Berezin-Toeplitz quantization in the Yau-Tian-Donaldson program

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Geometria em Lisboa seminar

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Plan

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1. **Yau-Tian-Donaldson program**

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2. **Balanced metrics**

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g_{FS} is not canonical : $GL(N+1) \circlearrowright \mathbb{CP}^N$ preserves $[\omega_{FS}] \in H^2(X, \mathbb{Z})$, but not ω_{FS}.

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Question : Among Kähler metrics $g \in [\omega_{FS}]$ (i.e. such that $\omega := g(J\cdot, \cdot) \in [\omega_{FS}]$), does there exist a canonical one ?

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- For $\lambda < 0$, there are obstructions

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Theorem [Matsushima,'57, Futaki, '83]

If there exists $g \in -c_1(X)$ Kähler-Einstein, then $\mathrm{Aut}(X)$ is reductive, and the **Futaki invariant** $\mathrm{Fut} : \mathrm{Lie}\,\mathrm{Aut}(X) \to \mathbb{C}$ vanishes.

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Conjecture [Yau,'90]

The existence and uniqueness of $g \in [\omega_{FS}]$ cscK on X is equivalent to some stability condition on its complex-algebraic geometry.

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Theorem [Chen-Donaldson-Sun, '15, Tian, '15]

The Conjecture is true for $\lambda[\omega_{FS}] = c_1(X)$, $\lambda < 0$.

Balanced metrics

- For $X \subset \mathbb{CP}^N$ projective, the dual **tautological line bundle** $\mathcal{O}(1) \to \mathbb{CP}^N$ restricts to an **ample** line bundle $L \to X$, and we have $[\omega_{FS}] = c_1(L)$.

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Definition

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L is called **ample** if for all $p \gg 0$ and any basis $\mathbf{s} := \{s_j\}_{j=0}^{N_p}$ of $\mathcal{H}_p$, the map

$$\begin{aligned} \iota_{\mathbf{s}} : X &\longrightarrow \mathbb{CP}^{N_p} \\ x &\longmapsto \left[s_0(x) : \cdots : s_{N_p}(x)\right], \end{aligned}$$

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- Furthemore, we have $L^p \simeq \iota_{\mathbf{s}}^* \mathcal{O}(1)$, so that $\frac{1}{p} \iota_{\mathbf{s}}^* g_{FS} \in c_1(L)$. (Recall $c_1(L^p) = p\, c_1(L)$.)

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The image of the **Fubini-Study map**

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- **Question :** Is there a finite-dimensional notion of a canonical Kähler metric ?

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Definition

A Hermitian metric h on $L \to X$ is **positive** if its Chern curvature R_h defines a Kähler metric $g_h \in \mathcal{H}_{c_1(L)}$ by $\omega_h := \frac{\sqrt{-1}}{2\pi} R_h \in c_1(L)$.

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- Uniqueness of balanced metrics follows from standard moment map techniques over $\mathrm{Herm}(\mathcal{H}_p)$.
- Key step in the resolution of Yau's conjecture.

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- For $\lambda = 0$, the measure $\nu := \nu_h$ is the measure induced by the holomorphic volume form of X as a Calabi-Yau manifold.

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Asymptotic estimate of the rate of convergence of Donaldson's approximation as $p \to +\infty$.

- This confirms the numerical prediction of Donaldson.

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Let $g_\infty \in \mathcal{H}_{c_1(L)}$ Kähler-Einstein. Then there exists a unique $g_p \in \mathcal{H}_{c_1(L^p)}$ canonically balanced for all $p \gg 0$, and

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- I.,'21 is based on **Berezin-Toeplitz quantization**.

Berezin-Toeplitz quantization

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- The Hilbert spaces $(\mathcal{H}_p, L^2(h^p, \nu))$ form a **holomorphic quantization** of the symplectic manifold (X, ω_h).

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Definition

The **Berezin-Toeplitz coherent state** $\Pi_p(x) \in \mathrm{Herm}(\mathcal{H}_p)$ at $x \in X$ is the unique orthogonal projector such that

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Proposition

There exists a unique function $\rho_p \in \mathcal{C}^\infty(X, \mathbb{R})$, called the **density of states** (or **Bergman kernel**), satisfying

$$\rho_p(x) \, \langle \Pi_p(x) s_1, s_2 \rangle_{L^2(h^p, \nu)} = \langle s_1(x), s_2(x) \rangle_{h^p} .$$

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Theorem [Boutet de Monvel-Sjöstrand,'75, Zelditch, Catlin,'98, Dai-Liu-Ma,'06]

There exists $b_r \in \mathcal{C}^\infty(X, \mathbb{R})$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ as $p \longrightarrow +\infty$,

$$\rho_p = p^n \sum_{j=1}^{k-1} p^{-r}\, b_r + O(p^{n-k})\,.$$

Furthermore, we have $b_0\, d\nu = dvol_{g_h}$, and in the case $d\nu = dvol_{g_h}$,

$$b_0 \equiv 1 \quad \text{and} \quad b_1 = \frac{1}{8\pi}\,\mathrm{scal}(\omega)\,.$$

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- We have $\dim \mathcal{H}_p = \int_X \rho_p\, d\nu = p^n\, \mathrm{Vol}(X, \omega) + O(p^{n-1})$: number of particles $\xrightarrow{p \to +\infty}$ volume of phase space.

