The geometric cobordism hypothesis

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with Dan Grady
- I Extended field theories are local arXiv:2011.01208 v3 41 pages
- II The geometric cobordism hypothesis arXiv:2111.01095 v2 53 pages

94 pages, with Dan Grady

in preparation
- III a) (w/ S. Stolz, P. Teichner) Differential cohomology via smooth field theories
  - b) (w/ D. Grady) Prequantum Chern–Simons theory & differential characteristic classes

IV, V, …

Outline:

1. Main definitions & statements
2. Examples and applications
3. Vistas: nonperturbative quantization, index theorems…

Theorem (The geometric cobordism hypothesis)

Input data:

- $d \geq 0$ (dimension)
- $\mathcal{C}$: smooth symmetric monoidal (∞, d)-category
- S: geometric structure (a simplicial presheaf on FEmb_d)

Part I: (Locality)

$$\underbrace{\mathrm{Fun}^{\otimes}}_{\substack{\text{smooth symmetric monoidal}\\ \text{functors of } (\infty,d)\text{-categories}\\ \text{(difficult to compute)}}}(\mathrm{Bord}_d^S, \mathcal{C}) \simeq \underbrace{\mathrm{Map}_{\mathrm{FEmb}_d}}_{\substack{\text{maps of simplicial presheaves}\\ \text{on } \mathrm{FEmb}_d \text{ (easy to compute)}}}(S, \mathcal{C}_d^{\times})$$

Part II: (Framed case)

$$\mathcal{C}_d^{\times}\left(\begin{matrix}\mathbb{R}^d \times \mathcal{U} \\ \downarrow \\ \mathcal{U}\end{matrix}\right) := \mathrm{Fun}^{\otimes}\left(\mathrm{Bord}_d^{\mathbb{R}^d \times \mathcal{U} \to \mathcal{U}}, \mathcal{C}\right) \simeq \mathcal{C}^{\times}(\mathcal{U})$$

Examples of geometric structures.

- Smooth maps to a fixed target manifold
- Riemannian metrics, Lorentzian metrics, conformal structures
- Principal G-bundles with connection (gauge fields)
- Bundle n-gerbes with connection (Kalb–Ramond B-field, SUGRA C-field)
- Geometric String structures
- Topological structures: orientation, spin, framing, etc.

(2) A bit of history and motivation:

Feynman 1942, 1948: Path integral formulation of quantum mechanics

→ uses integration over ∞-dimensional spaces of paths

1940s – 1980s: constructions of QFTs

→ uses integration over ∞-dim spaces like
$\mathrm{Map}(M, N)$, $\dim M > 1$; $\Omega^k N$; connections on M; etc.

mid 1980s: Witten $\rightsquigarrow$ Segal

Axioms for functional integrals:

(d-1)-manifold $M \mapsto$ a Hilbert space of states $\mathcal{F}(M)$
d-bordism $B: M_1 \to M_2 \mapsto$ a linear map $\mathcal{F}(B): \mathcal{F}(M_1) \to \mathcal{F}(M_2)$
gluing of bordisms $\mapsto$ composition of linear maps

That is (Segal): a symmetric monoidal functor $\mathrm{Bord}_d \to \mathrm{Hilb}$

Segal used <u>conformal</u> structures on bordisms

1988–: topological field theory (TQFT): Witten, Atiyah, Kontsevich ...

1992 (Freed, Lawrence): extended QFTs

1995 (Baez–Dolan): the (topological) cobordism hypothesis

2002 (Stolz–Teichner): smooth & supersymmetric FFTs

2004 (Costello): the topological cobordism hypothesis* for the $(\infty, 2)$-category of bordisms

2008 (Schommer-Pries): CH for (un)oriented 2-cat of Bord.

2009 (Hopkins, Lurie): the topological cobordism hypothesis for the (∞, d)-category of bordisms w/ topological structures

Missing: constructions of <u>fully extended</u> nontopological FFTs in dimension 2 and higher!

e.g., conformal field theories, gauged field theories, Riemannian / Lorentzian, etc.

3 Definition of FFTs

3a A crash course in higher category theory

Main idea: A smooth symmetric monoidal (∞,d)-category with duals has...

