Manifold diagrams

Outline

geometric

(1) Definition

(2) Duality

(3) Singularities

(4) Smoothness

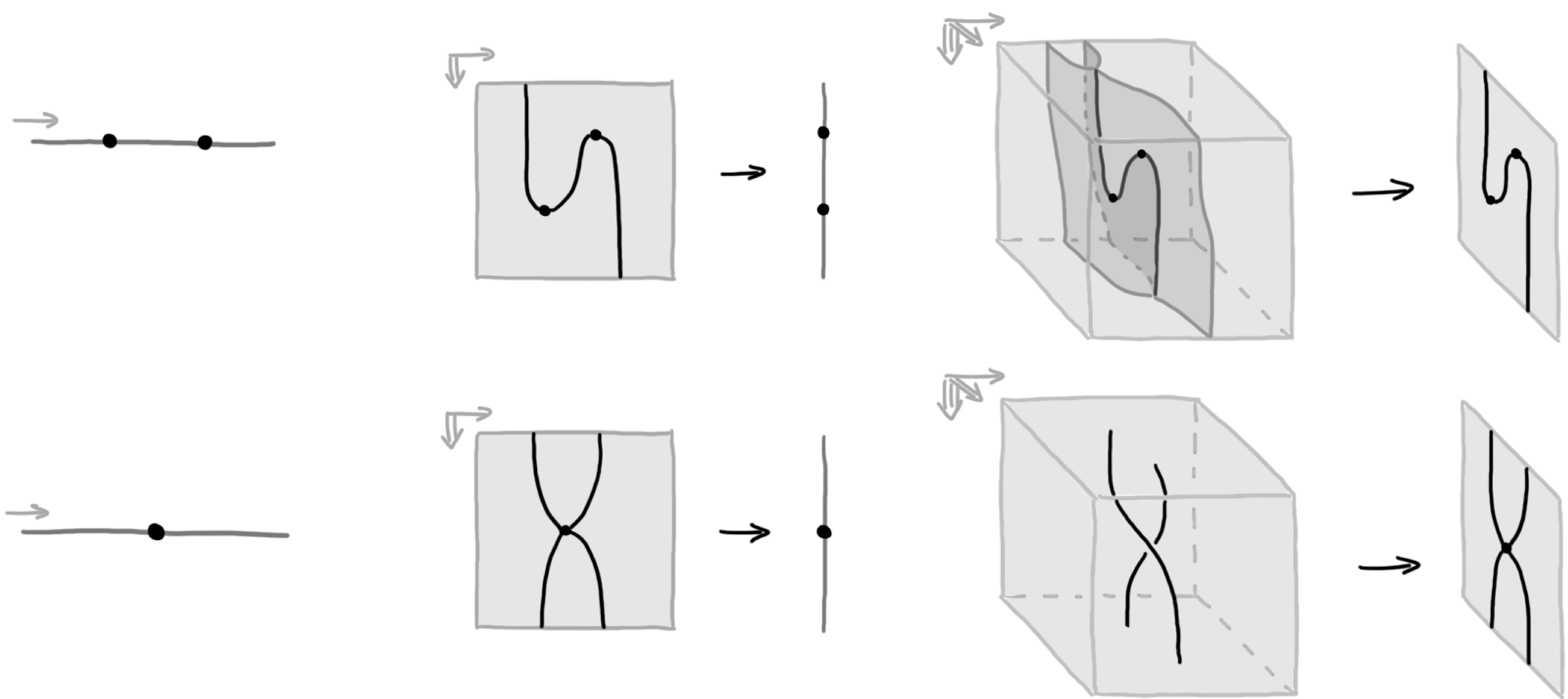
References

- much of the technology from "Framed combinatorial topology" arXiv 2112:14700 (today: focus on narrative!)
- introductory notes on the [web] = cxdorn.github.io

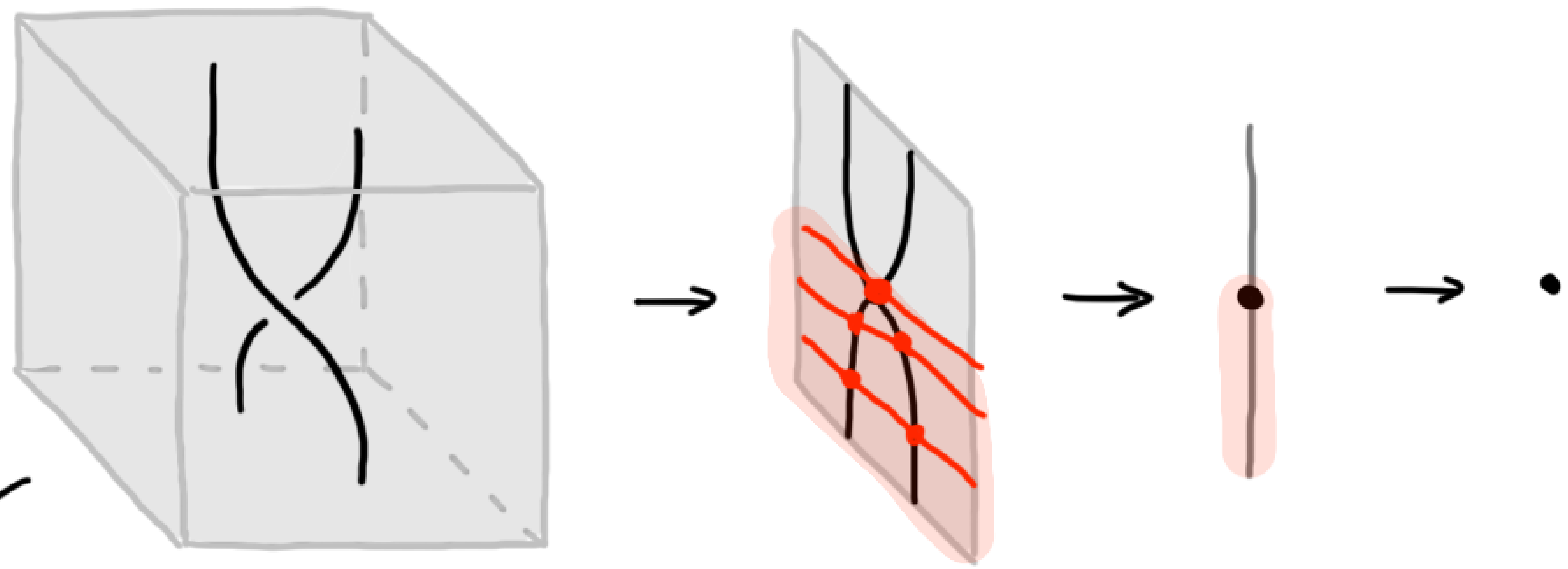
⇝ papers on (1), (2), (3) in preparation

