Symplectomorphism groups of irrational ruled 4-manifolds

(Based on joint work with Olguta Buse)

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  Biran-Giroux: fibered Dehn twists,
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Ruled surfaces. Diffeomorphic to one of the following:

$$\Sigma_g \times S^2,\ \Sigma_g \tilde{\times} S^2, (\Sigma_g \times S^2)\# n\overline{\mathbb{C}P^2}.$$

- McDuff 2000: For some symplectic forms on $\Sigma_g \times S^2$, a symplectomorphism is smoothly isotopic to identity iff it is symplectically isotopic to identity.

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Notation. $M_g^n := \Sigma_g \times S^2 \# n\overline{\mathbb{C}P^2}$.
Choose Base, Fiber, and Exceptional classes
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Partition of symplectic cone for ruled surfaces

Interior walls. For each $A \in H_2(M_g^n, \mathbb{Z})$ take $\text{cod}_A = 2(-A \cdot A - 1 + g)$.
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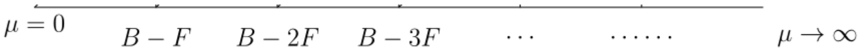

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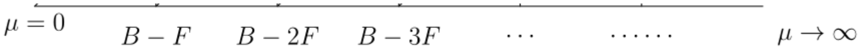

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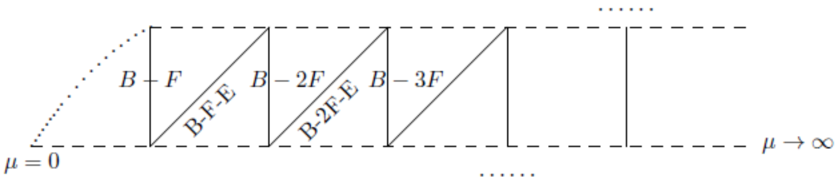

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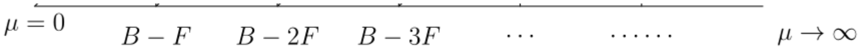

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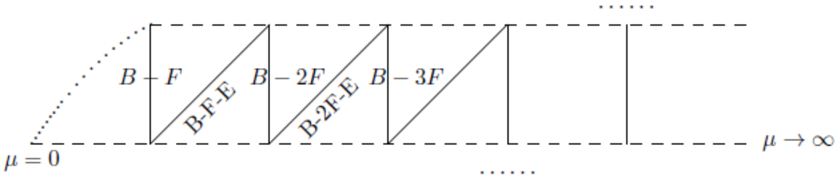

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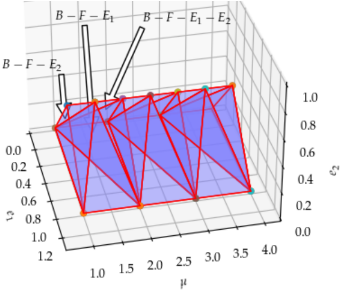

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Homotopic stability conjecture and stability results

Let $G^g_{u,n} = Symp(M^n_g, \omega) \cap \text{Diff}_0 M^n_g$, with $[\omega] = u$.

Conjecture (Homotopic stability in Chambers)

If the norm of u is sufficiently large, then $G^g_{u,n}$ has constant homotopy type throughout the chambers of the symplectic cone.

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Kronheimer's fibration Let $\mathcal{S}_{[\omega]}$ be the space of symplectic forms in the class $[\omega]$ and isotopic to a given form:
$Symp(M,\omega) \cap \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(M) \to \mathcal{S}_{[\omega]}$.

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$\mathcal{A}_u$ is stratified via nonregular J-holomorphic curves

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Many points blow-up irrational: $\mathcal{A}_u$ is stratified but more sectional classes curves

In this case we cannot control very well the degenerations of the exceptional curves.
To accommodate that, we take an ad-hoc splitting of $\mathcal{A}_u = \mathcal{A}_u^{top} \cup \mathcal{A}_{B_i}^2 \cup \mathcal{A}_{u,broken}^2 \cup \mathcal{A}^{high}$.

section class	embedded exceptional	mild broken deg	deg
$B + kF - \sum E_i, \text{cod} < 2$	$\mathcal{A}_u^{top}$	$\mathcal{A}_{u,broken}^2$	$\mathcal{A}_u^{high}$
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Tools: The inflation package, Anjos-Li-L-Pinsonnault

Inflation package, using McDuff, Buse, Li-Zhang

On M^4 with $b^+ = 1$, given a compatible pair (ω, J) and an embedded J-holomorphic curve Z, then exists an ω' in class
$[\omega'] = [\omega] + tP.D.[Z], t \in [0, \lambda)$ with
$\lambda = \infty$ if $[Z] \cdot [Z] \geq 0$ (McDuff 2000) and
$\lambda = \frac{\omega(Z)}{[Z]\cdot[Z]}$, if $[Z] \cdot [Z] < 0$, (Buse, 2011)
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Key input: J-holomorphic curve results on irrational ruled surfaces