- objects with $\otimes$; dualizable in the homotopy category
- k-morphisms for $0<k\leq d$; have adjoints if $k<d$.
- invertible k-morphisms for $k>d$.
- objects and k-morphisms can be organized into $\mathcal{U}$-indexed families ($\mathcal{U}\cong\mathbb{R}^n$, $n\geq 0$)
- these families can be pulled back along C^∞-maps $\mathbb{R}^m\to\mathbb{R}^n$ and also glued along open covers of $\mathbb{R}^n$.

Definition A smooth symmetric monoidal (∞,d)-category w/duals is a sheaf of sym. mon. (∞,d)-cat w/ duals on the site Cart

Cart: Ob: $\mathbb{R}^n$, Mor: C^∞-maps.

Definition A symmetric monoidal (∞,d)-category is a special Γ-object in (∞,d)-categories

Definition An (∞,d)-category is a globular complete Segal object in $(\infty,d-1)$-categories.

Definition An $(\infty,0)$-category is a simplicial set.

Definition A smooth symmetric monoidal (∞,d)-category with duals is a simplicial presheaf $\mathcal{C}:(\mathrm{Cart}\times\Gamma\times\Delta^d)^{op}\longrightarrow \mathrm{sSet}$, where

a) Cart: Ob $\cong\mathbb{R}^n$; Mor = smooth maps $\mathbb{R}^m\to\mathbb{R}^n$.
$\forall\langle\ell\rangle\in\Gamma\ \forall \mathbb{m}\in\Delta^d$: $\mathcal{C}(-,\langle\ell\rangle,\mathbb{m}):\mathrm{Cart}^{op}\to\mathrm{sSet}$ is an ∞-sheaf on Cart

b) $\Gamma=\mathrm{FinSet}_*^{op}$ encodes sym. mon. structure
$\forall U\in\mathrm{Cart}\ \forall\mathbb{m}\in\Delta^d$: $X=\mathcal{C}(U,-,\mathbb{m}):\Gamma^{op}\to\mathrm{sSet}$ is a special Γ-space: $X_0\xrightarrow{\sim}1$; $X_{a+b}\xrightarrow{\sim}X_a\times X_b$.

c) $\Delta^d=\Delta\times\cdots\times\Delta$ encodes d-cat. structure
$\forall U\in\mathrm{Cart}\ \forall\langle\ell\rangle\in\Gamma$: $X=\mathcal{C}(U,\langle\ell\rangle,-):(\Delta^d)^{op}\to\mathrm{sSet}$ satisfies:
- $X_0:(\Delta^{d-1})^{op}\to\mathrm{sSet}$ is a homotopy constant functor (globularity)
- $X:\Delta^{op}\to\mathrm{Fun}((\Delta^{d-1})^{op},\mathrm{sSet})$ is a complete Segal object (Rezk)
- $\forall[\ell]\in\Delta$: $X_\ell:(\Delta^{d-1})^{op}\to\mathrm{sSet}$ is a globular complete $(d-1)$-fold Segal space. (Barwick)

d) all objects of $\mathcal{C}$ have duals in $\mathrm{Ho}(\mathcal{C})$
all k-morphisms of $\mathcal{C}$ ($0<k<d$) have adjoints. (in the appropriate homotopy category)

Observation These conditions define a reflective localization of presheaves on $\mathrm{Cart}\times\Gamma\times\Delta^d$

$\rightsquigarrow$ easy to compute colimits of smooth sym. mon. (∞,d)-cat. in this model.

Geometric structures

Definition Fix $d \geq 0$. The site FEmb_d has

Objects: submersions $\begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix}$ with d-dimensional fibers

Morphisms:
$$\begin{tikzcd} T_1 \arrow[r] \arrow[d] & T_2 \arrow[d] \\ \mathcal{U}_1 \arrow[r] & \mathcal{U}_2 \end{tikzcd}$$
fiberwise (open) embeddings

Covering families: induced by the forgetful functor
$$\mathrm{FEmb}_d \longrightarrow \mathrm{Cart} \qquad \begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix} \longmapsto T.$$

Definition A *d-dimensional geometric structure* is an ∞-sheaf
$$S : \mathrm{FEmb}_d^{op} \longrightarrow \mathrm{sSet}$$

Examples a) M smooth manifold: $\begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix} \longmapsto C^\infty(T, M)$

b) $k \geq 0$: $\begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix} \longmapsto \underbrace{\Omega^k_{\mathcal{U}}(T)}_{\text{fiberwise}}$

c) G: a Lie group; $\mathbb{B}_\nabla G$: $\begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix} \mapsto$ groupoid of fiberwise principal G-bundles w/ connection ∇

Can always assume $T \cong \mathbb{R}^d \times \mathcal{U}$.

