Deformation classes of QFTs and the Freed-Hopkins conjecture.

Conjecture (Freed-Hopkins): 1-1 corresp.

$$\left\{\begin{array}{l}\text{def. classes of} \\ \text{refl. pos. invertible } n\text{-dim} \\ \text{field theories w/} \\ \text{symm. group } H_n\end{array}\right\} \longleftrightarrow [MTH, \Sigma^{n+1} I_{\mathbb{Z}(1)}]$$

- What is a field theory?

Ingredients:
1. Symmetry group $\underline{H_n}$
2. The dimension n
3. The background is dynamical
4. The theories are coupled to the background

Input: (n, H_n) Output: field theory

- We want a nice axiomatic system for field theories w/ fixed discrete invariants (like dim, symm.)

Idea: Wick rotate to Euclidean field theories ⟶ Segal's axioms for FFT.

Wick rotation: In QM, we have a t-par semigroup

$$t \longmapsto e^{-itH/\hbar}$$

- If H is positive definite, self adjoint, we can extend to

$$T = \mathbb{R} - i\mathbb{R}_{>0} \subset \mathbb{C}$$

- Restricting to $-i\mathbb{R}_{>0}$

$$\tau \longmapsto e^{-\tau H/\hbar}$$

- This is called <u>Wick rotation</u>.

<u>Reflection structure</u>

$\tau \longmapsto -\tau$

$e^{-\tau H/\hbar} \longmapsto \overline{e^{-\tau H/\hbar}}$

<u>Positivity</u>:

positive definite Hamiltonian.

$$\begin{tikzcd} H \arrow[d, mapsto] \\ \overline{H} \end{tikzcd}$$

- We will focus on EFT w/ symmetry type H_n.

<u>Def</u>: let $\rho_n : H_n \longrightarrow O(n)$ be a rep.
An <u>H_n-structure</u> on X is a principal bunde

$$\underline{P} \longrightarrow X, \qquad (\underline{P}, \theta)$$

w/ an iso

$$\rightarrow Fr(X) \xrightarrow[\underline{\theta}]{\cong} \rho_n(\underline{P}) = \underline{P} \times_{H_n} O(n)$$

Equivalently,

$$\begin{tikzcd} & BH_n \arrow[d] \\ X \arrow[ur, "\underline{P}"] \arrow[r, "Fr", "g"'] \arrow[ur, phantom, "\Downarrow \theta", near end, shift left=3] & B\underline{O}(n) \end{tikzcd}$$

<u>Def</u>: A <u>differential</u> H_n-structure is an H_n-structure along w/ a connection Θ on $\underline{P} \to X$ s.t.

$$\theta^* \Theta = \check{\Theta}_g \leftarrow \text{Levi-Civita connection}$$

<u>Example</u>: For EM, one considers connections on a $U(1)$-principal bundle

$$\underline{P} \to X.$$

Maxwells eq.

$$dF = 0, \quad d\overset{\text{connection}}{\underline{A}} = F$$

Symmetry group is

$$H_n = \underset{\text{gause group}}{\underline{U(1)}} \times \underset{\text{symmetries of oriented manifolds}}{\overline{SO(n)}}$$

w/ fermions, this is not right

$$H_n = Spin^c(n)$$

Segals axioms for FFT

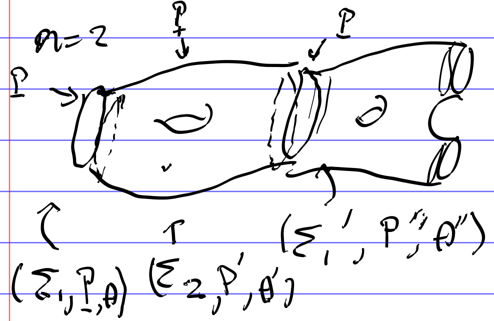
Def: A field theory is a symm. monoidal functor

topological

$$Z : \mathrm{Bord}_n^{(H_n)} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

obj: Σ_{n-1} w/ H_n-structure

Mor: Σ_n w/ $\partial\ \Sigma_{n-1} \sqcup \Sigma'_{n-1}$ w/ H_n-str.



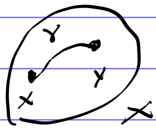
$$\begin{tikzcd} \longrightarrow Z(\Sigma, P, \theta) \in \mathrm{Vect}_{\mathbb{C}} \arrow[d, "\mathcal{L}(\Sigma_2, P', \theta')"] \\ Z(\Sigma_1', P'', \theta'') \end{tikzcd}$$

- A field theory is invertible if

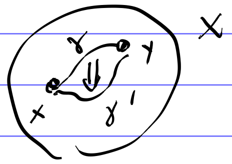
$$Z : \mathrm{Bord}_n^{H_n} \longrightarrow \mathrm{Line}_{\mathbb{C}} \hookrightarrow \mathrm{Vect}_{\mathbb{C}}$$

Example (Warm up): Fix a smooth man X. We claim that

$$\pi_0 \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^X, \mathrm{Line}_{\mathbb{C}}) \cong \underline{b\mathrm{Line}_{\nabla}(X)}$$

- Obj $\Sigma_0 \to X$, e.g. $x \in X \longmapsto \mathcal{L}_x$
- Mor $\Sigma_1 \to X$, e.g  $\longmapsto (\mathcal{L}_x \longrightarrow \mathcal{L}_y)$

Line bundle w/ connection!



