Resolving the rank two Hitchin system by comp. Jacobians of semistable curves

(jt. with Martin Möller)

1. Hitchin systems

G complex real. group

X Riem. surf. $g > 1$

$\Rightarrow \mathcal{M}_G(X)$ moduli space of semi-stable G-Higgs bundles on X

Ex: $G = GL(n, \mathbb{C})$: (E, Φ) E rank n holom. vb, $\Phi : E \to E \otimes K$

$G = SL(n, \mathbb{C})$: $+ \quad \Lambda^n E \cong \mathcal{O}_X$, $\operatorname{tr} \Phi = 0$

- holom. symplectic str., hyperkähler structure
- NAH: Diffeo. to moduli of reps $\pi_1(X) \to G$

Hitchin map: $\quad \mathrm{Hit}: \mathcal{M}_G(X) \longrightarrow \mathcal{B}_G(X)$ Hitchin base

$\mathbb{C}$-vsp: $\quad \dim \mathcal{B}_G(X) = \frac{1}{2} \dim \mathcal{M}_G(X)$

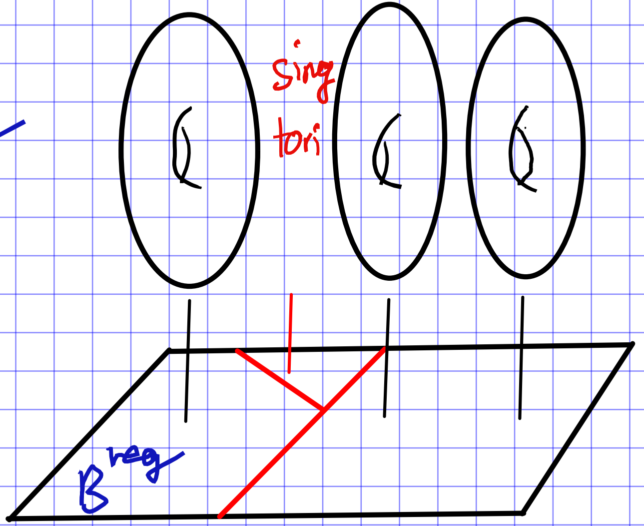
$G = SL(2,\mathbb{C})$: $\quad \mathcal{B}_G(X) = H^0(X, K^{\otimes 2})$ quadratic diff. on X

<u>Thm</u> (Hitchin, Scognamillo)

$\exists \; \underbrace{\mathcal{B}_G^{reg}(X)}_{\text{reg. locus}} \underset{\text{dense}}{\overset{\text{open}}{\subset}} \mathcal{B}_G(X)$:

$\mathrm{Hit}|_{\mathcal{B}_G^{reg}(X)}$ algebraically completely integrable system

abelian variety
||
compl. proj. tori



$$\begin{array}{c} \mathcal{M}_G(X) \\ \Big\downarrow \mathrm{Hit} \\ \mathcal{B}_G(X) \end{array}$$

Asymptotic of Hyperkähler structure

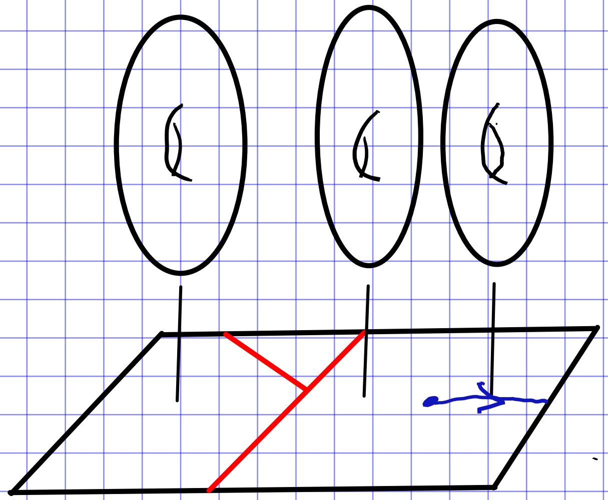
Freed 98: $\exists$ Hyperkähler metric on $\mathrm{Hit}^{-1}(B_G^{reg})$

"semi-flat hyperkähler metric h_{sf}"

Gaiotto-Moore-Neitzke 13,

Mazzeo-Swoboda-Weiss-Witt 18, Fredrickson 19

Hyperkähler metric of $\mathcal{M}_{SL(n,\mathbb{C})}$ is asymptotic to h_{sf} on $B^{reg}_{SL(n,\mathbb{C})}$



Open on singular locus.

