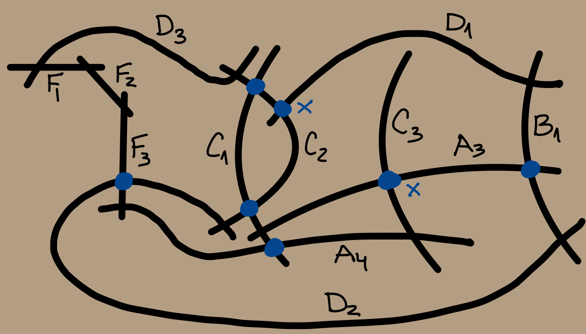
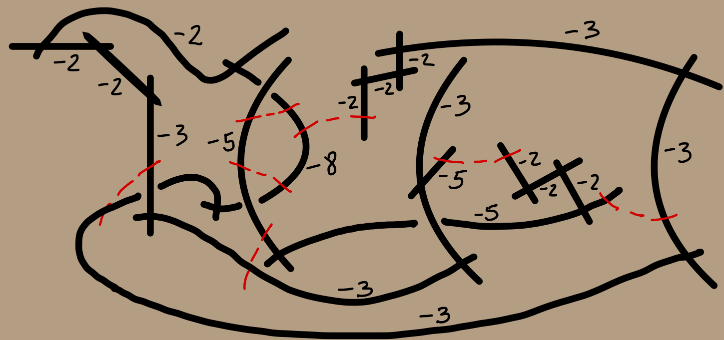
What is the right combinatorics for spheres in K3 surfaces?





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(13:30)

For details (in the ArXiv since october 2021):

«Rational configurations in K3 surfaces and simply-connected $p_g = 1$ surfaces for $K^2 = 1, 2, 3, 4, 5, 6, 7, 8, 9$»
(joint with Javier Reyes)

Preliminaries!

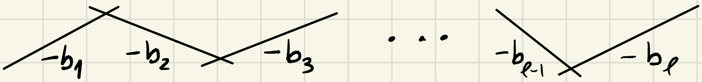
- A K3 surface is a nonsingular projective complex surface which is simply-connected and the canonical class is trivial.

Examples:
- nonsingular hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$ of degree 4 (e.g. $\{x^4 + y^4 + z^4 + w^4 = 0\}$)
- Doble covers of $\mathbb{P}^2_{\mathbb{C}}$ branched along a sextic (e.g. when sextic is formed by two cubics, producing an elliptic fibration)

- A $(-m)$-curve $\Gamma \subset X$ = nonsingular proj. surface is a $\mathbb{P}^1_{\mathbb{C}}$ such that $\Gamma \cdot \Gamma = -m$.

Examples:
- In the blow-up of a point a (-1)-curve is contracted.
- The only $(-m)$-curves in a K3 surface are (-2)-curves.

- A chain of $\mathbb{P}^1_{\mathbb{C}}$ is



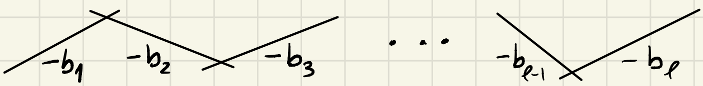
$\subset$ nonsingular surface

<u>Main example</u>: If $0 < q < m$ are coprime, then

$$\frac{m}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_\ell}}} =: [b_1, \ldots, b_\ell]$$

where $b_i \geq 2$ $\forall i$ (Hirzebruch-Jung cont. fract.).

Then,



$\longrightarrow$ $\bullet$ $\frac{1}{m}(1,q)$ singular point

where $\frac{1}{m}(1,q)$ is the germ at $(0,0)$ of the quotient of $\mathbb{C}^2$ by $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $(x,y) \mapsto (\zeta x, \zeta^q y)$ with ζ primitive m^{th} root of 1. (c.q.s.)

- A Wahl singularity is a c.q.s. of type $\frac{1}{n^2}(1, na-1)$ where $0<a<n$ are coprime. A Wahl chain is the chain of $\mathbb{P}^1_{\mathbb{C}}$ from the minimal resolution.

<u>Example</u>: $\frac{1}{2^2}(1, 2\cdot 1-1) = \frac{1}{4}(1,1)$ is the smallest Wahl singularity, where $\frac{4}{1} = [4]$.