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Let any $M^g_{u,n}$, $g \geq 1$. Then for any tamed J,

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Inflation process (McDuff): the minimal case

Inclusions of strata minimal case

Want:	$\mathcal{A}_{\mu,C} \subset \mathcal{A}_{\mu+l,C} \forall l$ $\mathcal{A}_{\mu}^{top} \subset \mathcal{A}_{\mu+l}^{top}$,	$\mathcal{A}_{\mu,C} \supset \mathcal{A}_{\mu+l,C}$, when possible	$\mathcal{A}_{\mu}^{top} \supset \mathcal{A}_{\mu+l}^{top}$.
Inflate along	**foliation** with J-holomorphic F	embedded negative curve $C = B - kF$	embedded $B + qF, q \leq g$

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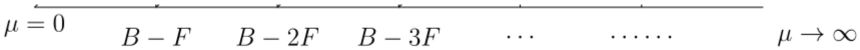

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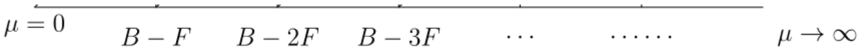

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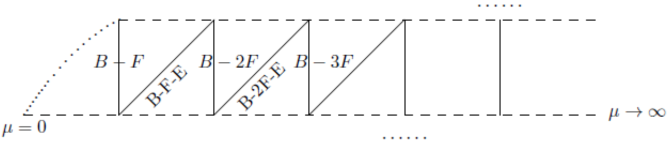

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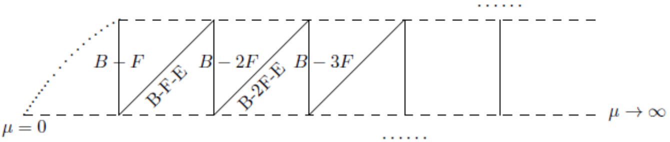

Inflation process, positively span of Zhang's curves

Direction	Strata	Class to inflate	Notes
In the same chamber	$\mathcal{A}^2_{u,\mathcal{C}}$	$B + xF - \sum E_i$, E_i, $F - E_i$	codim 2, emb. exp.
Within a chamber	$\mathcal{A}^2_{u,C,D}$	$B + xF - \sum E_i$, C, D, E_i, $F - E_i$	codim 2, broken exc.
Within a chamber	$\mathcal{A}_{u,open}$	$B + xF - \sum E_i$, $E_i, F - E_i$	top stratum, emb. exc.
Across to chambers with large μ	Any strata	F	foliation by F
Across to chambers with small μ	$\mathcal{A}_{u,\mathcal{C}}$ and $\mathcal{A}_{u,open}$	$B + xF - \sum E_i$	neg sect. class type inflation

Table: Inflation process for multiple-point blowup

Special case: many point blow-up with $\frac{1}{2}$ equal sizes

$\mathcal{A}_u^{high}$ is well understood in the $\left[\mu, 1, \frac{1}{2}, \cdots \frac{1}{2}\right]$ case since the exceptional curves cannot degenerate for homological reasons.

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Theorem (Recall from Main Theorem 1)

For $u = (\mu, 1, \frac{1}{2}, \ldots, \frac{1}{2})$, the homotopy type of $G^g_{\mu,n}$ is constant for $\frac{k}{2} < \mu \leq \frac{k+1}{2}$, for any integer $k \geq 2g$. Moreover as μ passes the half integer $\frac{k+1}{2}$, all the groups $\pi_i, i = 0, \cdots, 2k+2g-1$ do not change.

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1)**In all cases**: Foliations by F yield upwards inclusion $\mathcal{A}_u \subset \mathcal{A}_{u'}$ on any horizontal line; in turn these yield a topological limit when $\mu \to \infty$. This limit groups remember π_0 for any $G^g_{\mu,n}$ with $\mu > n, g$.

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Q1: What is the geometric description of the Elliptic diffeomorphism?

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Q1: What is the geometric description of the Elliptic diffeomorphism?

Q2: Is there any relation to the fibered Dehn twists?

The isotopy problem: seeking exotic symplectomorphisms

Theorem (Main Theorem 2)

For any non-minimal ruled surface, there is a symplectic form such that there are exotic symplectomorphisms. That is, $\exists \phi \in Symp$ smoothly but not symplectically isotopic to the identity.

Genus=1 case (Shevchishin-Smirnov, 2017): For some $(\mathbb{T}^2 \times S^2 \# \overline{\mathbb{C}P^2}, \omega)$, there exists an elliptic diffeomorphism, given by a loop in $\mathcal{A}_\omega$ generated by (-1) torus in $B - E$, that's not killed by $\pi_1 \mathrm{Diff}_0$.

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Thank You!