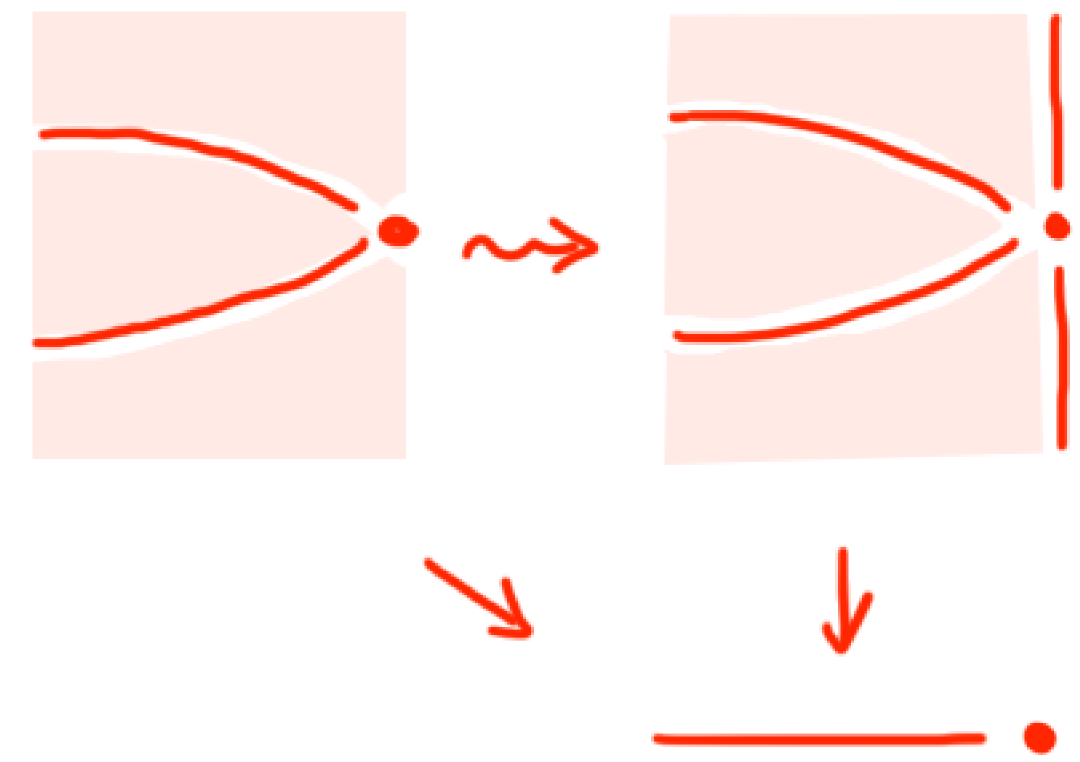
(1.1) Definition: Seeing the induction

dim 1 | dim 2 | dim 3



(1.2) Definition: understanding the induction



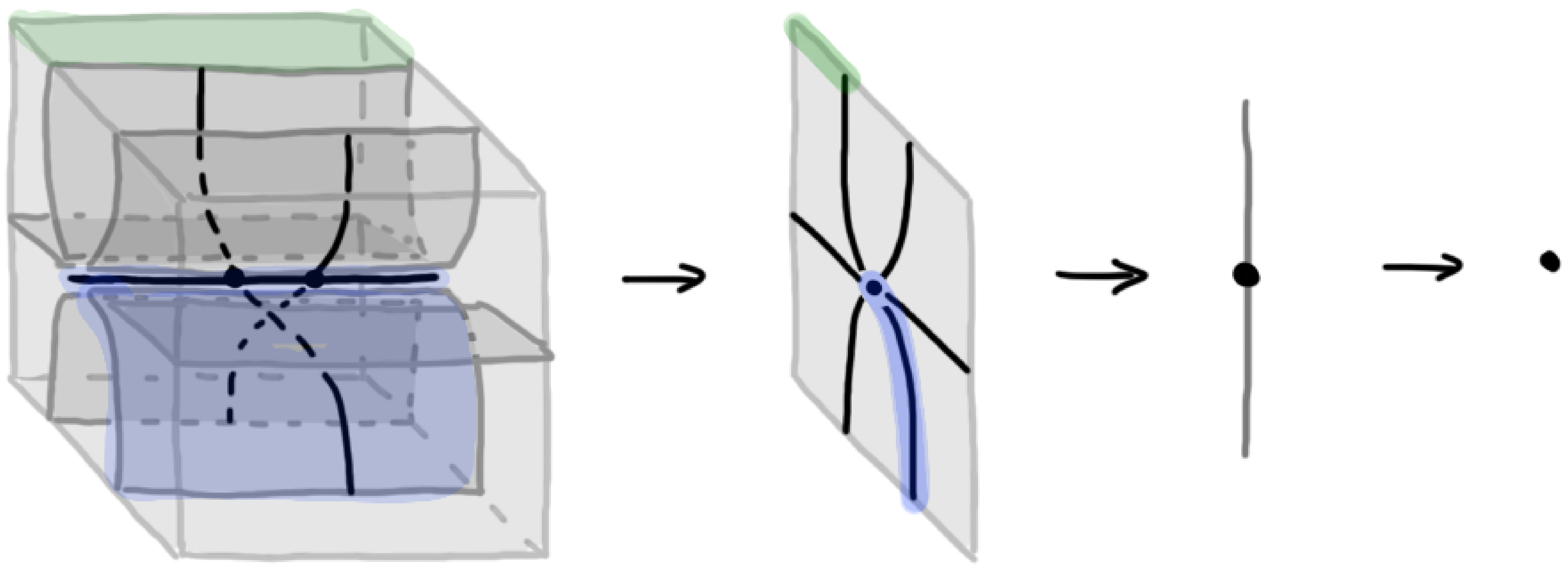


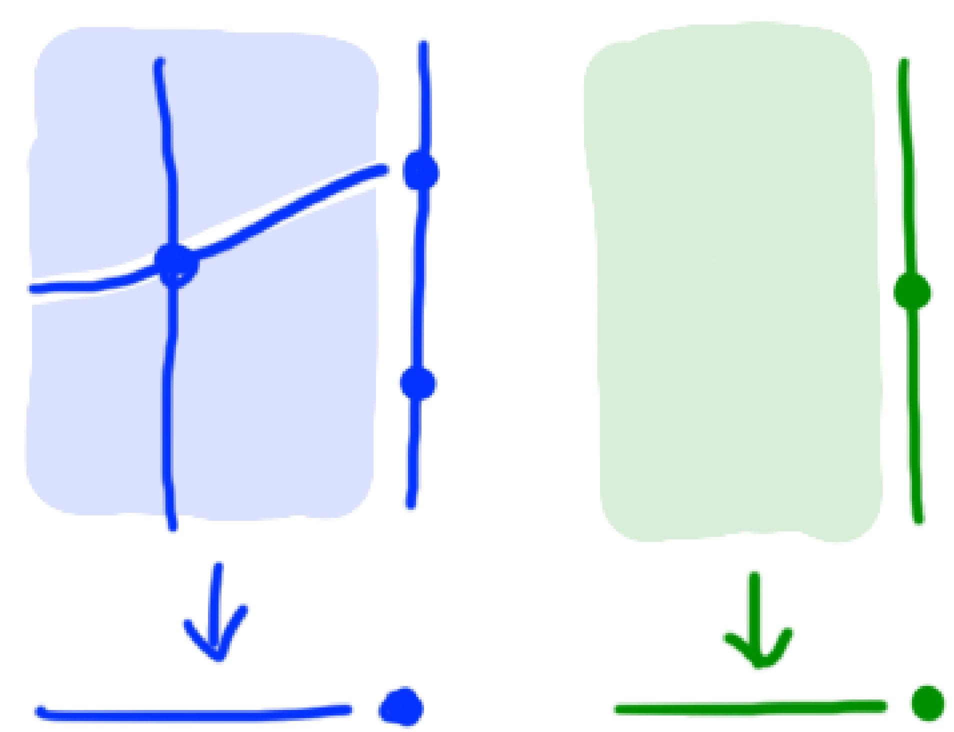
"stratified fiber bundle"

⇝ constructible bundle

- but what is a "projection" formally? ⇝ inspect fibers

⇝ (canonically) refine to constructible bundles





⇝ think of this refinement as a "triangulation" of the manifold diagram

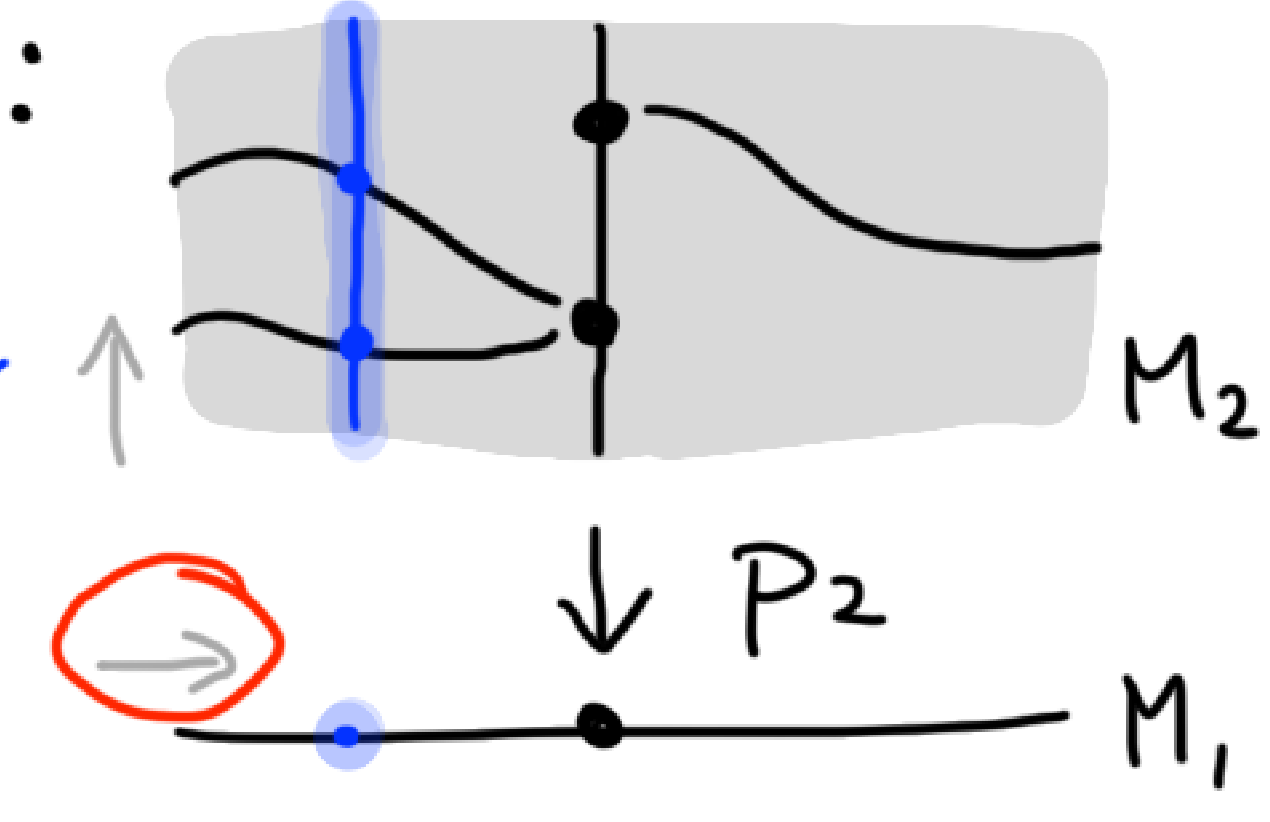
(yet better: a "cellulation")

