes}.

- $d=1$ (S. Pal) $\quad SN_X(2,1) = Q_1 \cap Q_2 \subset \mathbb{P}^5$

intersection of 2 quadrics

W = irreducible divisor cut out by hypersurface of degree 8 in $\mathbb{P}^5$

$$= \{ x \in Q_1 \cap Q_2 \ : \ \mathbb{T}_x(Q_1 \cap Q_2) \cap (Q_1 \cap Q_2) \text{ degenerate} \}$$

embedded tangent space

Characterization of wobbly bundles

E stable $\Rightarrow [E] \in \mathcal{N}_X(n,d)$ smooth point

$$V_E = T^*_{[E]}\mathcal{N}_X(n,d) = H^0(X, End(E)\otimes K) \subset Higgs_X(n,d)$$

Proposition (jt with A. Peón) $\quad E$ very stable $\iff V_E$ closed in $Higgs_X(n,d)$

$\iff h_{|V_E}$ proper

$$E \text{ wobbly} \iff \overline{V}_E \setminus V_E \neq \emptyset.$$

Sketch of proof for $n=2$ $\quad (E \text{ wobbly} \Rightarrow \overline{V}_E \setminus V_E \neq \emptyset)$

E wobbly $\Rightarrow \exists \varphi \in H^0(X, End(E)\otimes K)$ nilpotent of index 2

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] & E_\varepsilon \arrow[r, "\pi"] & M \arrow[r] \arrow[dll, "u"'] & 0 \\
0 \arrow[r] & LK \arrow[r, "i"'] & E_\varepsilon K \arrow[r] & MK \arrow[r] & 0
\end{tikzcd}$$

$\varepsilon \in Ext^1(M,L)$

$u \in Hom(M, LK)$

so we obtain a morphism

$$\Phi : \mathbb{P}(Ext^1(M,L)\oplus Hom(M,LK)) \longrightarrow Higgs_X(2,d)$$

$$(\varepsilon, u) \longmapsto \Phi(\varepsilon,u) = (E_\varepsilon, i\circ u\circ \pi) = (E_\varepsilon, \varphi)$$

Observation: $\Phi(t\varepsilon, tu) = \Phi(\varepsilon, u) \quad \forall t \in \mathbb{C}^*$

$$\Rightarrow \Phi(\varepsilon, tu) = \Phi(t^{-1}\varepsilon, u)$$

$$\Rightarrow \lim_{t\to+\infty}(E, t\varphi) = \left(L\oplus M, \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}\right) \notin V_E$$

__Aim:__ • classify all irreducible components of wobbly locus $W \subset \mathcal{N}_X(n,d)$
• check Drinfeld's conjecture

Review of results for $n=2$

(due to Drinfeld, Hitchin, Thaddeus, Pal)
with previous notation introduce the invariant

$$\delta = \deg(\mathrm{Hom}(M, LK))$$

then δ satisfies the 2 conditions

$$\begin{cases} \bullet\ 0 \leqslant \delta \leqslant 2g-2 \\ \bullet\ \delta \equiv d\ [2] \end{cases}$$

__theorem__ • For any δ satisfying these conditions, there exists an irreducible component of $h^{-1}(0)$ denoted $\mathcal{C}_\delta$, so.

$$h^{-1}(0) = \coprod_{\substack{0 \leqslant \delta \leqslant 2g-2 \\ \delta \equiv d\ [2]}} \mathcal{C}_\delta$$

• Any $\mathcal{C}_\delta$ satisfies $(*)$

• If $\pi : \mathrm{Higgs}_X(2,d) \dashrightarrow \mathcal{N}_X(2,d)$ rational forgetful map
the wobbly locus equals

$$W = \coprod_{\substack{0 < \delta \leqslant g \\ \delta \equiv d\ [2]}} \overline{\pi(\mathcal{C}_\delta)}$$

Remarks: • If $\delta > g$, then by Riemann-Roch
$$\dim \operatorname{Hom}(M, LK) > 1$$
so $\operatorname{codim} \pi(\mathcal{C}_\delta) > 1$.