$\longmapsto$ parallel transport is invariant.

In terms of moduli Stacks

$$\pi_0 \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{X}, \mathcal{L}ine_{\mathbb{C}}) \cong [X, \overset{\text{stack}}{b\mathcal{L}ine_{\nabla}}]$$

- $b\mathcal{L}ine_{\nabla} \simeq B\mathbb{C}^{\times}_{\delta}$.
- $B\mathbb{C}_{\delta} \xrightarrow{\exp} B\mathbb{C}^{\times}_{\delta} \xrightarrow{\beta} B^2\mathbb{Z}$
- $[X, B\mathbb{C}_{\delta}] \longrightarrow [X, B\mathbb{C}^{\times}_{\delta}] \longrightarrow [X, B^2\mathbb{Z}]$

$$\begin{tikzcd}
{[X, B\mathbb{C}_{\delta}]} \arrow[r] & {[X, B\mathbb{C}^{\times}_{\delta}]} \arrow[r] \arrow[d, phantom, "\parallel"] & {[X, B^2\mathbb{Z}]} \arrow[d, phantom, "\parallel"] \\
H^1(X, \mathbb{C}) \arrow[r] & H^1(X, \mathbb{C}^{\times}) \arrow[r, "\beta"] & H^2(X, \mathbb{Z})
\end{tikzcd}$$

Nontopological ? $\mathrm{im}(\beta) = H^2(X, \mathbb{Z})_{\mathrm{tor}}$ ✓

- Parallel transport is not flat!

✓

Replace $[X, b\mathcal{L}ine_{\nabla}] \rightsquigarrow [X, \mathcal{L}ine_{\nabla}]$

Deformation classes:

$\mathrm{Map}(X, \mathcal{L}ine_{\nabla}) \leftarrow$ Moduli stack of line bundles w/ conn. on X.

$$|\mathrm{Map}(X, \mathcal{L}ine_{\nabla})| = \underset{[n]\in\Delta}{\mathrm{hocolim}}\, \mathrm{Map}(X \times \Delta^n, \mathcal{L}ine_{\nabla})$$

- $|\mathrm{Map}(X, \mathcal{L}ine_{\nabla})| = \mathrm{Map}(|X|, |\mathcal{L}ine_{\nabla}|)$

$$H^2(X, \mathbb{Z}) \xrightarrow{=} \mathrm{Map}(X_{top}, B^2\mathbb{Z}).$$

$\mathcal{L}ine_{\nabla} \simeq B\mathbb{C}^{\times}_{\nabla} \longrightarrow \underset{\substack{\text{smooth}\\ \text{stack}}}{B\mathbb{C}^{\times}}$ $\quad |B\mathbb{C}^{\times}_{\nabla}| \simeq \underset{\substack{\simeq \\ B^2\mathbb{Z}}}{|B\mathbb{C}^{\times}|}$

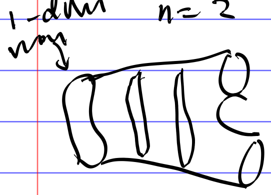
Fully extended field theories

- Bord_n^{th} is replaced by n-category (Freed, Baez-Dolan)
- n-categories have obj. mor. mor. between mor. etc.

1-cat

$\underline{\text{Bord}}_n$

$n=2$

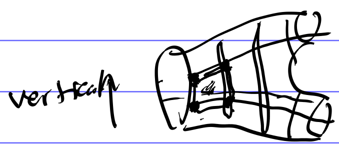


composable chain of morphisms.

n-cat

$\underline{\text{Bord}}_n$

$n=2$



objects are points

$\longrightarrow$

composition (horizontally)

Def: A fully extended field theory is a monoidal functor between n-cats

$$Z : \text{Bord}_n^{H_n} \longrightarrow \mathcal{C}$$

some higher cat.