(1.3) Definition: more on iterated constructible bundles

$\equiv$ 1-mesh bundle:

$$\underbrace{M_n \xrightarrow{p_n} M_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_3} \overbrace{M_2 \xrightarrow{p_2} M_1}^{\equiv\ 1\text{-mesh bundle}} \xrightarrow{p_1} *}_{\equiv\ n\text{-mesh}\ (\text{bundle} \leftarrow - - B)}$$

fibers



(open) 1-meshes vs. (closed) — " —

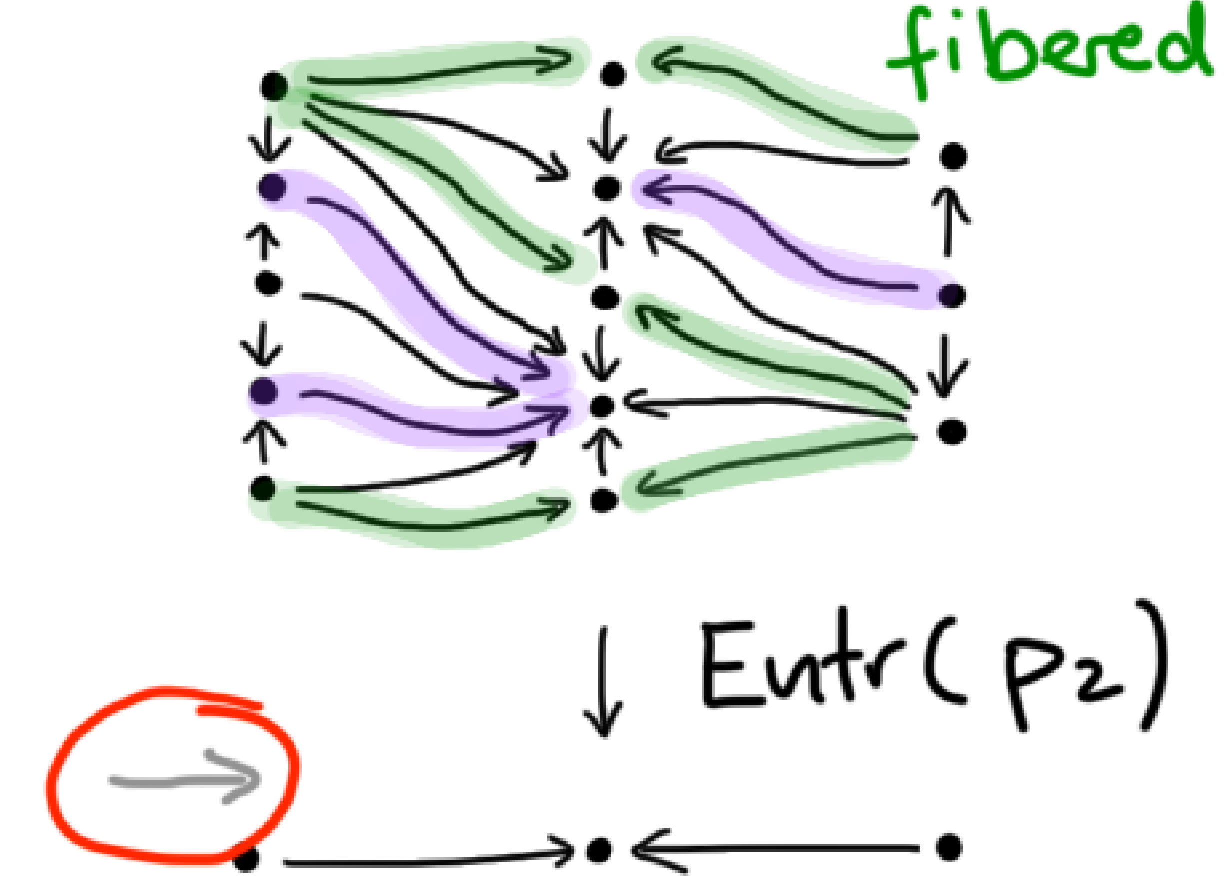
pass to entrance path ~~∞~~-categories
0-
$\equiv$ entrance path posets

$\Big\downarrow$ fund. $\sim$ "∞-gpd" for strat. spaces

$\equiv$ 1-truss bundle

$\rightsquigarrow$ n-truss

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow \overbrace{T_2 \longrightarrow T_1}^{\equiv\ 1\text{-truss bundle}} \longrightarrow *$$



∞-cat! 1-cat!

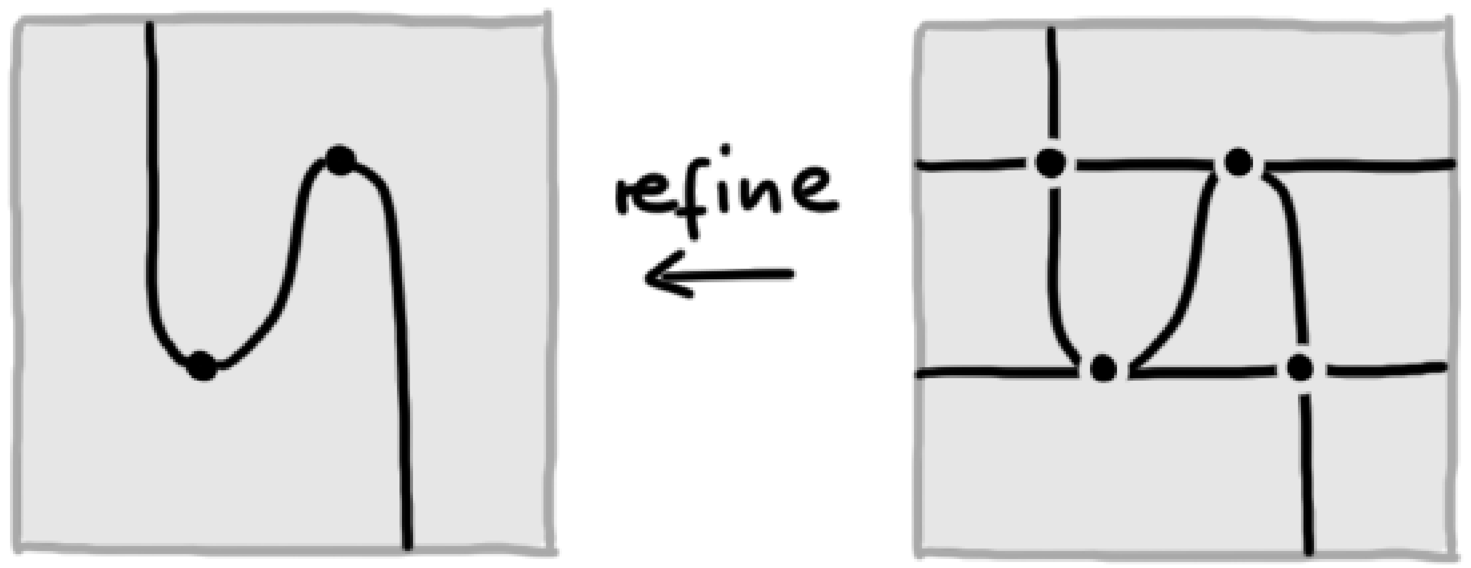
Thm (FCT): $\mathrm{Mesh}_n \underset{\mathrm{Entr}}{\simeq} \mathrm{Trs}_n$

(1.4) Definition: Putting things together

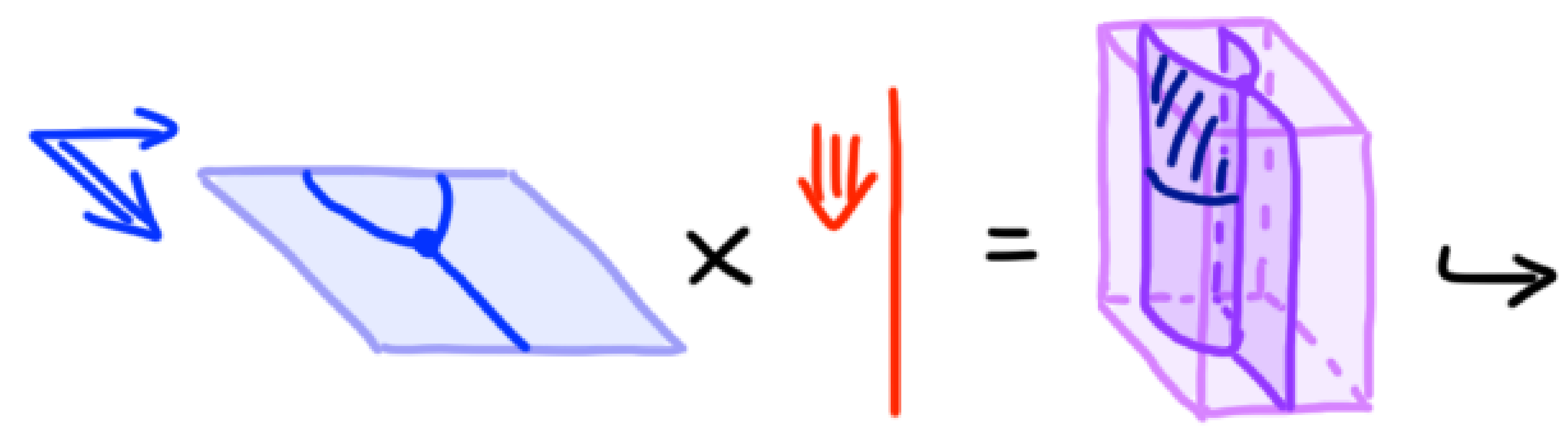
Def: A manifold n-diagram is a stratification of I^n which is meshable and framed conical.