• But $\pi(\mathcal{C}_\delta) \subset \overline{\pi(\mathcal{C}_g)}$ if $\delta > g$.

• Drinfeld is OK: W is of pure codimension 1.

Review of results for $n=3$

P. Gothen, (jt with A. Peon)

the Bialynicki-Birula decomposition of $h^{-1}(0)$

• $\mathbb{C}^*$-action on $h^{-1}(0)$ $\quad (E,\varphi) \mapsto (E, t\varphi)$ for $t \in \mathbb{C}^*$

• $h^{-1}(0)^{\mathbb{C}^*}$ = fixed point set of $\mathbb{C}^*$-action
$$= \left\{ (E_0 \oplus E_1 \oplus \cdots \oplus E_k,\ \varphi = \begin{pmatrix} 0 & u_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & u_k \\ 0 & & & 0 \end{pmatrix}) ;\ u_i : E_i \to E_{i-1}K \right\}$$

irreducible components are classified by type $(r_0, r_1, \dots, r_k)$ and degree $(d_0, \dots, d_k)$
$r_i = rk(E_i)$ $\quad d_i = \deg(E_i)$

<u>$n=3$</u> • $h^{-1}(0)^{\mathbb{C}^*} = F_{(3)} \cup F_{(2,1)} \cup F_{(1,2)} \cup F_{(1,1,1)}$

$F_{(3)} = N_X(3,d)$ $\quad$ note that $F_{(r_0,\dots,r_k)}$ not irr.

• $h^{-1}(0) = F^-_{(3)} \cup F^-_{(2,1)} \cup F^-_{(1,2)} \cup F^-_{(1,1,1)}$

where $F^- = \{ (E,\varphi) \in h^{-1}(0) : \lim_{t\to\infty} (E, t\varphi) \in F \}$

downward flow of F

<u>Remark</u>: If $d=0$ $\quad F^-_{(2,1)}$ and $F^-_{(1,2)}$ are interchanged by duality map $(E,\varphi)\mapsto(E^*,{}^t\varphi)$

<u>Results</u> (jt work with A. Peon)

$$F^-_{(2,1)} = \left\{ (\mathcal{E},\varphi) : 0 \to E_0 \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} F_1 \to 0. \right\}$$

$\operatorname{rk} E_0 = 2$, $\deg E_0 = d_0$ $\qquad$ $\operatorname{rk} F_1 = 1$, $\deg F_1 = d_1$

$$\varphi = i\circ u\circ\pi \qquad u : F_1 \to E_0 K$$

Introduce the invariant $\delta = \deg(\mathrm{Hom}(F_1, E_0K))$ then δ satisfies the two conditions.

$$\begin{cases} \cdot\ g-1 \leq \delta \leq 4g-4 \\ \cdot\ \delta \equiv -d+(g-1)\ [3] \end{cases}$$

<u>theorem</u>: • For any δ satisfying these conditions, there exists an irreducible component of $F^-_{(2,1)} \subset h^{-1}(0)$, so.

$$F^-_{(2,1)} = \coprod_{g-1\leq\delta\leq 4g-4} \mathcal{C}_\delta$$

• Any $\mathcal{C}_\delta$ satisfies (*)

• the wobbly locus of type $(2,1)$ equals

$$W_{(2,1)} = \coprod_{g-1\leq\delta\leq 2g-2} \overline{\pi(\mathcal{C}_\delta)}$$

Remark: . If $\delta > 2g-2$, then by Riemann-Roch.
$\operatorname{codim} \pi(\mathcal{C}_\delta) > 1$.

. But $\pi(\mathcal{C}_\delta) \subset \overline{\pi(\mathcal{C}_{2g-2})}$ if $\delta > 2g-2$

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similar results for $F^{-}_{(1,1,1)}$.
