The Fock Pseudomonad: Groupoidifying Second Quantization

(Based on work with Jamie Vicary)

Jeffrey C. Morton

SUNY Buffalo State College

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Outline

- The Heisenberg Algebra and the Fock Monad
- Groupoidification and the Span Construction
- The Fock Pseudomonad
- The Categorified Heisenberg Algebra
- 2-Hilbert Spaces and Representations of $Span_2(Gpd)$

Fock Representation of Heisenberg Algebra

The one-variable **Heisenberg algebra** is an algebra H given by two generators a ("annihilation") and $a^\dagger$ "creation"), satisfying the *canonical commutation relation*:

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

The general Heisenberg algebra has generators a_i and $a_i^\dagger$ for each $i = 1, \ldots, n, \ldots$.
There is only one nontrivial, irreducible representation (which is faithful) of the algbera, on **Fock space**, $H \mapsto Aut(\mathcal{F})$, where:

$$\mathcal{F} = \mathbb{C}[\![z]\!]$$

(The space of (formal) power series in z).

In this representation, the algebra is generated by:

$$\mathsf{a} f(z) = \partial_z f(z)$$

and

$$\mathsf{a}^\dagger f(z) = z f(z)$$

The commutation relation holds for a and $\mathsf{a}^\dagger$, since:

$$\partial_z (z f(z)) = z \partial_z f(z) + f(z)$$

If we define an inner product on $\mathcal{F}$ where $\{z^n\}$ is an orthogonal basis such that

$$\langle z^n, z^n \rangle = \frac{1}{n!}$$

then $\mathsf{a}^\dagger$ is the (linear) adjoint of a.

Fock Construction

Jamie Vicary has analyzed this representation construction in the general context $(\mathsf{C}, \dagger, \otimes, \oplus)$ of a category C with a †-structure (adjoint/dual arrows), *tensors* (a symmetric monoidal product), and *direct sums* (biproducts). (The usual setting is Hilb.)
It turns out the “Fock Space” construction relies on the existence of an adjunction:

$$\mathsf{C} \underset{R}{\overset{Q}{\rightleftarrows}} \mathsf{CMon}(\mathsf{C}) \qquad (Q \dashv R)$$

Here, CMon(C) is the category of commutative monoid objects in C, and R is the associated forgetful functor. The left-adjoint Q the “free commutative monoid object” functor.

Adjunctions induce *monads* on their underlying categories, so we have

$$F = RQ : \mathsf{C} \to \mathsf{C}$$

It acts on objects by:

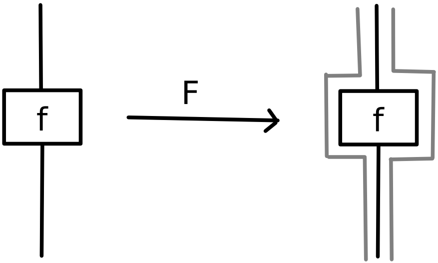
$$F_{\mathsf{C}}(X) = \bigoplus_{n} X^{\otimes_s n}$$

where $X^{\otimes_s n}$ is the symmetric n-fold tensor product. This is built from $X \otimes_s X$, the *equalizer* L in the diagram:

$$L \xrightarrow{\ s\ } X \otimes X \underset{id_{X \otimes X}}{\overset{\tau_{X,X}}{\rightrightarrows}} X \otimes X$$

When $\mathsf{C} = \mathrm{Hilb}$, this is just the symmetric part of the tensor product.

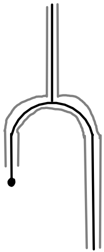
A graphical calculus for 1-cells in C can be extended to represent the Fock construction: an image under F is drawn as 'contained' within a pair of grey lines. The following image represents the 1-cell $F(f) : F(X) \to F(Y)$:



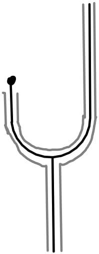
The creation and annihilation operators a and $a^\dagger$ are constructed using two facts:

- F is a monad, and has a unit $\eta : Id_C \Rightarrow F$ and counit $\epsilon : F \Rightarrow Id_C$ (which represent the "inclusion of the 1-particle state" into Fock space and projection onto it)
- F is symmetric monoidal relating biproduct and tensor product, so that

$$F(X \oplus Y) \cong F(X) \otimes F(Y)$$



a



$a^\dagger$

Categorification

We want a **categorification** of this construction. This means:

- an analogous structure replacing set-based structures with category-based structures
- not systematic: any inverse to some *decategorification* process, such as:
  - Degroupoidification (Baez-Dolan): a functor $D : \mathrm{Span}_1(\mathrm{Gpd}) \to \mathrm{Hilb}$
  - Khovanov-Lauda: $C \mapsto K_0(C)$, the Grothendieck ring (used for algebraic categorification of quantum groups)

In fact, we can find both - and show how they're related.