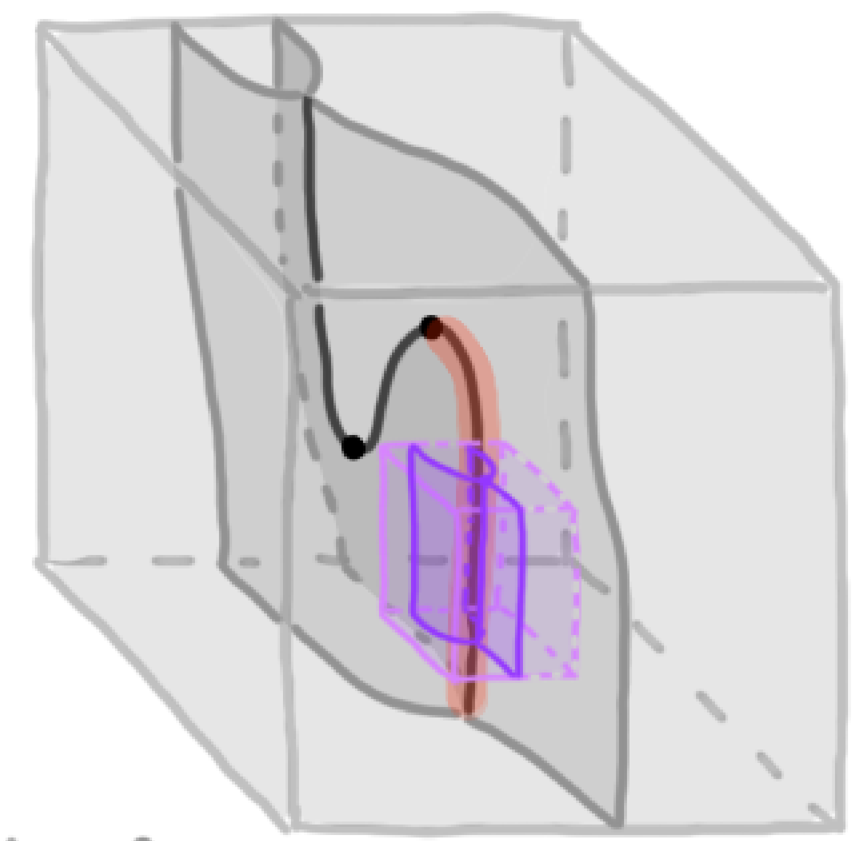
see [web]/research/manifold-diagrams

meshable $\equiv$ refinable by a mesh



framed conical $\equiv$ "conical", but compatibly w/ global trivial frame





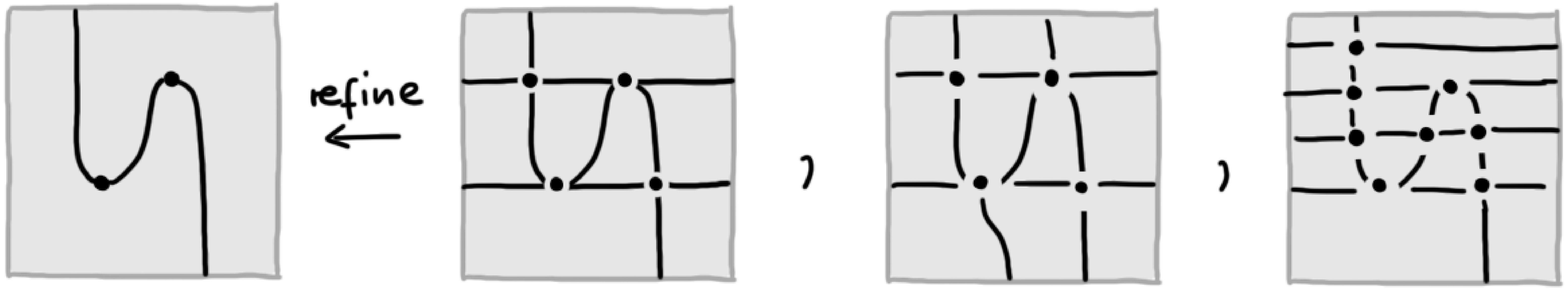
$$\underbrace{C \times U}_{\text{conical nbhd.}}$$

"normal" data (C) — "tangential" data (U)

Rmk (FCT)
- every compact PL stratification is meshable
- every meshable stratification has canonical PL struct. (up to framed stratified homeomorphism, see FCT)

(1.5) Definition: canonical combinatorialization

⇝ a priori, a given manifold diagram has many meshes:



TAKE-AWAY

THE GEOMETRIC THEORY HAS A FULLY COMBINATORIAL COUNTERPART!

Thm (FCT) $\forall$ manifold diagrams $\exists!$ coarsest refining mesh

Rmk this mesh can be constructed purely combinatorially via "normalization" of stratified trusses

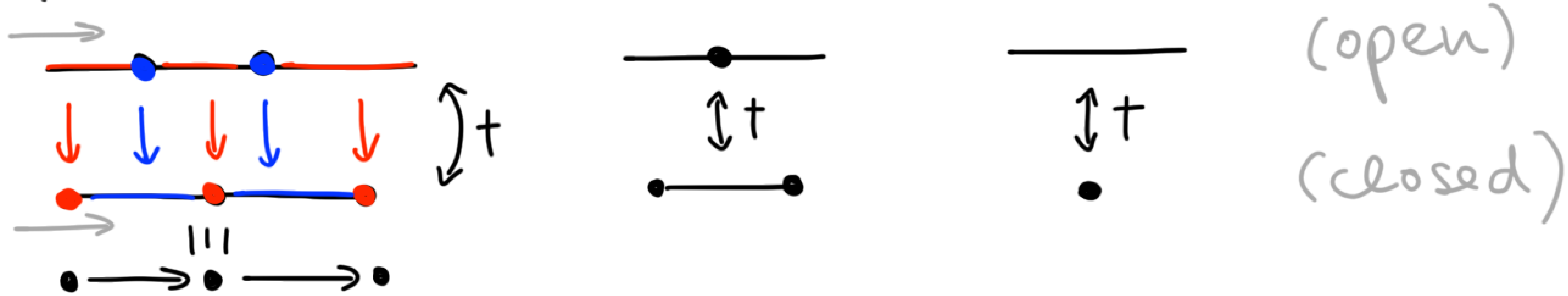
Cor

$$\left\{\begin{array}{l}\text{manifold diag}\\ \text{up to framed}\\ \text{stratified homeo.}\end{array}\right\} \cong \left\{\begin{array}{c}\text{normalized}\\ \text{conically strat.}\\ \text{trusses}\end{array}\right\}$$

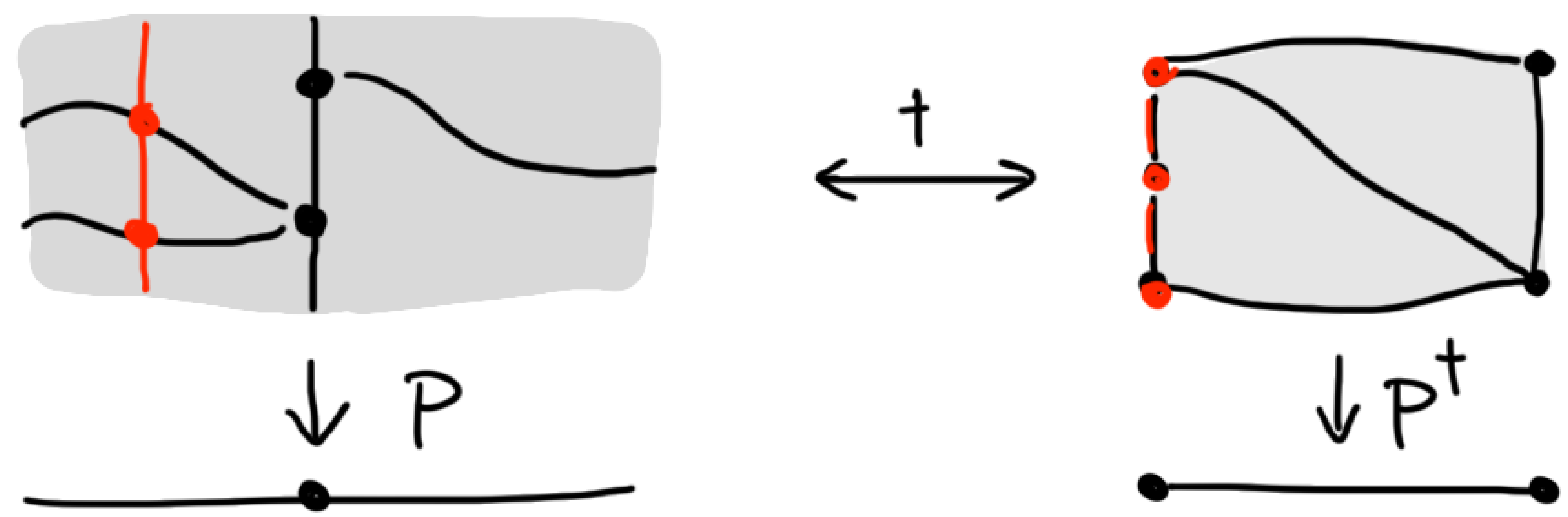
"combinatorial" manifold diagrams

(2.1) Duality: open and closed meshes

• 1-meshes



• 1-mesh bundles



• n-meshes

$$\left(M_n \xrightarrow{p_n} M_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} M_0\right) \overset{\dagger}{\longleftrightarrow} \left(M_n^\dagger \xrightarrow{p_n^\dagger} M_{n-1}^\dagger \xrightarrow{p_{n-1}^\dagger} \cdots \xrightarrow{p_1^\dagger} M_0^\dagger\right)$$

<u>Thm</u> (FCT)