$\rightsquigarrow \mathbb{B}_\nabla G$ can be defined as $\Omega^1_{\mathcal{U}}(T, \mathfrak{g}) // C^\infty(T, G)$

d) A: abelian Lie group (e.g., $A = U(1)$); $k \geq 0$

$\mathbb{B}^k_\nabla A$: fiberwise bundle $(k-1)$-gerbes w/ ∇

$\rightsquigarrow \mathbb{B}^k_\nabla A$ is the fiberwise Deligne complex.

e) $\begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix} \longmapsto$ fiberwise Riemannian metrics on $\begin{matrix} T \\ \downarrow \\ \mathcal{U} \end{matrix}$

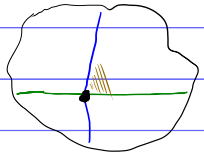
Also: Lorentzian, conformal, symplectic, complex, ...

3c Smooth symmetric monoidal (∞, d)-category of bordisms

Main idea: Given $d \geq 0$ and a d-dimensional geometric structure $\mathcal{S} \in sPSh(FEmb_d)$, the smooth symmetric monoidal (∞, d)-category of bordisms with geometric structure $\mathcal{S}$ has...

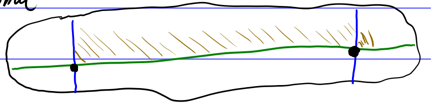
- objects: d-dimensional germs of points equipped with a germ of an $\mathcal{S}$-structure and a d-tuple of transversal cuts with normal orientations

$d=2$

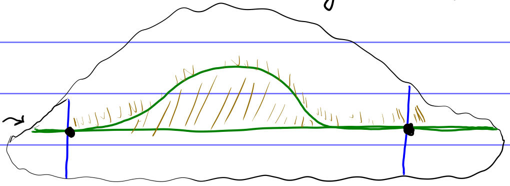


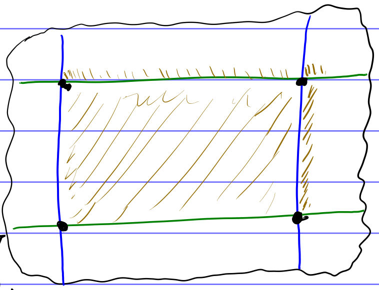
- k-morphisms $(0 < k \leq d)$: d-dimensional germs of k-dimensional bordisms with corners of all codimensions equipped with a germ of an $\mathcal{S}$-structure and a transversal grid of cuts.

$k=1$



globular





not globular

- k-morphisms for $k > d$ are $(k-d)$-homotopies in the space of transversal grids of cuts (keeping the bordism and the geometric structure fixed).

Practical consequence for field theories: intervals are invertible e.g., time evolution operators in quantum mechanics.

Warning There is a different type of homotopy for bordisms: a smooth $\mathbb{R}^1$-indexed family of bordisms, i.e., a concordance. Concordances can change geometric structures on bordisms.

Practical consequence for field theories: time evolution operators depend smoothly on time.

(3d) Functorial field theories

Definition Suppose $d \geq 0$, $\mathcal{S} \in \mathrm{sPSh}(\mathrm{FEmb}_d)$, $\mathcal{C} \in C^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$

Then $\mathrm{FFT}_{d,\mathcal{S},\mathcal{C}} := \underbrace{\mathrm{Hom}(\mathrm{Bord}_d^{\mathcal{S}}, \mathcal{C})}_{\text{Can discard structures we don't need.}} \in C^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$

Remark Can show using elementary arguments:
if $\mathcal{C}$ has duals, i.e., $\mathcal{C} \in C^\infty\mathrm{Cat}^{\otimes,\checkmark}_{\infty,d}$, then
$\underbrace{\mathrm{FFT}_{d,\mathcal{S},\mathcal{C}}}_{= \mathrm{FFT}^{\times}_{d,\mathcal{S},\mathcal{C}}} \in C^\infty\mathrm{Grpd}^{\otimes}_{\infty} = \mathrm{Fun}((\mathrm{Cart}\times\Gamma)^{op}, \mathrm{sSet})$.

Example $d = 1$, $\mathcal{S}$ = orientation + a smooth map to $X \in \mathrm{Man}$, $\mathcal{C} = \mathrm{Vect}$: fin-dim vect. sp. and linear maps

$$\mathrm{FFT}_{1,\mathrm{or},\mathrm{Vect}} \simeq \mathrm{Vect}_\nabla(X).$$

parallel transport

Observation The right side is a sheaf with respect to X.

