The size of Lagrangian complements

Richard Hind

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Motivation: Given a diffeomorphism Φ, a *wandering set* is an (open) subset U such that $\Phi^k(U) \cap U = \emptyset$ for all $k \neq 0$.

If Φ is a Hamiltonian diffeomorphism then wandering sets can exist only on manifolds of infinite volume.

Wandering sets are disjoint from any invariant tori.

In particular, if H defines an integrable system (with compact invariant submanifolds) then Φ_H^1 has no wandering sets.

Lazarrini, Marco and Sauzin studied perturbations of integrable systems on T^*T^n, that is, $\Phi^1_{\tilde{H}}$ where $\tilde{H}$ is a perturbation of an $H(p_1, \ldots, p_n)$.

They obtained upper bounds on the measure of bounded wandering sets in terms of a distance $\|\tilde{H} - H\|$ and also constructed examples for perturbations of $H = p_1^2 + \cdots + p_n^2$ (which by KAM theory have many invariant tori).

There are open questions about symplectic capacities and existence of wandering sets in the complements of arbitrary arrangements of Lagrangian tori. In particular, whether the invariant tori themselves impose bounds on the size of wandering sets.

Work in $\mathbb{R}^4 \equiv \mathbb{C}^2$ with the standard symplectic form $\frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2)$.

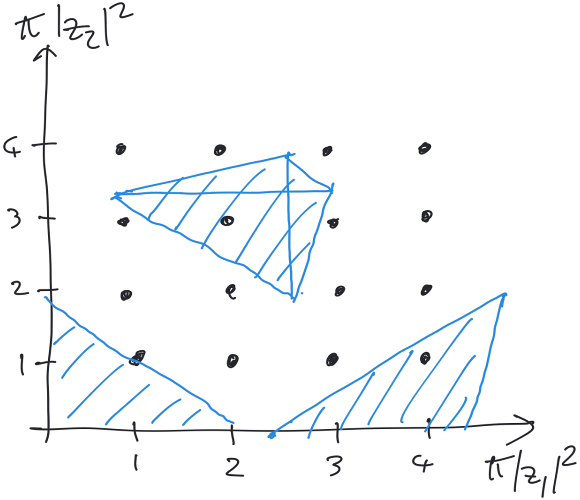
Let $L(k,l) := \{\pi|z_1|^2 = k,\ \pi|z_2|^2 = l\}$ be a product Lagrangian;

$B(a) = \{\pi(|z_1|^2 + |z_2|^2) < a\}$ is the ball of capacity a.

Recall the Gromov width $c_G(M,\omega) = \sup\{a | B(a) \hookrightarrow M\}$.

Problem: Compute $c_G(\mathbb{C}^2 \setminus \bigcup_{k,l\in\mathbb{N}} L(k,l))$. Is it greater than 2?

Images of Lagrangians and balls under the moment projection.



Let $\mathcal{L} = \bigcup_{k,l\in\mathbb{N}} L(k,l)$.

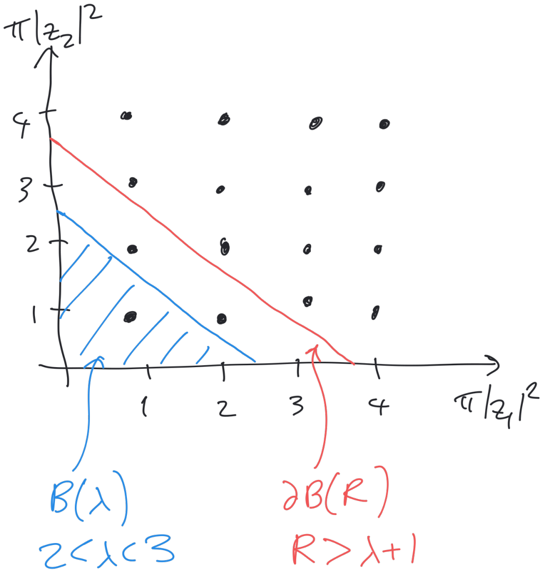
Theorem

$$c_G(B(R)\setminus\mathcal{L}) = \begin{cases} R, & R \le 2; \\ 2, & 2 \le R \le 3; \\ R-1, & 3 \le R \le 4. \end{cases}$$

Proof. For the lower bounds it suffices to produce symplectic embeddings

$$B(\lambda) \hookrightarrow B(\lambda+1+\epsilon)\setminus\mathcal{L}$$

for $2 < \lambda < 3$.