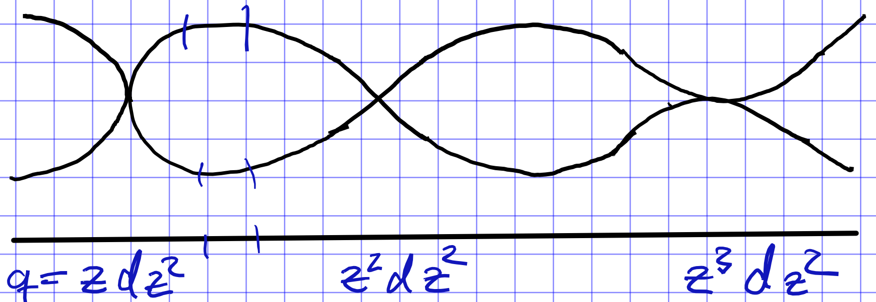
Hitchin fibers

$G = SL(2,\mathbb{C})$, $B_G(X) = H^0(X, K^2)$

Spectral curve:

$q \in H^0(X, K^2)$

$$\begin{array}{l} \Sigma_q = \{\lambda^2 - q = 0\} \subset \mathrm{Tot}(K) \\ \quad \downarrow 2:1 \\ \quad X \end{array}$$



$$B^{reg}_{SL(2,\mathbb{C})}(X) = \{ q \in H^0(X, K^2) \mid \Sigma_q \text{ smooth} \} = \{ q \mid q \text{ has simple zeroes} \}$$

$$q \in B^{reg}_{SL(2,\mathbb{C})} \subset B^{reg}_{GL(2,\mathbb{C})}$$

$$\mathrm{Hit}^{-1}_{SL(2,\mathbb{C})}(q) \cong \mathrm{Prym}(\Sigma_q \to X)$$

$$\mathrm{Hit}^{-1}_{GL(2,\mathbb{C})}(q) \cong \mathrm{Jac}(\Sigma_q)$$

Remark:

$\mathbb{C}^\times \curvearrowright H^0(X, K^2)$

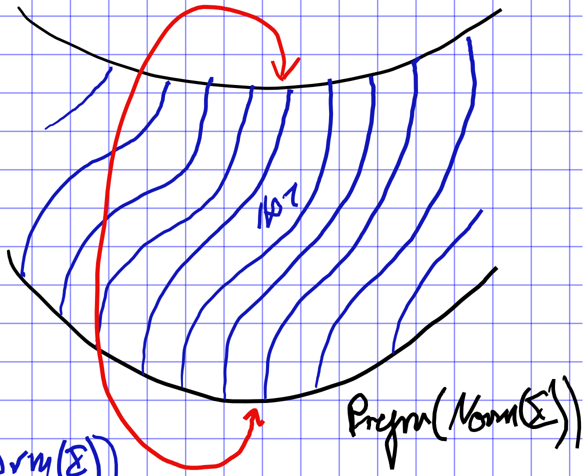
$q \mapsto t q$

abstract curve is inv.

Singular Hitchin fibers

$q \notin B^{reg}_{SL(2,\mathbb{C})} \Leftrightarrow \Sigma_q$ plane curve sing.

Structure of $Hit^{-1}(q)$ depends on sing. of Σ



$SL(2,\mathbb{C})$: (Gothen-Oliveira 13, H. 20)

Node: twisted $\mathbb{P}^1$-bundle over $\mathrm{Prym}(\underbrace{\mathrm{Norm}}(\Sigma))$

cusp: $\mathbb{P}^1$-bundle " "

A_3, $\lambda^2 - z^4$: twisted $\mathbb{P}(1,1,2)$-bundle over $\mathrm{Prym}(\mathrm{Norm}(\Sigma))$

A_4, $\lambda^2 - z^5$: $\mathbb{P}(1,1,2)$-bundle " "

$GL(n,\mathbb{C})$: all $\mathbb{P}^n$, $\mathbb{P}(1,\dots,1,2,\dots,2)$ appear.

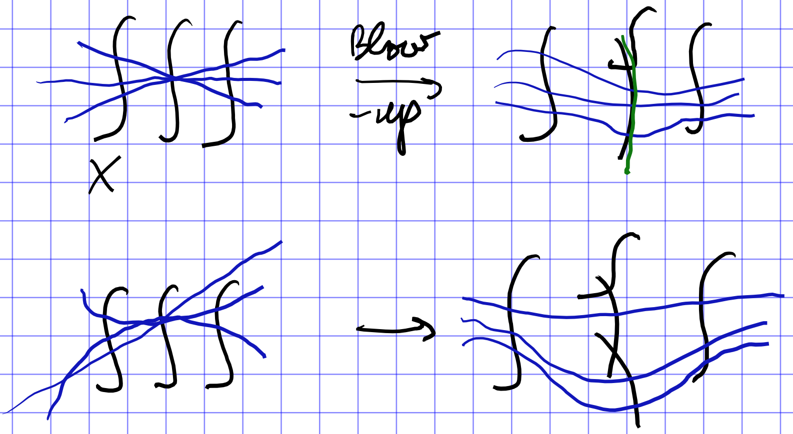
$$\begin{tikzcd} & \mathrm{Norm}(\Sigma) \arrow[dl] \arrow[d] \\ \Sigma \arrow[r] & X \end{tikzcd}$$

Q: Can we make the singular fibers simpler by modifying the Hitchin base s.th. the spectral correspondence extends?