Locality of fully extended FFTs:
(= GCH, Part I.)

Theorem (Grady–P., 2020) Given $d \geq 0$, $\mathcal{C} \in C^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$, the functor

$$\mathcal{S} \longmapsto \mathrm{FFT}_{d,\mathcal{S},\mathcal{C}} = \mathrm{Hom}(\mathrm{Bord}_d^{\mathcal{S}}, \mathcal{C})$$
$$\mathrm{sPSh}(\mathrm{FEmb}_d)^{op} \longrightarrow C^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$$

is an ∞-sheaf.

Proof (sketch): It suffices to show

$$\mathcal{S} \longmapsto \mathrm{Bord}_d^{\mathcal{S}}$$
$$\mathrm{sPSh}(\mathrm{FEmb}_d) \longrightarrow C^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$$

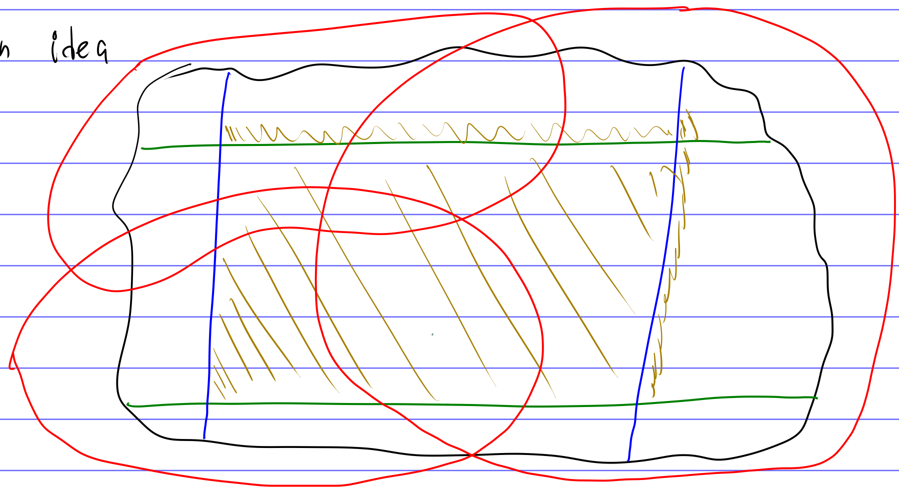
is an ∞-cosheaf, i.e., it preserves colimits.

Recall: $C^\infty\mathrm{Cat}^{\otimes}_{\infty,d} \subset \mathrm{sPSh}(\mathrm{Cart}\times\Gamma\times\Delta^d)$

Easy and formal: $\mathrm{Bord}_d : \mathrm{sPSh}(\mathrm{FEmb}_d) \longrightarrow \mathrm{sPSh}(\mathrm{Cart}\times\Gamma\times\Delta^d)$.

Remains to show: Bord_d sends Čech nerves of open covers in FEmb_d to local weak equivalences.

Main idea



Corollary (Grady – P., 2021)

$$\mathrm{FFT}_{d,\mathcal{S},\mathcal{C}}\underbrace{[X]}_{\text{concordance classes over } X} \simeq [X, \underbrace{B_{\mathcal{S}}\ \mathrm{FFT}_{d,\mathcal{S},\mathcal{C}}}_{\text{classifying space of FFTs}}]$$

conjectured by Stolz & Teichner ~ 2007 (?)

3e The geometric cobordism hypothesis

Definition Given $d\geq 0$, $\mathcal{C}\in C^\infty\mathrm{Cat}^{\otimes,\vee}_{\infty,d}$,

set $\mathcal{C}^\times_d : \mathrm{FEmb}_d^{op} \to \mathrm{sPSh}(\mathrm{Cart}\times\Gamma) = C^\infty\mathrm{Grpd}^\otimes_\infty$

$$\mathcal{C}^\times_d\begin{pmatrix}T\\ \downarrow\\ U\end{pmatrix} := \left(\mathrm{FFT}_{d,T\to U,\mathcal{C}}\right)^\times,$$

discards noninvertible k-morphisms for all $k>0$; automatic for targets w/duals

[Can also formulate a version without $\times$]

Corollary (of locality) (= GCH, Part I) Given $d\geq 0$, $\mathcal{C}\in C^\infty\mathrm{Cat}^\otimes_{\infty,d}$,

we have a weak equivalence

$$\mathrm{FFT}^\times_{d,S,\mathcal{C}} = \mathrm{Hom}(\mathrm{Bord}^S_d,\mathcal{C})^\times \xrightarrow{\sim} \mathrm{Map}(\underbrace{S}_{\substack{\infty\text{-sheaf}\\ \text{of spaces!}}}, \mathcal{C}^\times_d)$$

much easier to compute than FFTs!