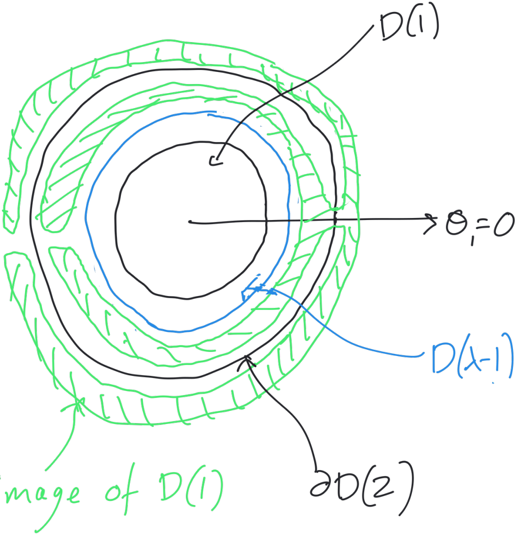

$\pi|z_2|^2$
1
2
3
4
$\pi|z_1|^2$
$B(\lambda)$
$2<\lambda<3$
$\partial B(R)$
$R>\lambda+1$

Note that the only integral Lagrangian intersecting $B(\lambda)$ for $2 < \lambda < 3$ is $L(1,1)$, and the only integral Lagrangians inside $B(\lambda + 1)$ are $L(1,1) \cup L(1,2) \cup L(2,1)$.

Therefore the goal is to displace $B(\lambda)$ from $L(1,1)$ (inside $B(\lambda + 1 + \epsilon)$), in the complement of $L(1,2) \cup L(2,1)$.

Equivalently we need to displace $L(1,1)$ from $B(\lambda)$ remaining in the complement of $L(1,2) \cup L(2,1)$.

Plane $z_2 = c$ with $\pi|c|^2 = 1$.



The ball $B(\lambda)$ intersects a plane $z_2 = c$ with $|c| = 1$ in the round disk $D(\lambda - 1)$ of area $\lambda - 1$.

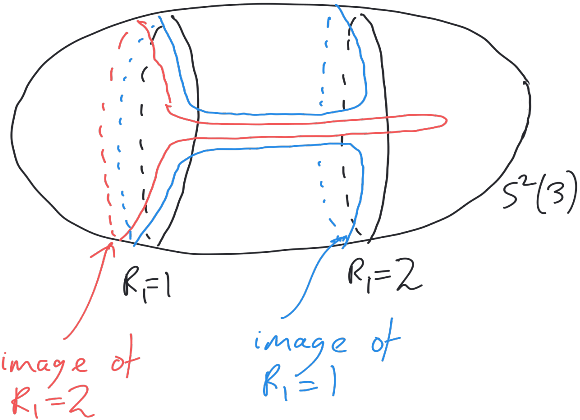
Therefore we can displace $L(1,1)$ from $B(\lambda)$ where the trace of the Lagrangian lies in a set

$$D((\lambda - 1) + 1 + \epsilon)) \times \partial D(1) = D(\lambda + \epsilon) \times \{|z_2| = 1\} \subset B(\lambda + 1 + \epsilon).$$

It suffices to displace $L(1,2)$ and $L(2,1)$ from this trace, and we will do this in a neighborhood of $S = \partial B(3)$.

Using symplectic polar coordinates $R_k = \pi |z_k|^2$, θ_k we can arrange that this trace intersects S in a neighborhood of $R_1 = 2, R_2 = 1, \theta_1 = 0$.

Symplectic reduction of $S = \partial B(3)$.



Visualize $S = \partial B(3)$ using the symplectic reduction $p : S \to \Sigma = S^2(3)$.

Fibers are the characteristic Reeb circles, and Lagrangians project to circles in Σ. The product Lagrangians $L(a, b)$ project to circles bounding disks of areas a and b.

Hamiltonian isotopies of a circle in Σ lift to exact Lagrangian isotopies in S.

Note the only intersections occur near $\{\theta_1 = 0\}$ on a particular Reeb orbit, say $\{\theta_1 - \theta_2 = 0\}$, hence near $\theta_1 = \theta_2 = 0$.

The final step is to apply a Hamiltonian $H(\theta_2)$ to (the image under our isotopy of) $L(2, 1)$.

Choose H so that $H' = -1$ near $\theta_2 = 0$ and small elsewhere.

To complete the theorem we need an embedding obstruction.