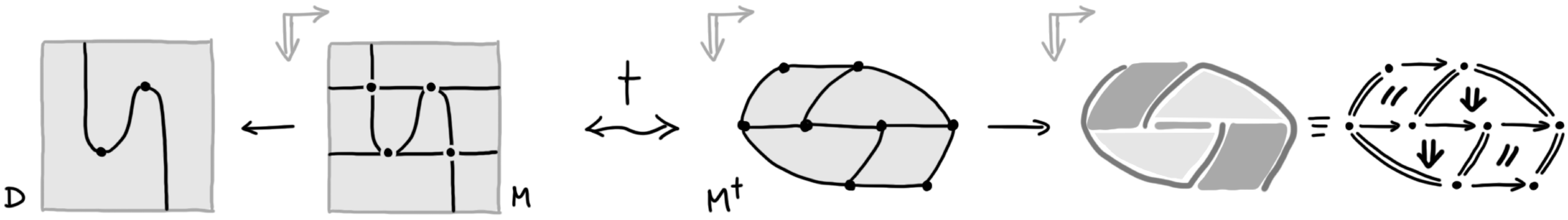
closed n-meshes

open n-meshes

$$\overline{\mathrm{Mesh}}_n \overset{\dagger}{\simeq} \mathring{\mathrm{Mesh}}_n$$

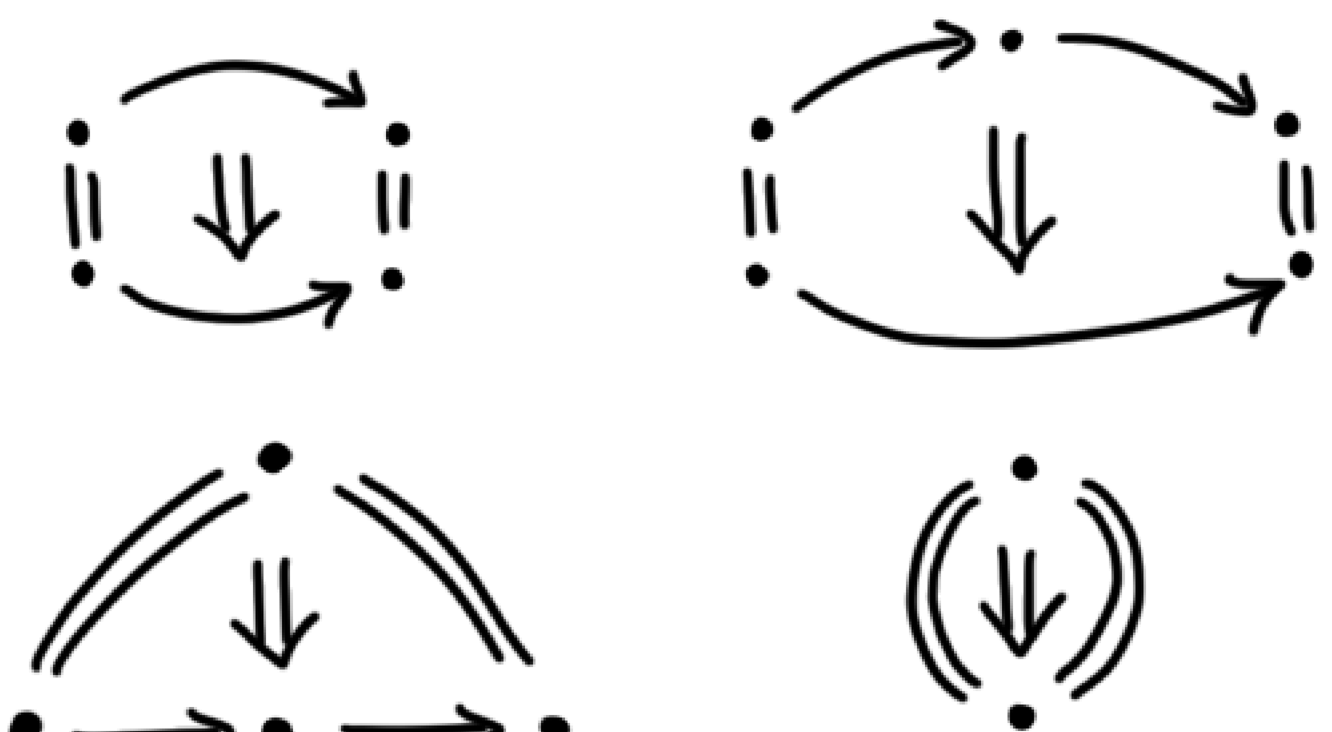
∞-categories

(2.2) Duality: manifold vs pasting diagrams



$$\mathrm{Entr}(D) \longleftarrow \mathrm{Entr}(M) \qquad \mathrm{Entr}(M^\dagger) = \mathrm{Entr}(M)^{op} \longrightarrow \mathrm{Entr}(D)^{op}$$

$\rightsquigarrow$ get computadic cell shapes



[web]/research/computadic-shapes

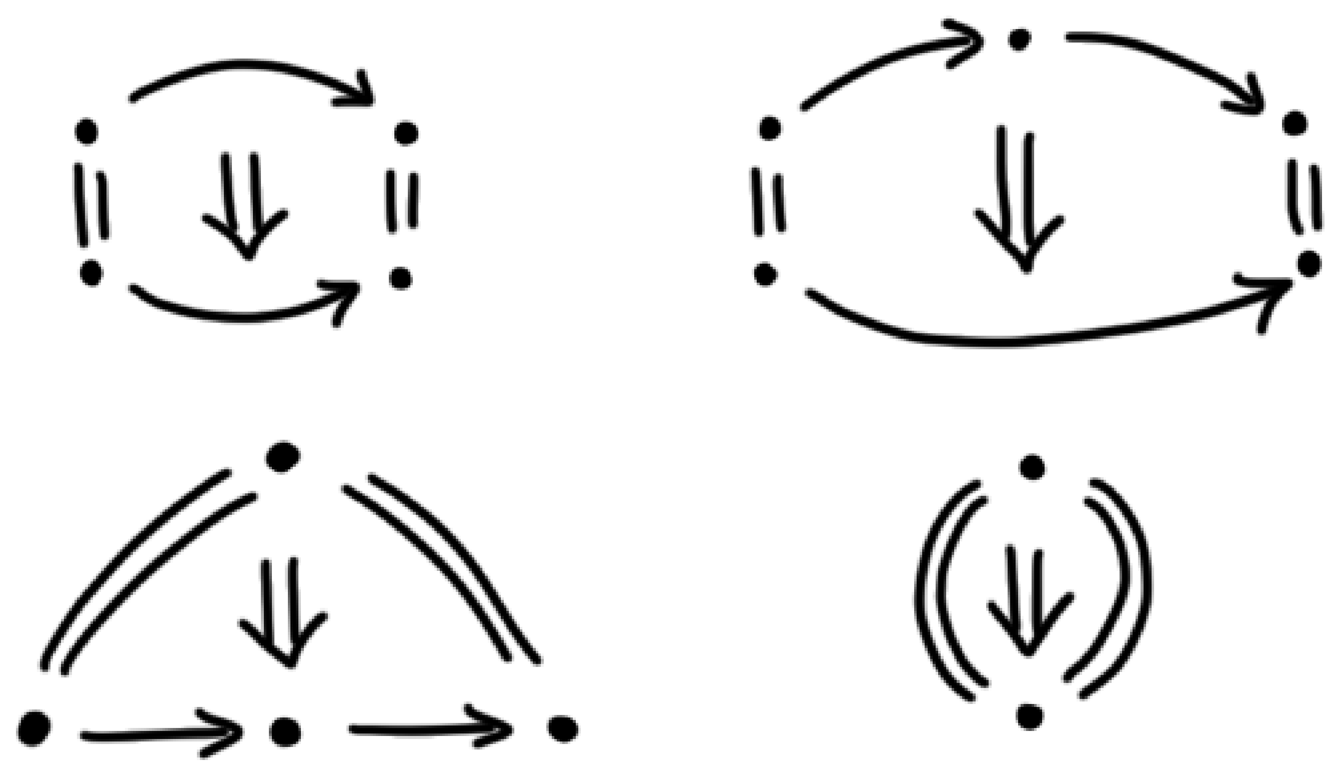
$\rightsquigarrow$ get pasting diagrams describing non-trivial paths in spaces of composites (homotopies)

defn: a manifold diagram without point strata.