This is $\frac{1}{2}$ of GCH!

Question: How to compute $\mathcal{C}^\times_d$?

Definition Given $d\geq 0$, $\mathcal{C}\in C^\infty\mathrm{Cat}^\otimes_{\infty,d}$, $U\in\mathrm{Cart}$, the evaluation map

$$\mathcal{C}^\times_d\begin{pmatrix}\mathbb{R}^d\times U\\ \downarrow\\ U\end{pmatrix} = \mathrm{FFT}^\times_{d,\mathbb{R}^d\times U\to U,\mathcal{C}} \overset{\sim}{=} \mathrm{Hom}(\mathrm{Bord}_d^{\mathbb{R}^d\times U\to U},\mathcal{C})^\times \longrightarrow \underbrace{\mathcal{C}^\times}_{\mathcal{C}^\times\in\mathrm{sPSh}(\mathrm{Cart}\times\Gamma)}(U)$$

evaluates at 0-bordisms $\{0\}\times U$.

Theorem (Grady–P., 2021) (= GCH, Part II) The evaluation map is a weak equivalence.

Proof (sketch)
1) Induction on $d\geq 0$.
2) Filter $\mathrm{Bord}_d^{\mathbb{R}^d\times U\to U}$ using the Morse index k
3) Present individual steps in the filtration as cobase changes of much simpler bordism categories (e.g., handles), using the machinery of the locality theorem.
4) Connect $k=-1$ to Bord_{d-1}.
5) Assemble 2)–4) together.

④a A recipe for computing (spaces of) FFTs

① Compute $\mathcal{C}_d^\times$ Tools: (higher) connections on ∞-bundles

② Compute $\mathrm{Hom}(\mathcal{S}, \mathcal{C}_d^\times)$ Tools: natural operations in differential geometry

Examples

How to compute $\mathcal{C}_d^\times$ in practice?

easy to write down because the "topological sector" is trivial!

① Guess a map $\mathcal{F} \longrightarrow \mathcal{C}_d^\times$.

In practice: throw in some connection forms

② Prove: for any $U \in \mathrm{Cart}$, the composition

$$\begin{tikzcd} & \mathcal{C}_d^\times\left(\begin{smallmatrix}\mathbb{R}^d \times U\\ \downarrow \\ U\end{smallmatrix}\right) \arrow[dr] & \\ \mathcal{F}\left(\begin{smallmatrix}\mathbb{R}^d \times U\\ \downarrow \\ U\end{smallmatrix}\right) \arrow[ur] \arrow[rr, "\sim"] & & \mathcal{C}^\times(U) \end{tikzcd}$$

is a weak equivalence.

does not involve field theories!

Example $d \geq 0$, A: abelian Lie group, $\mathcal{C} = \underbrace{\mathbb{B}^d A}$

a single k-morphism for all $k \neq d$

d-morphisms $= A$.

⓪ $\mathbb{B}^d A$ classifies bundle $(d-1)$-gerbes w/ band A.

① Guess: $\mathcal{F}(T \to U)$ is the fiberwise Deligne complex $\Omega_U^{\leq d}(-, A)$

$$\Omega_U^d(-,\mathfrak{a}) \leftarrow \Omega_U^{d-1}(-,\mathfrak{a}) \leftarrow \cdots \leftarrow \Omega_U^1(-,\mathfrak{a}) \leftarrow C^\infty(-,A).$$

The map $\mathcal{F} \longrightarrow (\mathbb{B}^d A)_d^\times$ integrates an $\mathfrak{a}$-valued d-form over a d-bordism in $\mathbb{R}^d$, and applies $\mathfrak{a} \xrightarrow{\exp} A$.