Theorem (Schommer-Pries) Formally inverting all morphisms and objects gives a spectrum

$$|\text{Bord}_d^{H_n}| \simeq \Sigma^{ⓝ} MTH_n \quad ✓$$

Madsen-Tillmann

Theorem (Freed-Hopkins): 1-1 corresp.

$$\left\{ \begin{array}{l} \text{def classes of refl. pos. topological field} \\ \text{theories, } n\text{-dim invertible} \end{array} \right\} \longleftrightarrow \left[\overset{\text{spectrum}}{MTH}, \Sigma^{n+1} I_{\mathbb{Z}(1)} \right]_{\text{tor}}.$$

Proof: We consider $\mathbb{Z}/2$ equivariant functors

$$Z : \text{Bord}_n^{H_n} \longrightarrow \Sigma^n I_{(\mathbb{C}^\times)^\sigma} \quad \text{definition}$$

$\circlearrowleft$ reflection $\qquad$ $\circlearrowleft$ conjugation

- $I_{(\mathbb{C}^\times)^\sigma}$ is a picard ∞-groupoid

$$\Rightarrow \quad Z : |\text{Bord}_n^{H_n}| \xrightarrow{\mathbb{Z}/2\text{-equi}} \Sigma^n I_{(\mathbb{C}^\times)^\sigma}$$

- By Schommer-Pries,

$$\begin{tikzcd} Z : \Sigma^n MTH_n \arrow[r] \arrow[loop below, "\text{refl.}"] & \Sigma^n I_{(\mathbb{C}^\times)^\delta} \arrow[loop below, "\text{conj}"] \end{tikzcd}$$

$$\Rightarrow \left\{ \begin{array}{l} \text{def. classes of} \\ \text{refl. pos. } n\text{-dim} \\ \text{theories (topological)} \end{array} \right\} \leftrightarrow [\Sigma^n MTH_n, \Sigma^n I_{(\mathbb{C}^\times)^\delta}]_{\underline{\mathbb{Z}_2}}$$

Remarkably, (Freed-Hopkins)

no equiv.

$$\begin{tikzcd} & J_n(H_n)_{\text{refl.pos}} \arrow[dl] \arrow[dr, "\cong"] & \\ {[\Sigma^n MTH_n, \Sigma^n I_{(\mathbb{C}^\times)^\delta}]^{\mathbb{Z}/2}} & & {[MTH, \Sigma^n I_{(\mathbb{C}^\times)^\delta}]} \end{tikzcd}$$

(**) $$[\underset{\text{ker}}{\underline{MTH, \Sigma^n}} I_{(\mathbb{C}^\times)^\delta}] \xrightarrow[\underline{\beta}]{\cong} [MTH, \Sigma^{n+1} I_{\mathbb{Z}(1)}]_{\underline{tor}}$$

$\square$

<u>Construction</u> (G. Pavlov): There is a fully extended bordism category

geometric structures

$$\underline{\underline{\mathrm{Bord}}}_n^S \in PSh_\Delta(\mathrm{Cart} \times \Gamma \times \Delta^{\times d})$$

<u>Theorem</u> (G. Pavlov): There is (*) a commutative diagram

$$\begin{tikzcd} \text{sheafy} & \mathrm{Bord}_n^{\textcircled{S}} \arrow[r, mapsto, "\text{invert}"] \arrow[d] & \Sigma^n MT(S) \arrow[d, "\Leftarrow \text{concordify}"] & \text{sheafy} \\ \text{topological} & \underline{\underline{\mathrm{Bord}}}_n^{\textcircled{B_f(S)}} \arrow[r, "\text{invert}"] & \underline{\Sigma^n MT(B_f(S))} & \text{topological} \end{tikzcd}$$

- Horizontal maps are w.e. after inverting all morphisms and objects.

Proof of main theorem:

We consider equivariant functors

$$\underline{\mathrm{Bord}_n^{(H_n,\nabla)}} \longrightarrow \Sigma^n \underline{I}_{\mathbb{C}^\times} \leftarrow \text{sheaf!}$$

knows about $\underline{\text{topology}}$ on $\mathbb{C}^\times$!

Using $(\star)$, we have

$$\begin{tikzcd}
\mathrm{Bord}_n^{(H_n,\nabla)} \arrow[r] \arrow[d, "\text{deformations}"'] & \Sigma^n I_{\mathbb{C}^\times} \arrow[d, "\text{deformations}", "\simeq"'] \\
\mathrm{Bord}_n^{H_n} \arrow[r] & \Sigma^{n+1} \underline{I}_{\mathbb{Z}(1)}
\end{tikzcd}$$

$$\left| \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{(H_n,\nabla)}, \Sigma^n I_{\mathbb{C}^\times}) \right|^{\mathbb{Z}/2}$$

$$\simeq \mathrm{Map}(\underline{\mathrm{Bord}_n^{H_n}}, \Sigma^{n+1} I_{\mathbb{Z}(1)})^{\mathbb{Z}/2}$$

$$\simeq \mathrm{Map}(\Sigma^n MTH_n, \Sigma^{n+1} I_{\mathbb{Z}(1)})^{\mathbb{Z}/2}$$

$\Rightarrow$ result, using (Freed-Hopkins) $(\star\star)$.

$\square$

<u>Remark</u>: The positivity part of reflection positivity is encoded as a <u>structure</u> in the fully extended case.

In fact, it is shown in (Freed-Hopkins) that positivity can be encoded as a <u>trivialization</u> of a reflection structure, so one never has to talk about positivity structure directly.

Hence the only additional structure on the bordism side is reflection - positivity is taken care of in the target.