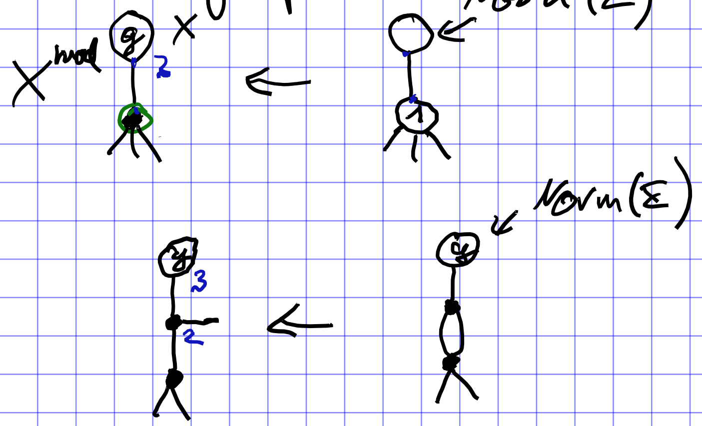
Multi-scale quadratic differentials with simple zeroes

$\mathbb{P}\, B^{reg}_{SL(2,\mathbb{C})} \setminus \{0\}$

"Don't zeroes to collide"



dual graphs



$\overline{\mathcal{M}}_{g,4g-4}$

$B_{SL(2,\mathbb{C})} \to \mathcal{M}_g$ 2nd Hodge bundle, $B^{reg}_{SL(2,\mathbb{C})} \subset B_{SL(2,\mathbb{C})}$ quadr. diff. with simple zeros.

<u>Thm</u> (Bainbridge - Chen - Gendron - Grushevsky - Möller 19, Constantini - Möller - Zachhuber 20)

$\exists$ $\mathbb{C}$-orbifold: $\overline{Q_g}$ moduli space of quadratic multi-scale diff. with simple zeroes, s.th.

i) $B^{reg}_{SL(2,\mathbb{C})} \subset \overline{Q_g}$ dense

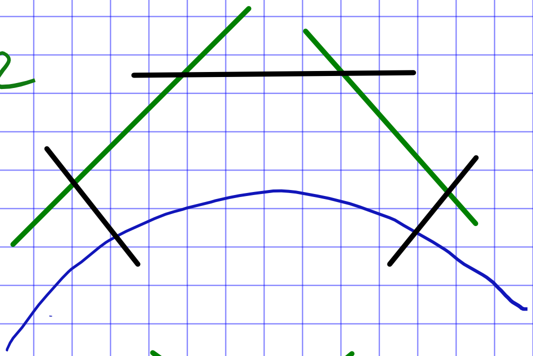
ii) $\overline{Q_g} \setminus B^{reg}_{SL(2,\mathbb{C})}$ normal crossing divisor

iii) $\mathbb{C}^* \curvearrowright B^{reg}_{SL(2,\mathbb{C})}$ extends to $\overline{Q_g}$

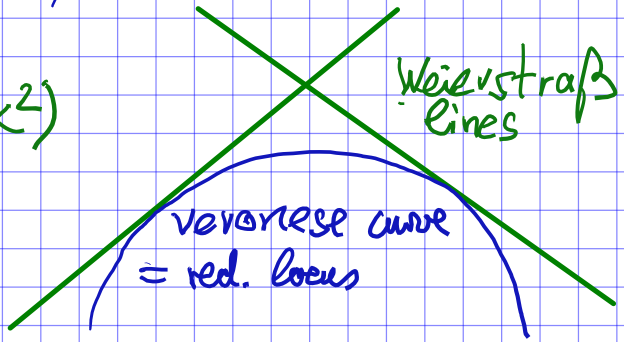
$\mathbb{P} B^{reg}_{SL(2,\mathbb{C})} \subset \mathbb{P}\overline{Q}_g$ comp.

