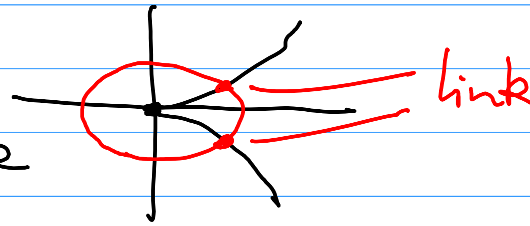
Symplectic Cohomology of Compound Du Val Singularities

Joint with Yanki Lekili

$$x^2 + y^3 + z^5 + w^7 = 0 \qquad \mathbb{C}$$

hypersurface singularity

"Link"



$y^2 = x^3$ in $\mathbb{C}^2$

Link is a trefoil knot in S^3

$x_1^2 + \cdots + x_4^2 + x_5^3 = 0$ exotic 9-sphere

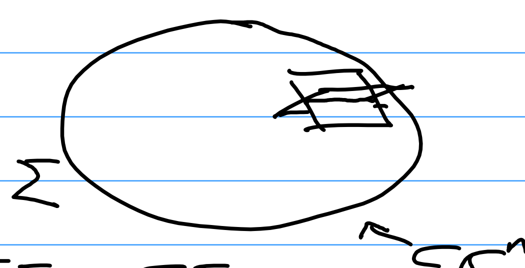
Brieskorn

$$x^2 + y^2 + z^2 + w^N = 0$$

$$\text{link} = \begin{cases} S^2 \times S^3 & N \text{ even} \\ S^5 & N \text{ odd} \end{cases}$$

Thm (Varchenko) The field of $\mathbb{C}$-tangencies of the link is a contact structure on the link.

$\nwarrow$ distribution of hyperplanes in tangent spaces to link.



$T\Sigma \cap iT\Sigma$

$\subseteq T\Sigma$

Thm (McLean): If $p \in X$ ($\mathbb{C}$ 3-fold) then link of p is contactomorphic to (S^5, ξ_{std}) iff p is a smooth point.

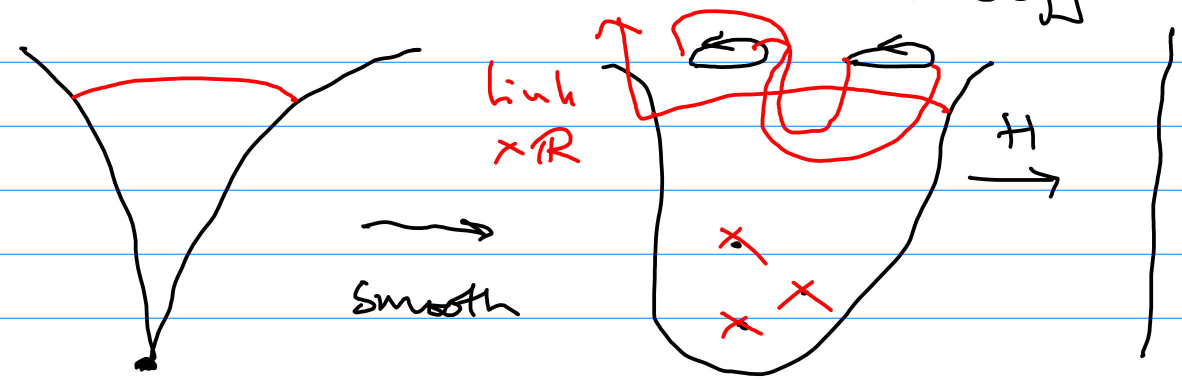
Thm (Ustilovsky) Family of contact str.s on S^5 from before are all different.

Contact homology

Thm (Van Koert) $x^2 + y^2 + z^2 + w^{2n} = 0$ (contact str. on $S^2 \times S^3$) have same contact homology.

Thm (Uebele) ... but they are pairwise distinct.

Using SH^+ positive symplectic cohomology



Milnor fibre

Chain complex generators = closed orbits of H γ

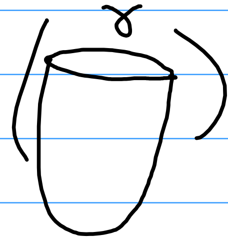
differential $\partial \gamma = \sum_{\delta \text{ orbits}} \# \square \; \delta$

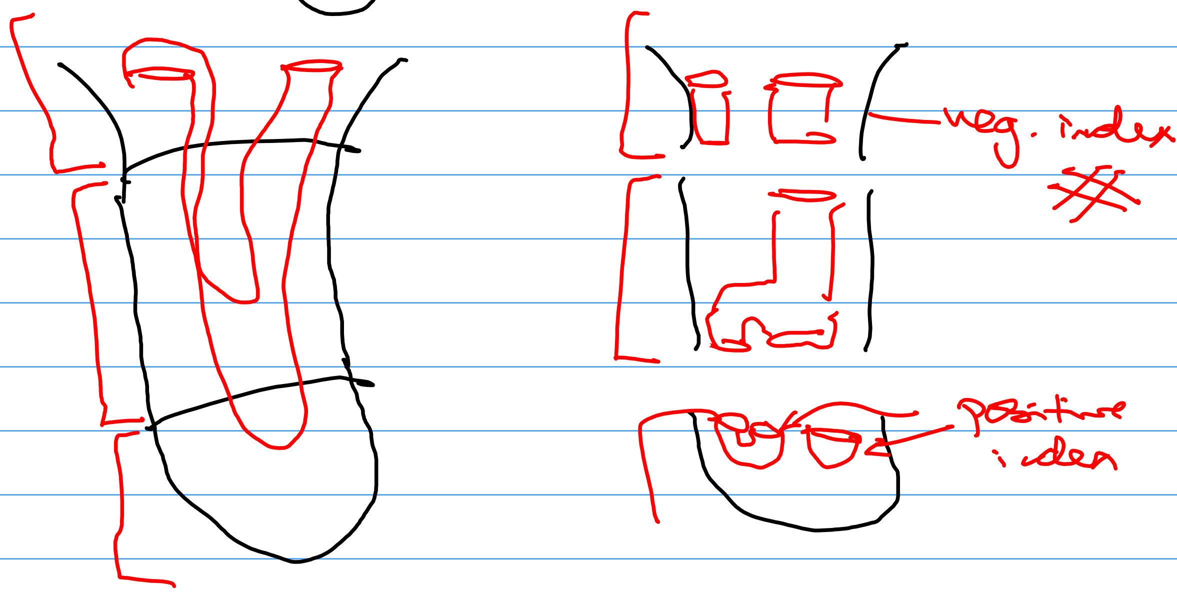
$\xrightarrow{\text{cohomology}}$ $SH(\text{Milnor fibre})$