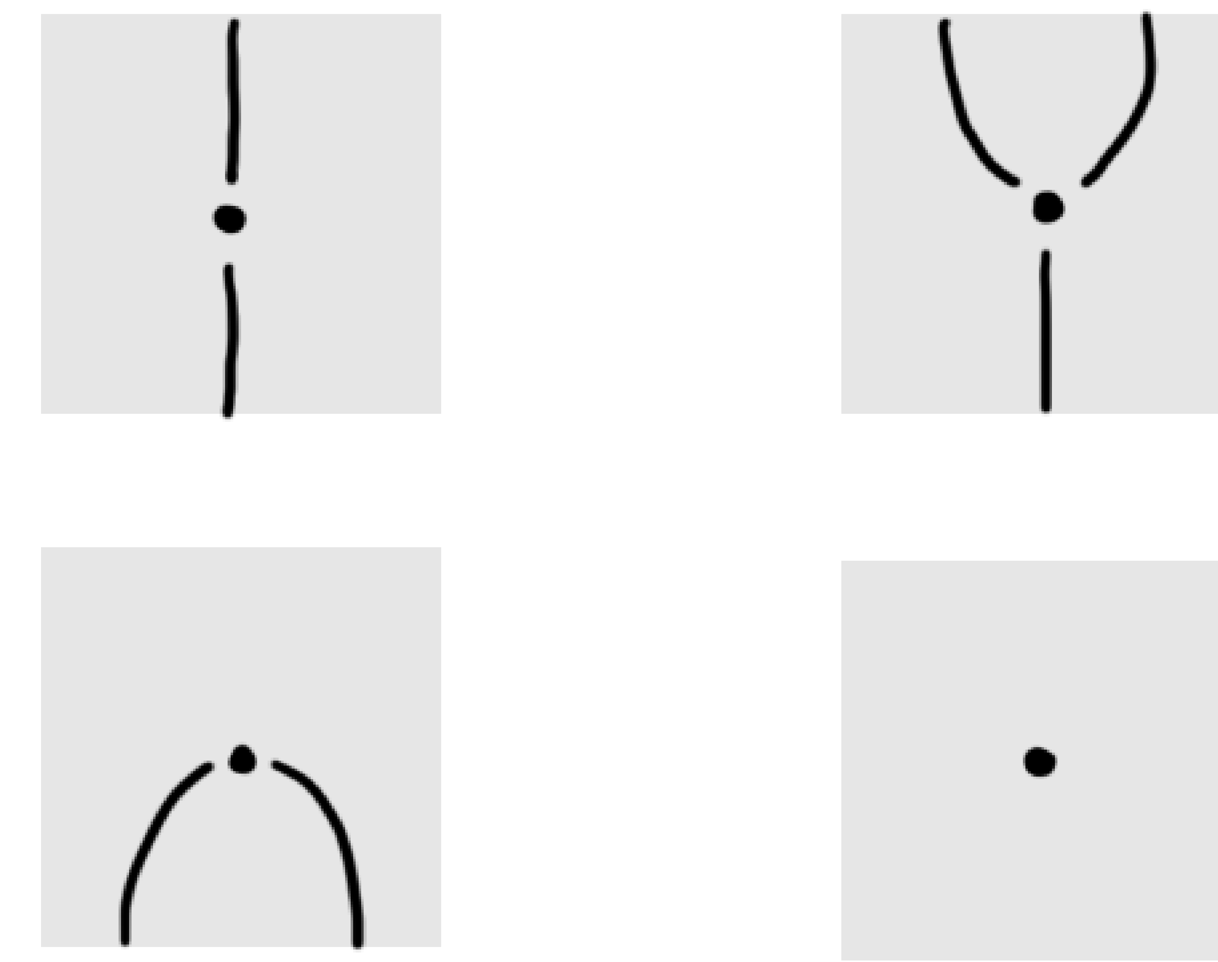
eg:  $\overset{\dagger}{\rightsquigarrow}$ exercise!

[web]/research/paradigms-of-higher-cats

(2.3) Duality: cells vs singularities



$\overset{+}{\leftrightsquigarrow}$



cells

singularities

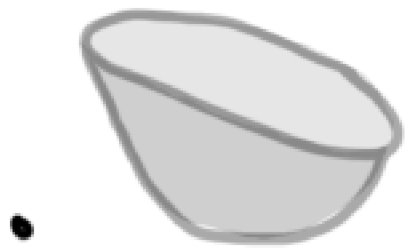
- consider presheafs on cells/singularities
  $\rightsquigarrow$ get a notion of free (semistrict) higher categories
  $\equiv$ computads

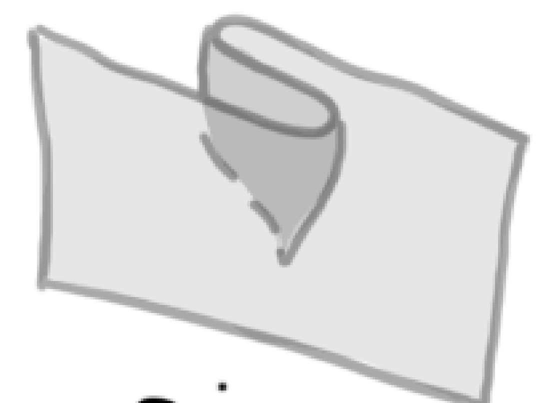
technically, "sheafs"...

[web]/research/trusses

(3.1) Singularities: the classical conundrum

- singularity theory studies the "stability" of (k-parameter families of) functions under perturbation

  ⇝ $k=0$ ("Morse function"): saddle  max.  min. 

  ⇝ $k=1$ ("Morse-Cerf"): cusp  (easy dimension argument)

  ⇝ $k \geq 6$: uncountably many singularities

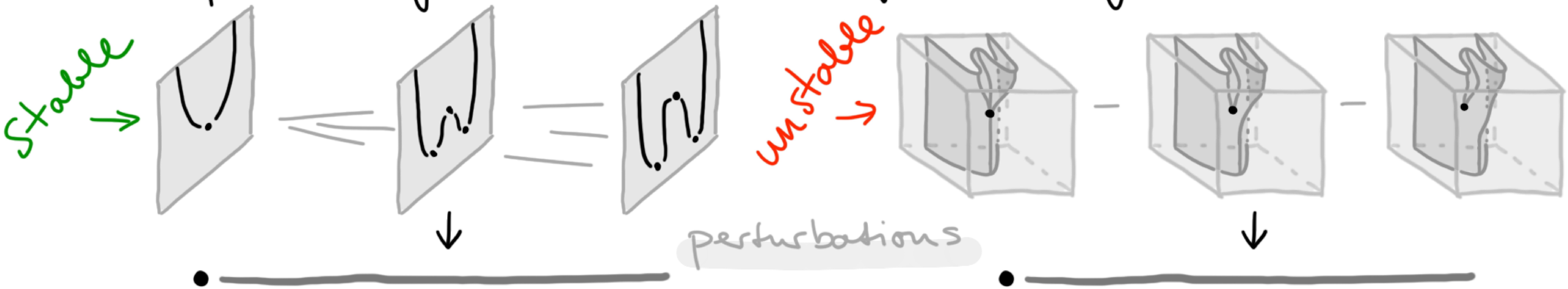
- via tangle hypothesis: sing $\equiv$ dualizability laws ⇝ ??

- manifold diagram "tailored" to resolve this, since can study diagrams in families

  ⇝ one-size-fits-all solution

(3.2) Singularities: tangles in families

- n-meshes $\rightsquigarrow$ n-mesh bundles
- manifold diagrams $\rightsquigarrow$ manifold diagram bundles



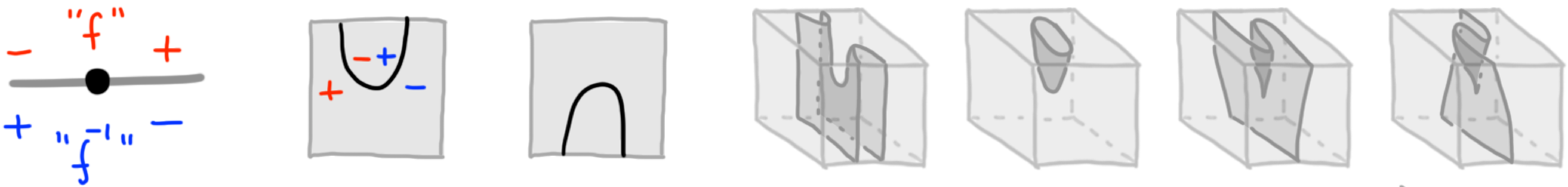
Defn: A tangle (singularity) is a manifold $M \hookrightarrow I^n$ refinable by a manifold diagram (singularity).

eg:

Defn: A tangle singularity is stable if it cannot be perturbed to a tangle with (strictly) simpler singularities.

(3.3) Singularities: examples and conjectures

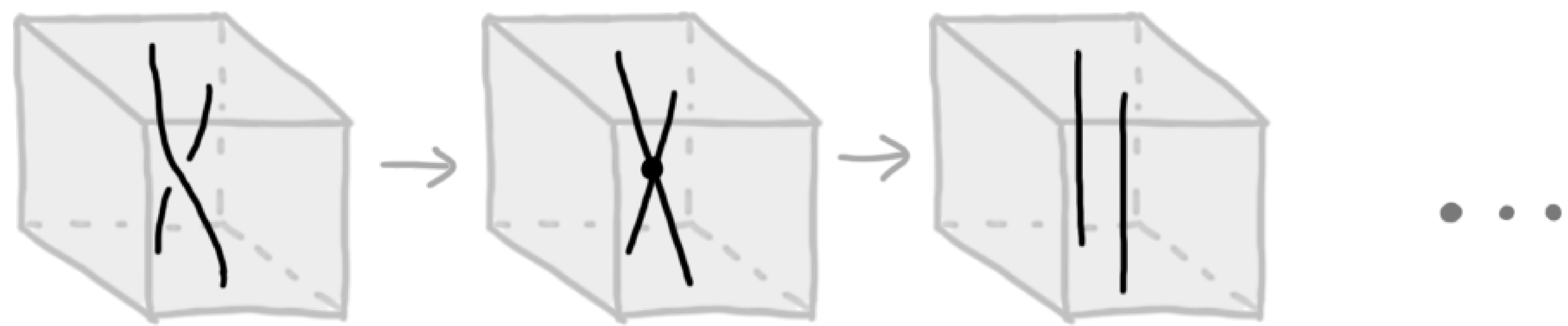
eg (codim 1)



~ theory of dualizable morphism f

(+ vertical reflections)

eg (codim > 1)



conj: • class of stable sing. is finite (countability is trivial)

(FCT) • class has constructive classification

• in codim 1, sings have unique smooth struct ($\leftrightarrow$ SPC4)

therefore speculative

Question: What about smoothness in other codimensions?