Groupoidification

Definition

There is a 2-category Gpd, with:

- **Objects**: Groupoids (as categories, possibly internal to spaces, with all morphisms invertible)
- **Morphisms**: Functors (internal)
- **2-Morphisms**: Natural Transformations (internal)

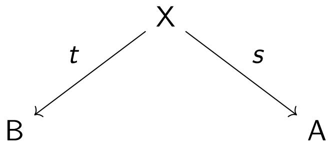
Facts: Gpd has a number of properties which make the construction which follows possible:

- Gpd has products, coproducts, and a terminal object
- Gpd has (homotopy) pullbacks

Definition

The monoidal category $\mathrm{Span}_1(\mathrm{Gpd})$ has:

- **Objects**: ("tame") groupoids.
- **Morphisms** are isomorphism classes of ("tame") spans of groupoids, so that $A \xrightarrow{F} B$ is a span of the form:

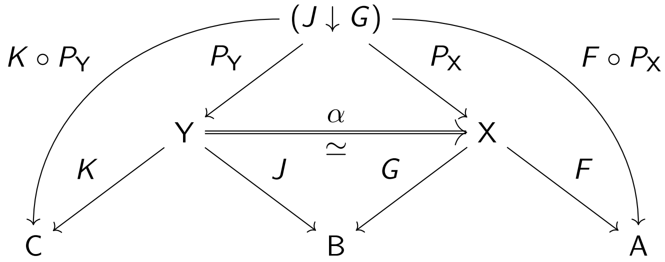


  up to isomorphism of spans
- **Monoidal (tensor) Product**: Disjoint union of groupoids (and spans).

(Note: the "Span Construction" taking a category C to *Span*(C) amounts to *freely adjoining adjoints* for all morphisms.)

Definition - cont'd

Composition of spans is given by a *pseudo-pullback* (a.k.a. *homotopy pullback*) groupoid $(J \downarrow G)$:



Based on the universal property described above, a standard construction for $(J \downarrow G)$ is the following groupoid:

- **Objects** are triples $\big(x \in \mathrm{Ob}(\mathsf{X}), y \in \mathrm{Ob}(\mathsf{Y}), G(x) \xrightarrow{f} J(y)\big)$.
- **Morphisms** $(x_1, y_1, f_1) \to (x_2, y_2, f_2)$ are pairs of morphisms $x_1 \xrightarrow{a} x_2$ and $y_1 \xrightarrow{b} y_2$ satisfying the following commuting diagram:

$$\begin{array}{ccc} G(x_1) & \xrightarrow{f_1} & J(y_1) \\ {\scriptstyle G(a)}\downarrow & & \downarrow{\scriptstyle J(b)} \\ G(x_2) & \xrightarrow[f_2]{} & J(y_2) \end{array}$$

Representation in Hilb

The category $\mathrm{Span}_1(\mathrm{Gpd})$ has a representation into Hilbert spaces (alternatively: inner product spaces) which is the basis of Groupoidification:

Definition

The degroupoidification functor $D : \mathrm{Span}_1(\mathrm{Gpd})) \to \mathrm{Hilb}$ takes:

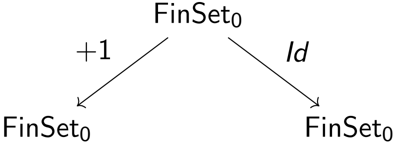
- **Objects**: $D(\mathrm{A})$ goes to $\mathbb{C}^{\underline{\mathrm{A}}}$, the space of invariant (and, if relevant, L^2) functions on objects of A, with the inner product where
$$\langle \delta_a, \delta_b \rangle = \delta_{a,b} \#(Aut(a))$$
- **Morphisms**: The morphism $\mathrm{A} \xrightarrow{F} \mathrm{B}$ above goes to $t_\star \circ s^\star$, the pullback along s followed by the adjoint of the pullback aloing t

This functor is †-monoidal: it preserves the (free!) adjoints and the monoidal product.