Quotient by subcomplex of low action orbits

$SH^+(\text{Milnor fibre})$

Index positivity: Each orbit γ has Conley-Zehnder index $\mu_{CZ}(\gamma)$.

dim () $= \boxed{\mu_{CZ}(\gamma) + \underbrace{n}_{\dim_{\mathbb{C}}} - 3} > 0 \quad \forall \gamma$



Thm: (McLean) Link is index positive $\Leftrightarrow$ Singularity is terminal.

Thm: (Reid) A 3-fold hypersurf. sing. is terminal iff it's

$$\left.\begin{array}{ll} A_\ell & x^2+y^2+z^{\ell+1} \\ D_\ell & x^2+y(z^2+y^{\ell-2}) \\ E_6 & x^2+y^3+z^4 \\ E_7 & x^2+y(y^2+z^3) \\ E_8 & x^2+y^3+z^5 \end{array}\right\} + w\, g(x,y,z,w)$$

Du Val

Compound Du Val (ADE)

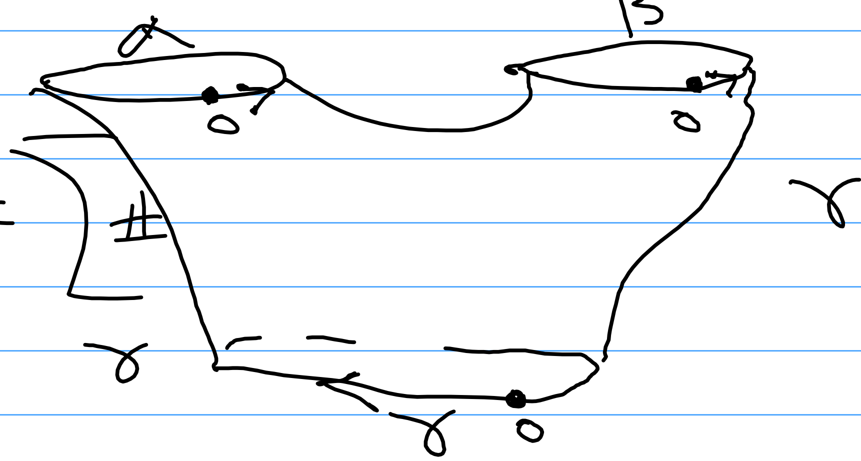
Th: We compute SH^+ for some cDV singularities. (over $\mathbb{Q}$)
Some patterns emerge.

		R	Link
cA_ℓ	$x^2+y^2+z^{\ell+1}+w^{k(\ell+1)}$	ℓ	$\#_\ell(S^2\times S^3)$ $\ell=1$
cA_ℓ	$x^2+y^2+zw(z^{\ell-1}+w^{k(\ell-1)})$	ℓ	$\#_\ell(S^2\times S^3)$
cD_4	$x^2+y^3+z^3+w^{6k}$	4	$\#_4(S^2\times S^3)$
cD_4	[$x^3+yz^{2k+1}+yz^2+w^2$ "Laufer"	1	$S^2\times S^3$ *]
cE_6	$x^2+y^3+z^4+w^{12k}$	6	$\#_6(S^2\times S^3)$
cE_8	$x^2+y^3+z^5+w^{30k}$	8	$\#_8(S^2\times S^3)$

For all these rank $SH^d = \begin{cases} 0 & \text{if } d>3 \text{ or } d=2 \\ \text{Milnor number} & d=3 \\ R & d=1,\ 0 \text{ or } <0 \end{cases}$

Conj. If X is 3-fold cDV then X admits a small res. by ℓ curves iff SH^+ has rank ℓ in every negative degree.

__Thm:__ Can distinguish these using Gerstenhaber bracket.

$[\alpha, \beta] = \pm$ 

$$SH^i \times SH^j \longrightarrow SH^{i+j-1}$$

$$SH^1 \times SH^1 \longrightarrow SH^1$$ Lie subalgebra

$$SH^+ \circlearrowleft SH^1$$

contact invariant

$$\boxed{\mu_{\mathbb{Z}} \geq \max(5-n, n-1) = 2}$$

$$x^2 + y^2 + z^2 + w^2 \qquad \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}$$

$$xyzw^2 \qquad 1\ 1\ 1\ 2$$

Ballard – Favero – Katzarkov HH

Futaki-Ueda

Hakemann-Smith

Polischuk-Varolgunes

Ganatra

SH = HH