(4.1) Smoothness: observations and aims

i.e. framed homeo. to smooth submanifold

- in general, stable sings in codim > 1 not smoothable

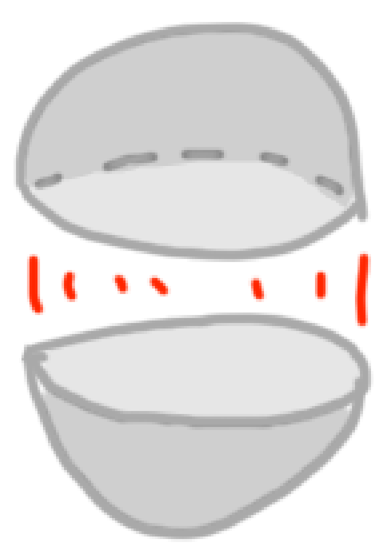
  ⤳ e.g. embed a compact, non-smoothable PL manifold and perturb it to contain only stable singularities…

<u>conj</u> (FCT): if tangle is smoothable then uniquely so.

<u>conj.</u> (FCT): Every smooth (compact) M can be realized as a tangle $M \hookrightarrow I^n$.

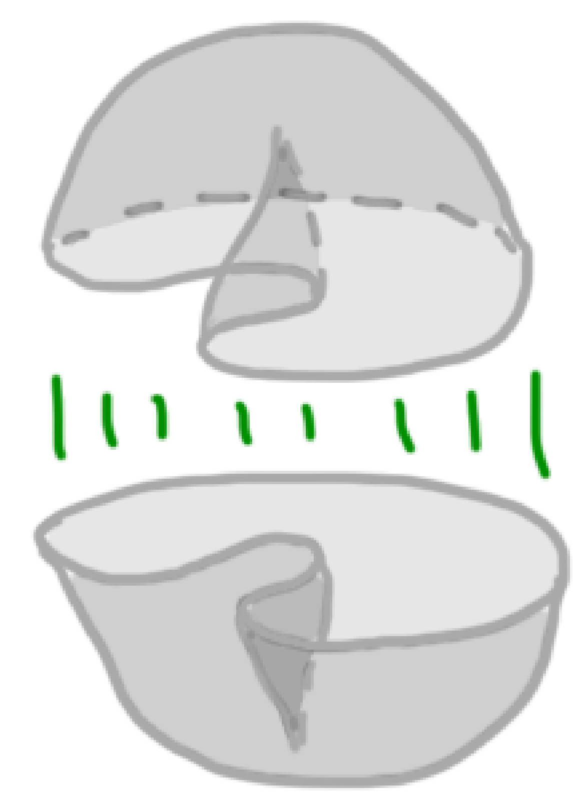
… but why? <u>heuristic 1</u>:

1-Morse theory:



exotic attachment of fiber sphere ($S^6 \to S^6$)

n-Morse theory



inductively, fiber spheres are dim 0

(4.2) Smoothness: The PT heuristic

- PT construction : $\Omega_n(\mathbb{R}^{n+k}) \cong \pi_{n+k} MO(k)$

  $\rightsquigarrow$ model $MO(k)$ as a computad ("Thom computad")

  (of course, all cells will be dualizable / invertible !)

  $\rightsquigarrow$ consider functors $S^{n+k} \longrightarrow MO(k)$

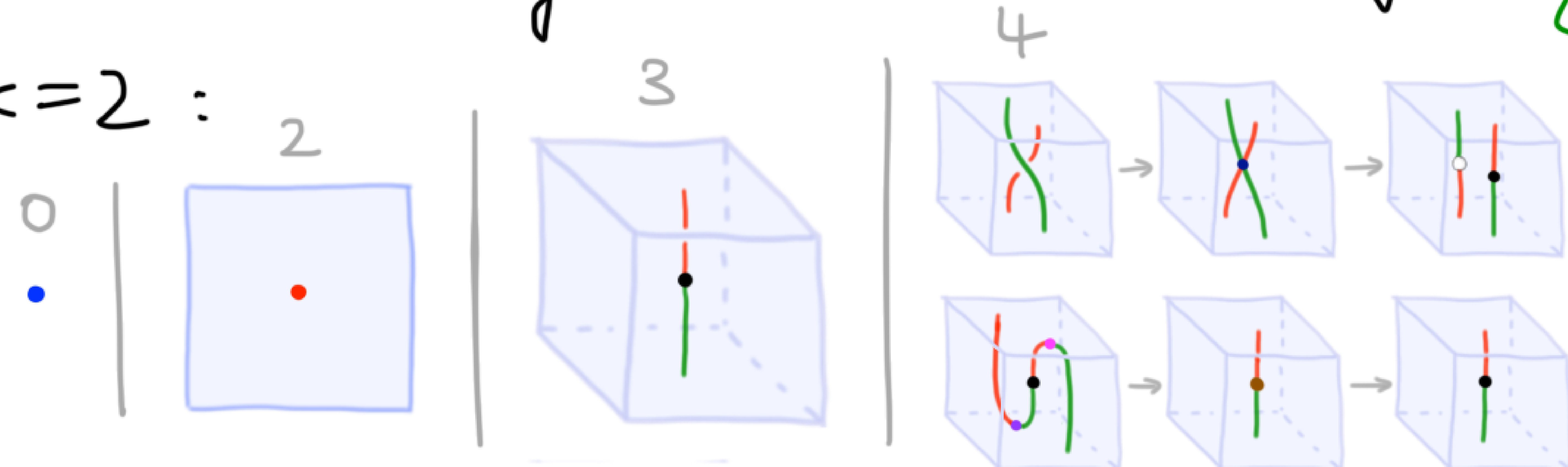
  $\Rightarrow$ the generating $(n+k)$-cell maps to a pasting diagram in $MO(k)$

  $\Rightarrow$ dualize to get a manifold diagram, which (combinatorially) describes a tangle $M \hookrightarrow I^n$

  $\hookrightarrow$ in classical PT, all tangles are obtained this way

e.g. $k=2$:

$MO(2)$



Summary

FCT allows to

- → define manifold diagrams, and work with them both geometrically and combinatorially
- → define pasting diagrams dual to manifold diag. (⇝ introduce "computads" as presheafs on cells)
- → uniformly treat structural stability of singularities (⇝ "combinatorial" singularity and higher Morse theory)

FCT aims to

- → combinatorialize smooth structure (⇝ "differential geometry without $\mathbb{R}$")

...

Appendix A: FCT book overview

Chapter 1

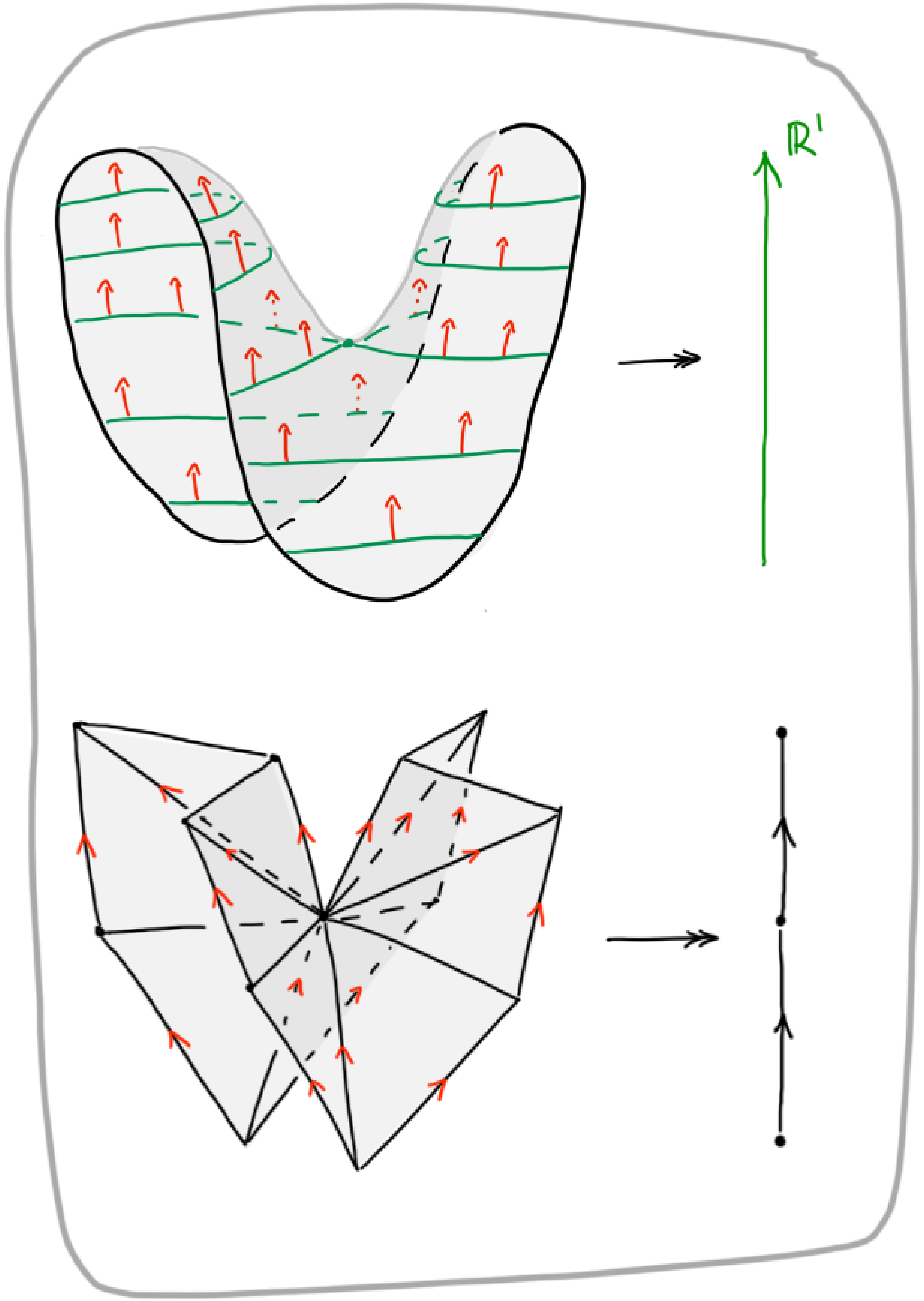
Chapter 2 trusses

Fr Δ

Fr Cell

Chapter 3

classification



Chapter 4 meshes ($\simeq$ trusses)