All Wahl chains can be constructed from [4]...

$$[4]$$
$$\downarrow$$
$$[5,2]$$
$$\swarrow \qquad \searrow$$
$$[6,2,2] \qquad [2,5,3]$$
$$\swarrow \quad \searrow \qquad \swarrow \quad \searrow$$
$$[7,2,2,2] \quad [2,6,2,3] \quad [3,5,3,2] \quad [2,2,5,4]$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

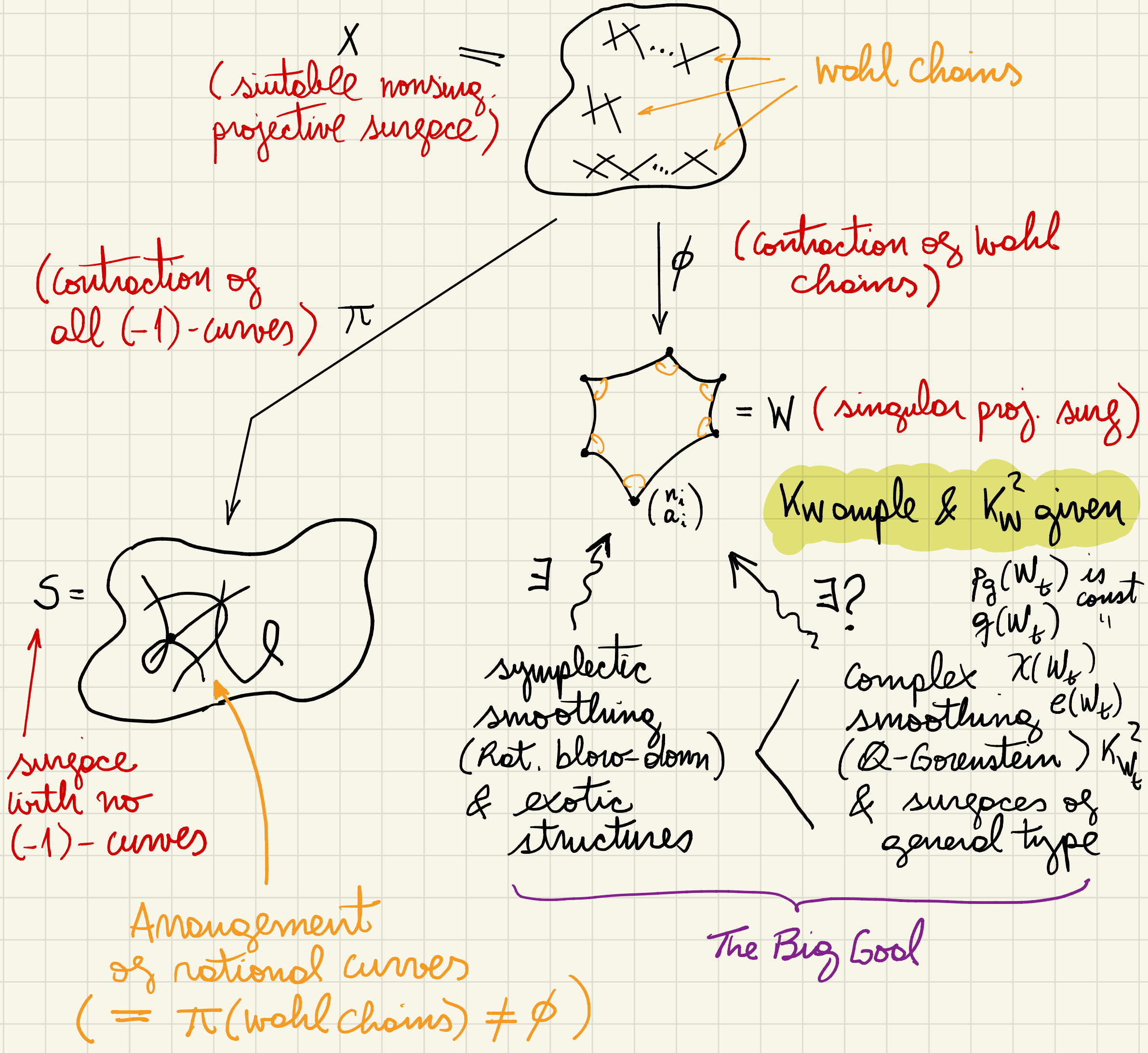
$[b_1, \dots, b_s]$

$\Downarrow$

- $[2, b_1, \dots, b_{s-1}, b_s+1]$
- $[b_1+1, b_2, \dots, b_s, 2]$

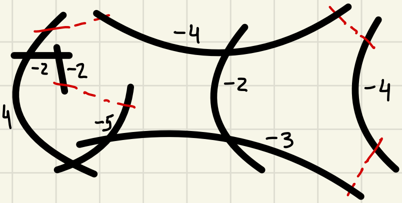
Wahl singularities are the c.q.s. which admit a smoothing whose Milnor fiber has 2nd Betti number equal to zero.