Groupoidification of the Fock Representation

Baez and Dolan introduced a groupoidification of the Fock space representation of H. The correspondence is:

- **Representation Space**: $\mathbb{C}[\![z]\!]$ is represented by the groupoid FinSet_0 of finite setse and bijections
- **(State) Vectors**: A vector in $\mathbb{C}[\![z]\!]$ corresponds to a "stuff type", i.e. a span $1 \leftarrow \mathsf{G} \rightarrow \mathsf{FinSet}_0$
- **Generators**: The operators a and $a^\dagger$ are dual spans:



The functor $+1$ is the "disjoint union with a one-element set" functor.

The Fock monad F can be defined in any †-monoidal category with †-biproducts, including both Hilb and $\mathrm{Span}_1(\mathrm{Gpd})$. Denoting these by F_S and F_H respectively, we have:

Theorem

The following diagram commutes up to natural equivalence:

$$\begin{array}{ccc} \mathrm{Span}_1(\mathrm{Gpd}) & \xrightarrow{D} & \mathrm{Hilb} \\ \Big\downarrow F_S & & \Big\downarrow F_H \\ \mathrm{Span}_1(\mathrm{Gpd}) & \xrightarrow[D]{} & \mathrm{Hilb} \end{array}$$

Indeed: the Fock representation of the Heisenberg algebra in Hilb is the *degroupoidification* of the same construction in $\mathrm{Span}_1(\mathrm{Gpd})$.

The spans A and $A^\dagger$ have interpretations as processes performed on on "sets with extra structure" (formally, *combinatorial species*):

- A : "Remove an element from set S" (before defining the species)
- $A^\dagger$: "Add a new element to set S"

The commutation relation becomes:

$$A \circ A^\dagger \simeq (A^\dagger \circ A) \oplus \mathrm{id}_{\mathsf{FinSet}_0},$$

(Where $\oplus$, the biproduct in $\mathsf{Span}_1(\mathsf{Gpd})$, is just disjoint union.)

This has an interpretation in terms of *combinatorial histories*: there is one more way to *add an element* to a set S, and then *remove an element* than there is to first *remove an element* from S and then *add an element*. This extra way is equivalent to the identity.

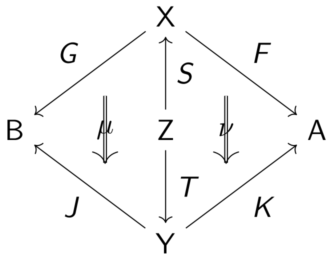
Lifting to a 2-category

Since Gpd is a 2-category, the span construction can be extended, to give 2-morphisms between spans. We can use this extension to find a 2-categorical analog of constructions in $\mathrm{Span}_1(\mathrm{Gpd})$.

Definition

The 2-category $\mathrm{Span}_2(\mathrm{Gpd})$ has:

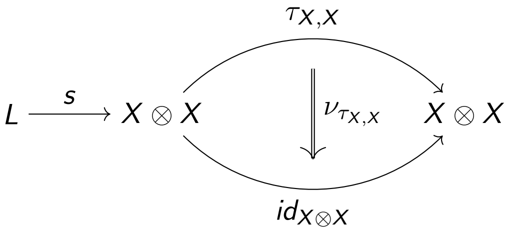
- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids
- **2-Morphisms**: Isomorphism classes of spans of spans:



- For Cartesian C, *Span*(C) is the *universal* 2-category containing C, for which every morphism in C has a (two-sided) adjoint.
- In fact, *Span*(C) is a †-monoidal, †-abelian 2-category.
- There is a **Fock PSEUDOMONAD** for any †-monoidal †-abelian 2-category
- It is associated to a pseudoadjunction and gives the "free symmetric pseudomonoid object":

$$F_C(X) = \bigoplus_n X^{\otimes_s n}$$

This uses the pseudoequalizer, the universal triple $(L, s, \nu_{\tau_{X,X}})$:



Theorem

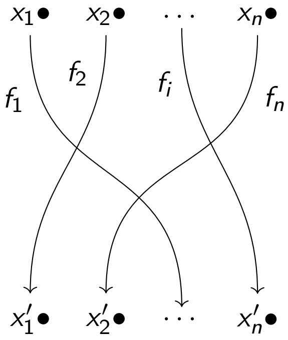
The pseudoequalizer in $\mathsf{Span}_2(\mathsf{Gpd})$ *consists of* $(L, s, \nu_{\tau_{X,X}})$*, where:*