App A

App B

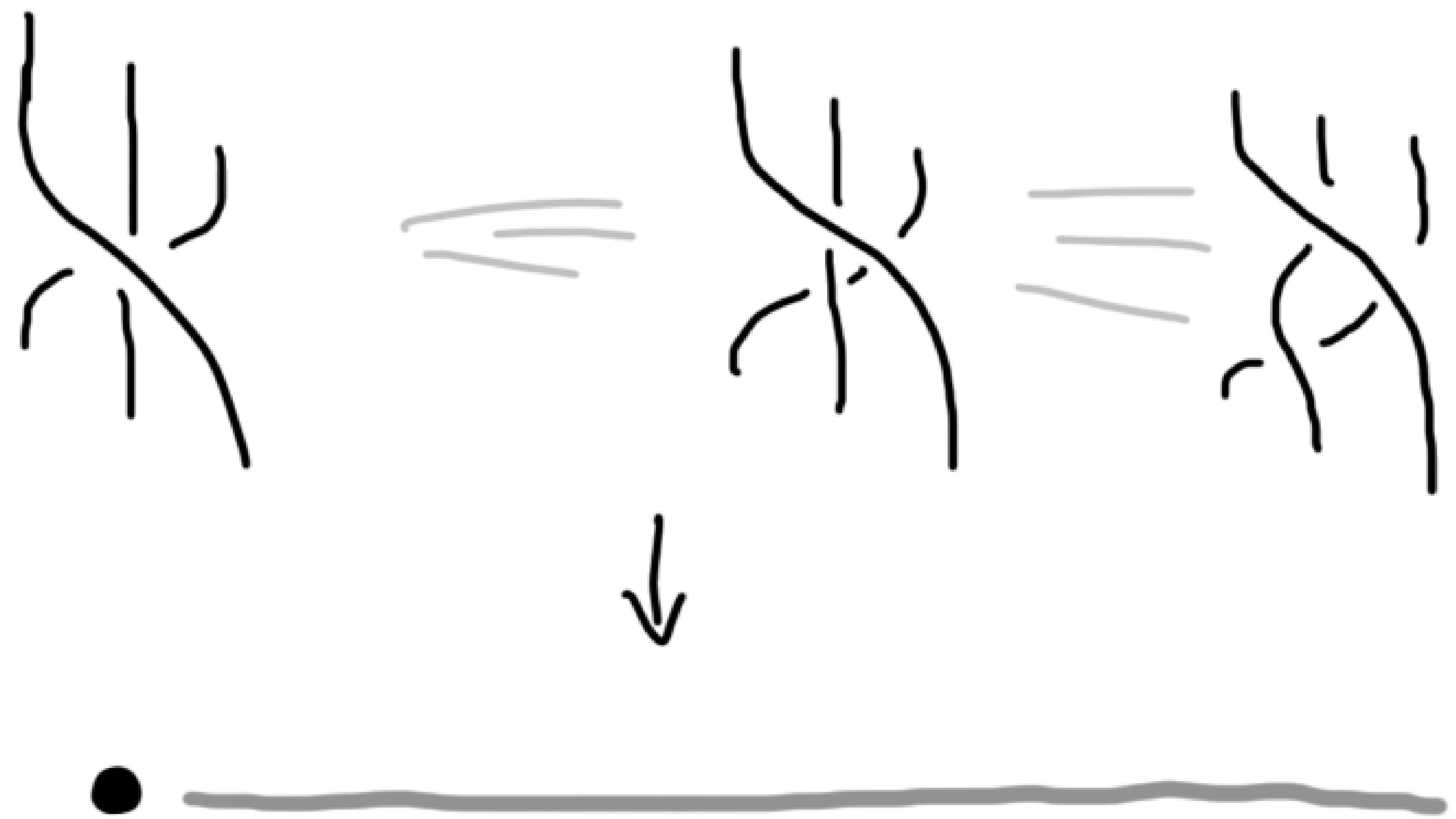
Chapter 5

Fr Strat

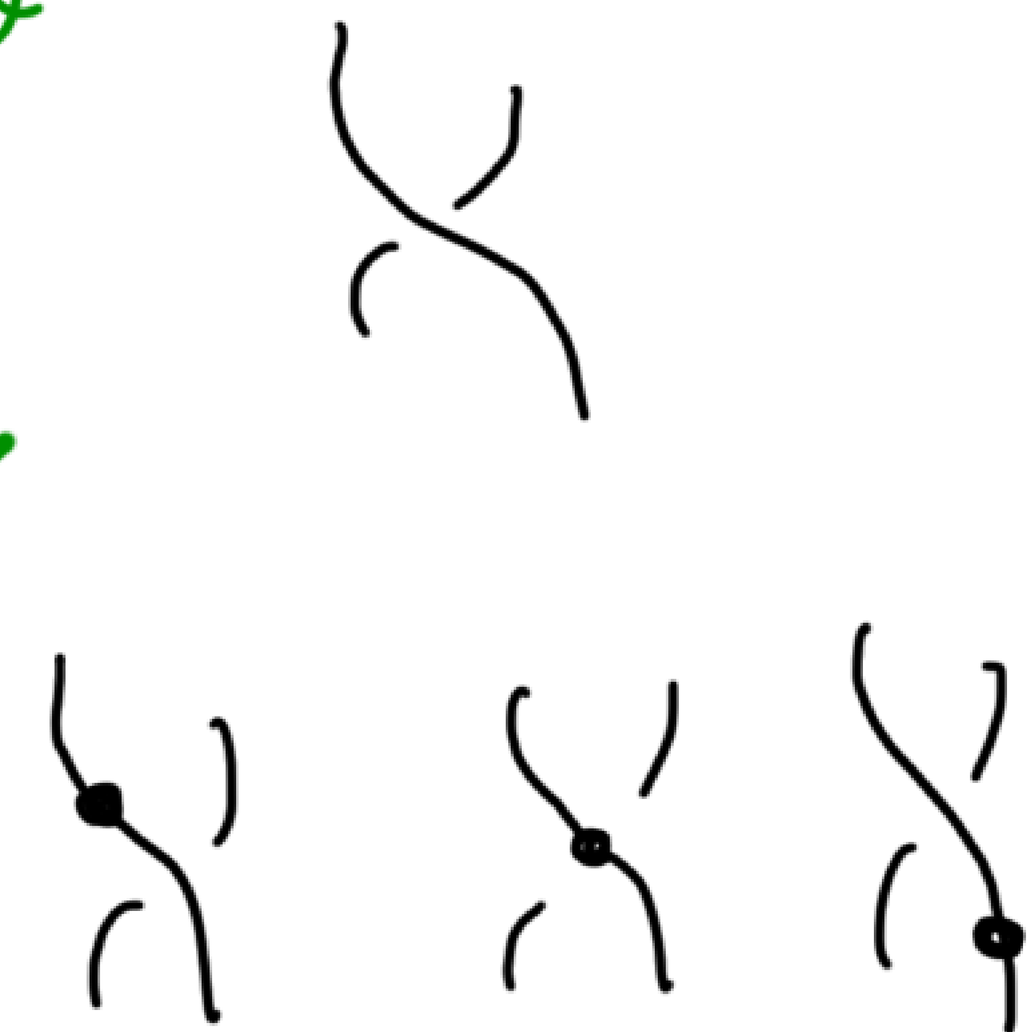
Appendix B: Elementary homotopies

($\equiv$ structurally stable homotopies)

unstable



stable



Other stable homotopies: those in Manuel's talk!

(e.g. Yang-Baxterator)

(Appendix C) The issue w/ singularities up to $\mathrm{Diff}(\mathbb{R}^n)$

given $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathrm{Diff}(\mathbb{R}^n)$ acts by Gl_n around $0 \in \mathbb{R}^n$
($\hookrightarrow$ on all J^k)

- if $n \geq 3$, then $\mathrm{corank}_{\geq 3} \subset J^2$ has codim 6

$\Longrightarrow$ w/ 6+ parameters can assume $f = \underbrace{\text{cubic in 3 var.}}_{\dim (10)}$ $(= J^3)$

$9 = \dim Gl_3 < \dim(10)$

$\dim J^1 = n$ "uses up" $\mathbb{R}^n$ itself

- if $n = 2$, then $\dim J^2 + \dim J^3 = 7$

$\Longrightarrow$ w/ 7+ parameter family can assume $f = \underbrace{\text{quartic in 2 var}}_{\dim 5}$

$4 = \dim Gl_2 < \dim 5$

($\bullet$ $n = 1 \rightsquigarrow A_k$ sing)