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Proposition

If $\mathbf{s} \in \mathcal{B}_p$ is orthonormal with respect to $L^2(h^p, \nu)$, then we have

$$h^p = \rho_p \, \iota_{\mathbf{s}}^* h_{FS} \, .$$

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Basic link with Donaldson,'01 : Assume that we have $g_p \in \mathcal{H}_{c_1(L^p)}$ balanced for all level $p \gg 0$ and that there exists $g_\infty \in \mathcal{H}_{c_1(L)}$ such that $\frac{1}{p} g_p \xrightarrow{p \to +\infty} g_\infty$.

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$$p^{-n} \rho_p = 1 + \frac{1}{8\pi p} \operatorname{scal}(\omega_p) + O(p^{-2}) \equiv \text{constant} \, ,$$

so that $\operatorname{scal}(g_p) \xrightarrow{p \to +\infty} \operatorname{scal}(g_\infty) \equiv$ constant.

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The **Berezin-Toeplitz quantization** $T_p : \mathcal{C}^\infty(X, \mathbb{R}) \to \mathrm{Herm}(\mathcal{H}_p)$ is defined on $f \in \mathcal{C}^\infty(X, \mathbb{R})$ by

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The **Berezin symbol** (or **dequantization map**) $\sigma_p : \mathrm{Herm}(\mathcal{H}_p) \to \mathcal{C}^\infty(X,\mathbb{R})$ is defined on $A \in \mathrm{Herm}(\mathcal{H}_p)$ and $x \in X$ by

$$\sigma_p(A)(x) := \mathrm{Tr}[A\Pi_p(x)]\,.$$

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The **Berezin transform** is

$$\mathcal{B}_p := \sigma_p \circ T_p : \mathcal{C}^\infty(X, \mathbb{R}) \to \mathcal{C}^\infty(X, \mathbb{R}) .$$

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- Hence $\mathcal{B}_p$ measures the speed of convergence towards a fixed point of the map $FS \circ L^2$ acting on $\mathrm{FS}(\mathrm{Herm}(\mathcal{H}_p))$, i.e. a ν-balanced metric.

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- Write $\mathrm{Spec}(\mathcal{B}_p) =: \{1 = \gamma_{0,p} \geqslant \gamma_{1,p} \geqslant \cdots \geqslant \gamma_{k,p} \geqslant \cdots \geqslant 0\}$.

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- Write $\mathrm{Spec}(\mathcal{B}_p) =: \{1 = \gamma_{0,p} \geqslant \gamma_{1,p} \geqslant \cdots \geqslant \gamma_{k,p} \geqslant \cdots \geqslant 0\}$.
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- When $\nu = \nu_h$ depends on h, the operator S_p defined by $\frac{d}{dt}\Big|_{t=0} \iota_{\mathbf{s}_t}^* h_{FS} = S_p(f)\ h_{FS}$ as before does not coincide anymore with the Berezin transform.

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- I.-Polterovich,'21 : Case of stable vector bundles endowed with **Hermite-Einstein metrics**, giving a new proof of Wang,'05.

Quantum noise

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For any $L > 0$ and $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all eigenfunction $f \in \mathcal{C}^\infty(X, \mathbb{R})$ of B_p with associated eigenvalue $\mu \in [1, 1 - L/p]$ and all $p \in \mathbb{N}$, we have

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- This allows to apply KS,'01, to estimate the eigenvalues ■

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- **Case** $d\nu_h = dvol_{g_h}$ (IP,'21), : we use an estimate of Ma-Marinescu,'12,

$$\mathcal{B}_p(f) = f - \frac{\Delta}{4\pi p} f + \frac{1}{8\pi p^2}\left(\frac{\Delta^2}{2\pi} - D\right) f + O(p^{-3})|f|_{\mathcal{C}^6} .$$

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- **Case of a vector bundle** E (IP,'21), : we introduce a Berezin transform acting on $\mathcal{C}^\infty(X, \mathrm{End}(E))$ by interpreting the Berezin-Toeplitz quantization for vector bundles of Ma-Marinescu,'07,'12, as the **quantization of a symplectic fibration**.

The End

Thank you !