- $L = S_2 \ltimes X^2$*, the semidirect product where* S_2 *acts on* X^2 *by permutations, so that the morphism* $(-1, Id)$ *takes* (x_1, x_2) *to* (x_2, x_1)
- s *is the span* $S_2 \ltimes X^2 \overset{i \circ \Delta}{\leftarrow} X \overset{\Delta}{\rightarrow} X^2$*, where* i *is the inclusion map* $x \mapsto (1, x)$
- $\nu_{\tau_{X,X}}$ *is the identity 2-cell (up to canonical choice of composite* $\tau_{X,X} \circ s$*)*

Corollary

The symmetric monoidal product $X^{\otimes_s n}$ is the groupoid $S_n \ltimes X^n$.

this is the "wreath product", whose objects are n-tuples of X-objects, and whose morphisms are permutations whose strands are labelled by X-morphisms:



Theorem

The free pseudomonoid object

$$F(G) = \bigoplus_n S_n \ltimes G^n$$

on an object $G \in \mathrm{Span}(\mathrm{Gpd})$ *is the free symmetric monoidal category generated by* G*.*

Some special cases are:

$$F(1) \simeq FinSet_0$$

and

$$F(n) = nColFinSet_0$$

In each case, we get "creation/annihilation" pairs A_x and $A_x^\dagger$ for each object x.

Khovanov's Categorification

Khovanov described a categorification of the Heisenberg algebra

Definition

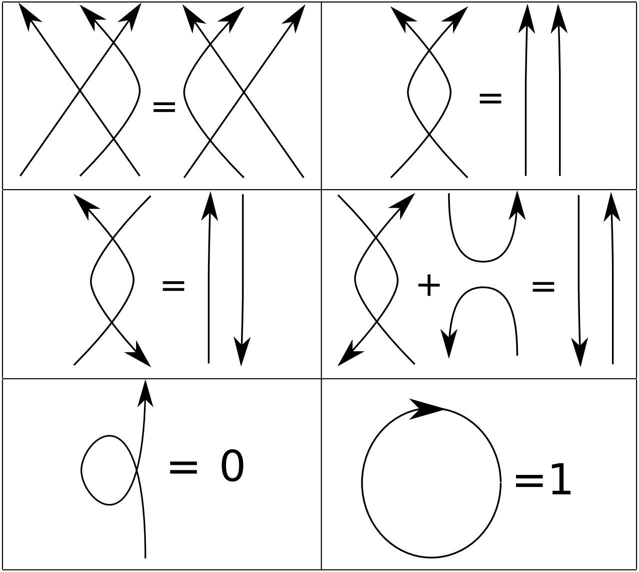
The monoidal category H has:

- Objects: generated by points labelled Q_+ ("up") and Q_- ("down")
- Morphisms: linear combinations of (string diagrams, agreeing with orientations at endpoints, taken up to isotopy and certain local moves):

The monoidal category H′ is the *Karoubi envelope* $\mathrm{H} = \mathit{Kar}(\mathrm{H}')$.

(The Karoubi envelope H′ makes all idempotents split - i.e. creates new objects such that idempotent morphisms can be seen as projections onto them. It includes symmetric and antisymmetric powers of the objects.)

Local Moves for morphisms of H:



Commutation relations become specified isomorphisms, which are described by such diagrams.

Proposition (Khovanov)

There is a surjective map $K_0(\mathrm{H}') \to H_+$ *(onto the positive integer form of the Heisenberg algebra).*

(Khovanov conjectured it is an isomorphism.)
This correspondence works by:

- Objects yield generators of the algebra:

$$Q_+ \mapsto \mathsf{a}^\dagger$$

  and

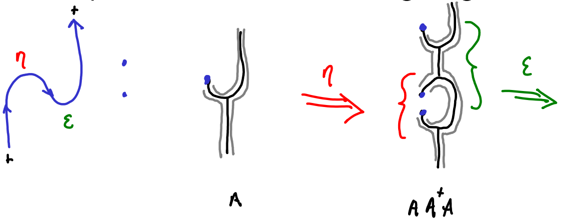
$$Q_- \mapsto \mathsf{a}$$

- Isomorphisms impose equivalence relations on objects
- All other morphisms are ignored

Take Khovanov's monoidal category H as a bicategory with one object, $\bullet$. Compare this to our $\mathrm{h} \subset End_{\mathrm{Span}_2(\mathrm{Gpd})}(\mathrm{FinSet}_0)$.