② The composition is

$$\Omega_U^d(-,\mathfrak{a}) \leftarrow \Omega_U^{d-1}(-,\mathfrak{a}) \leftarrow \cdots \leftarrow \Omega_U^1(-,\mathfrak{a}) \leftarrow C^\infty(-,A).$$

$$\uparrow$$

$C^\infty(U, A)$ includes as fiberwise locally constant functions

Corollary (P., Stolz, Teichner, ~2012)

$$\mathrm{FFT}_{d,X,\mathbb{B}^d A} \simeq \left\{\begin{matrix}\text{bundle } (d-1)\text{-gerbes over } X \\ \text{with connection} \\ \text{and band } A\end{matrix}\right\}$$

Example $(\mathbb{B}G)_1^\times \simeq \mathbb{B}_\nabla G$ (G can be nonabelian)

Conjecture: $(\mathbb{B}G)_d^\times \simeq \underbrace{\mathbb{B}_{\nabla_d} G}$ connections with d-dimensional holonomy

for any Lie ∞-group G.

Example $d \geq 0$, $\mathcal{S} = \mathrm{Riem}$, $\mathcal{C} = \mathbb{B}^d\mathbb{R}$

$$\mathrm{FFT}_{d,\mathcal{S},\mathcal{C}} \simeq \mathrm{Map}(\mathrm{Riem}, \Omega^{\leq d}(-,\mathbb{R})).$$

Gilkey: $\mathrm{Riem} \to \Omega^{\leq d}$ are given by Pontrjagin forms!

$\rightsquigarrow \exists$ Riemannian FFT of dimension $4k=d$

closed $4k$-manifold $\mapsto \int p_k$

$\mathcal{C} = \mathbb{B}^d U(1)$: differential Pontrjagin classes

Example $d \geq 0$, $\mathcal{S} = \mathbb{B}_\nabla G$, G Lie group, $\mathcal{C} = \mathbb{B}^d\mathbb{R}$; $\mathbb{B}^d U(1)$. Freed–Hopkins: $\mathbb{B}_\nabla G \to \mathcal{C}^\times_d = (\mathbb{B}^d\mathbb{R})^\times_d = \Omega^{\leq d}$

$\cong$ G-invariant polynomials p on $\mathfrak{g}$ (via Chern–Weil)

$\rightsquigarrow$ gauged FFT of dimension $d \geq 0$ $\forall p$ of degree d

closed d-manifold $\mapsto \int p$

$\mathcal{C} = \mathbb{B}^d U(1)$: differential characteristic classes

Kolář, Michor, Slovák:

Natural operations in differential geometry

Example (Smooth extended GMTW).

(Grady–P., 2021) $d \geq 0$, $\mathcal{S}$, $\mathcal{C}$: invertible obj & mor

$$\mathrm{FFT}_{d,\mathcal{S},\mathcal{C}} \simeq \mathrm{Hom}(\underbrace{MT(\mathcal{S})}_{\text{smooth Madsen–Tillmann spectrum of } \mathcal{S}}, \mathcal{C})$$

Conjecture 8.37 (Freed, Hopkins)

$$\left\{\begin{array}{l}\text{Deform. classes of} \\ \text{refl. positive inv.} \\ d\text{-dim ext FFT} \\ \text{with sym. type } (H_d, \rho_d)\end{array}\right\} \simeq [MTH, \Sigma^{d+1} I_{\mathbb{Z}(1)}].$$

Theorem (Grady, 2022) Conjecture 8.37 is true.

5) What is nonperturbative quantization?

Joint work in progress with Daniel Grady.

$f: X \to Y$ smooth map w/ add. struct.

Hard! ∞-dim $\int$

only finite-dimensional objects!

$$\begin{tikzcd} \mathrm{FFT}(X) \arrow[r, "\sim", "\mathrm{GCH}"'] \arrow[d] & \mathrm{Map}(X, \mathcal{C}_d^\times) \arrow[d] \\ \mathrm{FFT}(Y) \arrow[r, "\sim", "\mathrm{GCH}"'] & \mathrm{Map}(Y, \mathcal{C}_d^\times) \end{tikzcd}$$

Conjecture For $d=1$, $S=$ Spin-structure recover pushforwards in differential K-theory

Remark We know: shape$(\mathrm{FFT}_{d,S(-),\mathcal{C}})$ is a (connective) spectrum;
cohomology theory $\equiv$ concordance classes of FFTs
differential coh. thy $\equiv$ geometric c. c. of FFTs.

Conjecture shape(pushforward quantization) = topological pushforward (= index)

For $d=1$ recover the Atiyah—Singer index theorem

What kind of index theorem do we get for $d>1$?