Proposition

If $\lambda > 2$ there does not exist a symplectic embedding

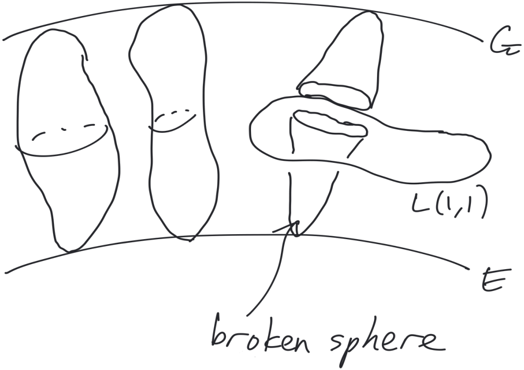
$$\overline{B(\lambda)} \hookrightarrow B(\lambda + 1) \setminus L(1, 1).$$

Proof. By a classification due to McDuff, $B(\lambda + 1) \setminus B(\lambda)$ can be compactified to the nontrivial S^2 bundle over S^2 with a standard symplectic form, namely $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

There are two natural sections, G, the symplectic reduction of $\partial B(\lambda + 1)$, and E, the symplectic reduction of $\partial B(\lambda)$.

Note G has area $\lambda + 1$ and E has area λ.

Stretching holomorphic spheres



For generic J, there is a foliation by holomorphic spheres in the class of the fiber $F = G - E$. These spheres have area 1.

The idea is to stretch the almost complex structure along $L = L(1,1)$. Certain spheres split along L into two Maslov 2 planes.

Let P be such a plane disjoint from G, and suppose its boundary lies in a homology class (m, n). Set $e = P \bullet E \in \{0, 1\}$.

We compute

$$\mu(P) = 2(m+n) - 2e = 2; \qquad \mathrm{area}(P) = m + n - \lambda e.$$

Therefore

$$\mathrm{area} = 1 + e - \lambda e = 1 - e(\lambda - 1) > 0$$

which means $e = 0$ and $\mathrm{area} = 1$. This is a contradiction as the other plane must also have positive area.

When $\lambda > 3$ the construction does not apply because $B(\lambda)$ also contains $L(1,2)$ and $L(2,1)$. We can displace these by a similar method but only in a ball of size at least 5.

Question: Is $c_G(B(R) \setminus \mathcal{L}) = 3$ for all $4 \leq R \leq 5$?

In progress: $c_G(B(R) \setminus (\mathcal{L} \cup \{z_1 z_2 = 0\})) = 3$.

Question: What is $c_G(\mathbb{R}^4 \setminus \mathcal{L})$? Is it finite? Is there a staircase?

Another measure of size is the existence of Lagrangian tori with given area class (Lagrangian capacities).

Question: Which Lagrangian submanifolds exist in a Lagrangian complement?

Eventually would like a theory about Lagrangian intersections of disjoint unions (of possibly non Hamiltonian isotopic tori), but can anyway again see rigidity in a bounded region.

Related work here of Polterovich-Shelukhin and Mak-Smith.

Definition. Say a Lagrangian is *integral* if the area class takes integer values.

Define a cube $P(c, c) = \{\pi|z_1|^2 < c, \pi|z_2|^2 < c\}$.

Theorem (H., Kerman)

Let $c \leq 2$. Then any integral $L \subset P(c, c)$ intersects $L(1, 1)$.

Let $c > 2$. Then there exists
$\overline{P(1, 1)} \hookrightarrow P(c, c) \setminus \bigcup_{1 \leq k, l \leq \lfloor c \rfloor} L(k, l)$.

Talk about rigidity result. Let $L_C = L(1,1)$. Suppose $L \subset S^2(2) \times S^2(2)$ is monotone and $L \cap L_C = \emptyset$.

(Work with Opshtein implies that an integral Lagrangian must be monotone.
In particular, if $k \geq 2$ there are no Hamiltonian diffeomorphisms $L(1,k) \hookrightarrow P(2,2)$.)

Denote $S_0 = S^2 \times \{0\}$; $S_\infty = S^2 \times \{\infty\}$; $T_0 = \{0\} \times S^2$; $T_\infty = \{\infty\} \times S^2$.

Proposition

Up to Hamiltonian diffeomorphism, we may assume L is disjoint from S_0, S_∞, T_0, T_∞.