$\mathrm{Span}_2(\mathrm{Gpd})$	Khovanov
h	H
FinSet_0	$\bullet$
A, $A^\dagger$	Q_-, Q_+
Id_A, $Id_{A^\dagger}$	$\downarrow, \uparrow$
$\circ$	$\otimes$
η, ϵ	$\cap, \cup$

So Khovanov's morphisms are 2-morphisms between the 1-morphisms of our earlier string diagrams:



2-Hilbert Spaces

There is a representation of the 2-category $\mathrm{Span}_2(\mathrm{Gpd})$ into 2-Hilbert spaces, just as there is a degroupoidification of $\mathrm{Span}_1(\mathrm{Gpd})$ into Hilbert spaces. More precisely:

Definition

2Hilb is the 2-category of (finitely semisimple) 2-Hilbert spaces, which consists of:

- Objects: $\mathbb{C}$-linear abelian categories, generated by simple objects
- Morphisms: **2-linear maps**: $\mathbb{C}$-linear (hence abelian) functor.
- 2-Morphisms: Natural transformations

Theorem

There is an ambiadjunction-preserving 2-functor ("2-linearization"):

$$\Lambda : \mathrm{Span}_2(\mathrm{Gpd}) \to \mathrm{2Hilb}$$

Definition

Define the 2-functor Λ as follows:

- Objects: $\Lambda(\mathrm{B}) = \mathit{Rep}(\mathrm{B}) := \mathit{Hom}(\mathrm{B}, \mathrm{Hilb})$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\mathrm{A}) \longrightarrow \Lambda(\mathrm{B})$
- 2-Morphisms:
  $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \to (t')_* \circ (s')^*$

(The map N is the *Nakayama isomorphism* between the left and right adjoints of s^* or t^*.)
(This is a "pull and push" of functors through the 2-morphism: it uses adjunctions to add or delete "pull-push" pairs.)

The general construction of the Fock pseudomonad means, as before, that there are Fock pseudomonads for $\mathrm{Span}_2(\mathrm{Gpd})$ and 2Hilb, which we again denote F_S and F_H again. They are compatible in the following sense:

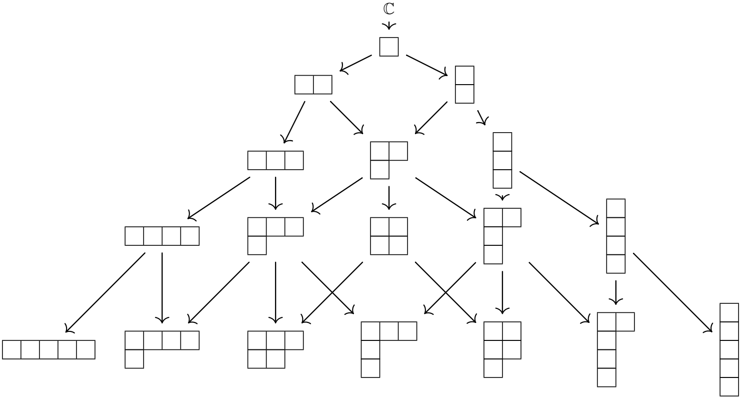
Theorem

The following diagram commutes up to pseudonatural equivalence:

$$\begin{array}{ccc}
\mathrm{Span}_2(\mathrm{Gpd}) & \xrightarrow{\Lambda} & \mathrm{2Hilb} \\
{\scriptstyle F_S}\downarrow & & \downarrow{\scriptstyle F_H} \\
\mathrm{Span}_2(\mathrm{Gpd}) & \xrightarrow[\Lambda]{} & \mathrm{2Hilb}
\end{array}$$

Thus, we have a natural analog in 2Hilb of the Fock representation of the Heisenberg algebra: we call this the *categorified quantum harmonic oscillator*.

The 2-vectorial "Fock space" is $\Lambda(\mathrm{FinSet}_0) \cong \prod_n Rep(\Sigma_n)$. $\Lambda(A)$ and $\Lambda(A^\dagger) = \oplus_n(- \otimes \mathbb{C}^n)$ give representations counting paths in this lattice:



Prospects and Questions

- Categorification has proved useful in extending TQFT to higher codimension
- 2-Hilbert spaces can be attached to boundary conditions for regions in space
- The "2-Fock space" for these 2-Hilbert spaces should be relevant to extending Quantum Field Theory in general
- The representations of S_n between the totally symmetric (bosonic) and totally antisymmetric (fermionic) representations play no role in *unextended* QFT
- Is this still true in extended QFT?
- What relation do they have to "paraparticle statistics"?