<u>Ex:</u> $g=2$

$\mathbb{P}\overline{Q_g}(X)$



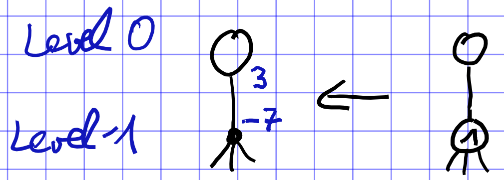
$\mathbb{P}H^0(X,K^2) = \mathbb{P}^2$

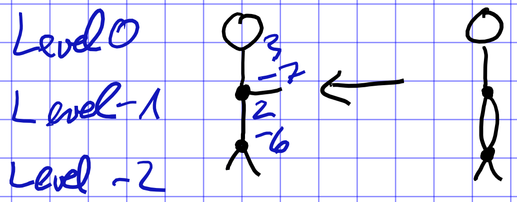


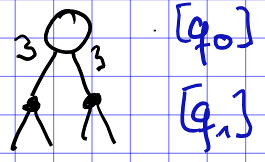
Multi-scale quadratic differentials

$(X, z) \in \overline{\mathcal{M}}_{g,4g-4}$

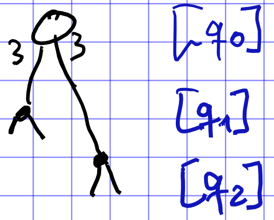
+ level str on dual graphs of X
+ levelwise quadr. diff. sat. a residue cond.







boundary divisor



higher codim.

+ prong matchings

mod scaling diff on lower levels

Back to Hitchin systems

universal spectral curve

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ B^{reg}_{SL(2,\mathbb{C})}(X) \arrow[r] & \overline{Q}_g(X) \end{tikzcd}$$

smooth curves

family of nodal curves

universal compactified Jacobian

$$\begin{tikzcd} \mathrm{Jac}(\Sigma) \arrow[r] \arrow[d] & \overline{\mathrm{Jac}}^{\varphi}(\mathcal{S}) \arrow[d] \\ B^{reg}_{SL(2,\mathbb{C})}(X) \arrow[r] & \overline{Q}_g(X) \end{tikzcd}$$

comp. Jacobians of $\mathcal{S}$ resp to a polarisation φ

(Caporaso, Esteves, Melo, ...)

Thm (H. – Möller)

$GL(2,\mathbb{C})$ – spectral correspondence extends:

$SL(2,\mathbb{C})$

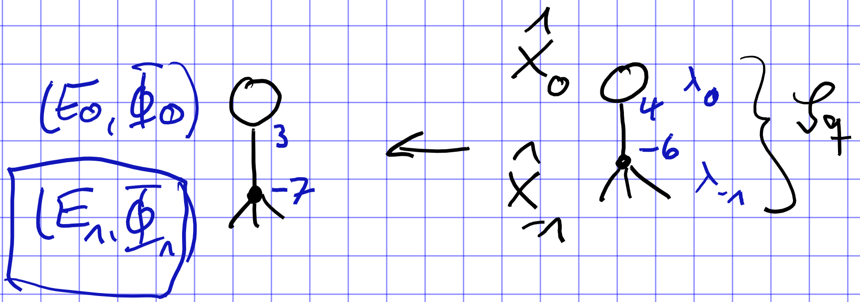
$q \in \overline{Q}_g(X)$

$$\left\{ \begin{array}{l} \mathcal{F}\ \varphi\text{-semistable} \\ \text{tors-free sheaf} \\ \text{on } \mathcal{S}_q \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \check{\varphi}\text{-semistable Higgs pairs} \\ (E,\Phi) \text{ on } X \text{ augmented} \\ \text{by tree of rational curves} \\ \text{with } \det(\Phi) = q \end{array} \right\}$$

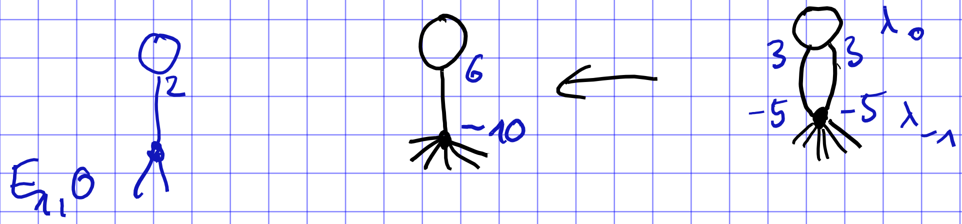
Prym cond.

det condition

Examples:



$$\overline{\mathrm{Jac}(S_q)} \cong \mathrm{Jac}(\hat{X}_0) \times \mathrm{Jac}(\hat{X}_{-1})$$



2 irred. comp., each isom. to twisted $\mathbb{P}^1$-bundle over $\mathrm{Jac}(X_0) \times \mathrm{Jac}(X_{-1})$

Achievment: + Extends the Hitchin system by comp. Jac. of nodal curves
+ Spectral correspondence extends
+ Works universally on $\overline{\mathcal{M}}_g$