Note that $S^2 \times S^2 \setminus \{S_0 \cup S_\infty \cup T_0 \cup T_\infty\}$ can be identified with a subset of T^*T^2, say with zero section $L(1,1)$.

Theorem (Arnold)

Homologically nontrivial Lagrangian tori in T^*T^2 are homologous to $L(1,1)$.

Theorem (Dimitroglou-Rizell, Goodman, Ivrii)

Any exact $L \subset T^*T^2$ is Hamiltonian isotopic to the zero section.

Corollary

Exact $L \subset T^*T^2$ intersect the zero section.
Homologically nontrivial Lagrangian tori which are monotone in $S^2 \times S^2$ intersect the zero section. (See also Cieliebak, Schwingenheuer.)

But not all monotone Lagrangian tori are Hamiltonian isotopic to L_C (Chekanov) and so not all tori can be displaced from the axes to become homologically nontrivial.

The plan is to remove a different configuration of curves.

Example. Suppose $G, H \subset S^2 \times S^2$ holomorphic spheres in the class $(1, d)$.

They intersect in $2d$ points; let $F_1, \ldots, F_{2d}$ be spheres in the class $(0, 1)$ through these points.

Blow-up the points and let $\hat{G}, \hat{H}, \hat{F}_i$ be the proper transforms.

Inflate $\hat{G}$ and $\hat{H}$ with capacity d.

Blow down $\hat{F}_1, \ldots, \hat{F}_{2d}$.

The result is symplectomorphic to a monotone $S^2(2d+2) \times S^2(2d+2)$ with $\hat{G}, \hat{H}$ mapping to $(1, 0)$ curves.

Aim to do this keeping L, L_C monotone, such that they become homologically nontrivial in the complement of $\hat{G} \cup \hat{H} \cup T_0 \cup T_\infty$.

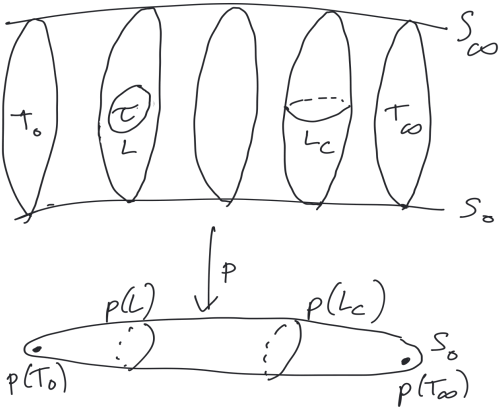
Finding ‘linking’ spheres seems easier in high degree.

Lagrangian tori and foliation by holomorphic spheres.

Foliation by holomorphic curves in the class (0,1) defines a projection p from $S^2 \times S^2$ onto S_0.

Stretching the neck the foliation in the class (0,1) becomes compatible with L and L_c. Then the tori project to circles.

Want sections intersecting the family *T*.

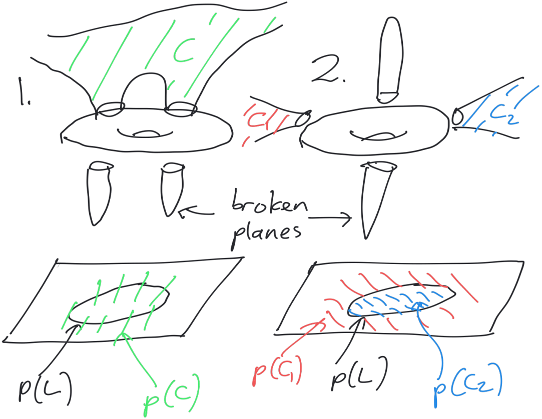


Holomorphic building limits of (1,d) curves.

Foliation defines a map p from $S^2 \times S^2$ to S_0.

Top level curves map to $S^2 \times S^2 \setminus (L, L_c)$ and p is injective on the union.

This implies two possible pictures for degenerations along a Lagrangian.



How to find curves of Type 2 with respect to L and L_C?

Fix $d+1$ points on L_C and d points on L. There exists a unique J-sphere in class $(1, d)$ through the points. Then stretch the neck along $L \cup L_C$.

1. Building is Type 2 with respect to L_C. Will see d planes covering broken leaves along L_C.

2. By contradiction, if building is Type 1 for L then see d planes covering disks in τ. (Otherwise the planes result in $2d$ zeros and poles.)

3. Remaining curves have area $(2d+2) - d - d = 2$, but have d total zeros and poles. Can exclude this by monotonicity.

Thank-you!