End of preliminaries!

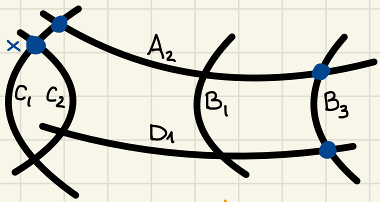


X =
(suitable nonsing. projective surface)
wahl chains
(contraction of wahl chains)
φ
(contraction of all (-1)-curves)
π
= W (singular proj. surf)
$\binom{n_i}{a_i}$
K_W ample & K_W^2 given
S =
surface with no (-1)-curves
Arrangement of rational curves
(= π(wahl chains) ≠ ∅)
∃
∃?
$p_g(W_t)$ is const
$q(W_t)$
$\chi(W_t)$
$e(W_t)$
$K_{W_t}^2$
symplectic smoothing (Rat. blow-down) & exotic structures
<
complex smoothing (Q-Gorenstein) & surfaces of general type
The Big Goal

$X \supset$



$\pi \swarrow \qquad \downarrow \phi$



$= W$

K_W ample

$K_W^2 = 2$

$S \,\cap$ (sic: rotated $\subset$)

(A K3 surface)

Configuration of (−2)-curves

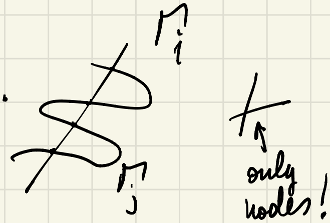
Complex surfaces of general type with $p_g = 1$, $\pi_1 = 1$, $K^2 = 2$

A brief historical account: (missing many names here!!!)

- J. Wahl (1981) and sing. with smoothings $\mu=0$.
- Kollár-Shepherd-Barron (1988): use of Mori theory to define compactification of moduli of surfaces and Def$\left(\frac{1}{m}(1,q)\right)$ via P-resolutions.
- Fintushel-Stern (1997): Rational blow-down [J. Park (1997)].
- M. Manetti (2001): Components of moduli with fixed diff. type.
- J. Park (2005): Exotic $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$.
- J. Park (2007): Exotic $3\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$.
- Y. Lee - J. Park (2007): $\pi_1=1$, $p_g=0$, $K^2=2$ sing. gen. type
- Lee-Park-H. Park-D. Shin -etc: many others for $p_g=0,1$, Horikawa, etc.
- P. Hacking - J. Tevelev - __ (2013): MMP and other developments.
- ETC, etc ...

Say $\{\overset{\mathbb{P}^1_{\mathbb{C}}}{\overset{\shortparallel}{\Gamma_1}}, \ldots, \overset{\mathbb{P}^1_{\mathbb{C}}}{\overset{\shortparallel}{\Gamma_r}}\} \subset S = $ K3 surface.

$\underbrace{\{\Gamma_1, \ldots, \Gamma_r\}}_{\text{SNC arrangement of } \mathbb{P}^1_{\mathbb{C}}}$



At first, what can we say about the arrangement in relation to our goal W ?

r = # (−2)-curves
t_2 = # of nodes
P = # Wahl chains = # singularities in W
$K^2 = K^2_W$

Then one can show :

$$2t_2 - 2r = \bar{c}_1^2 = 2K^2 \qquad \text{(1st log chern \#)}$$

$$24 + t_2 - 2r = \bar{c}_2 = 24 - P - K^2 \qquad \text{(2nd log chern \#)}$$

$$K^2 \leq 14 - \frac{1}{5}(3P - 2) \qquad \text{(log BMY inequality)}$$

and so : $\quad r = P + 2K^2 \qquad t_2 = 3K^2 + P$

High $K^2 \Rightarrow$ high $\bar{c}_1^2 / \bar{c}_2$, r , t_2 and $K^2 \leq 13$

To ensure complex deformations by no local-to-global obstructions we need $\{\Gamma_1, \ldots, \Gamma_r\}$ independent in the Neron-Severi group

$\Rightarrow$ Picard number of K3 surface $\geq r = P + 2K^2$

Then $K^2 \leq 9$ in this case, since $\rho(K3) \leq 20$.

Hence increasing K^2 makes the arrangement and the K3 surface more special.

CAUTION! Existence of the arrangement in a K3 surface does not guarantee the existence of a W.

$\boxed{\text{What is the right combinatorics for spheres in K3 surfaces?}}$

I.e. given K^2 and P, which are the arrangements of SNC spheres (or more general!) which produce W with K_W ample, $K_W^2 = K^2$, P wohl sing?

We do not know an answer, but:

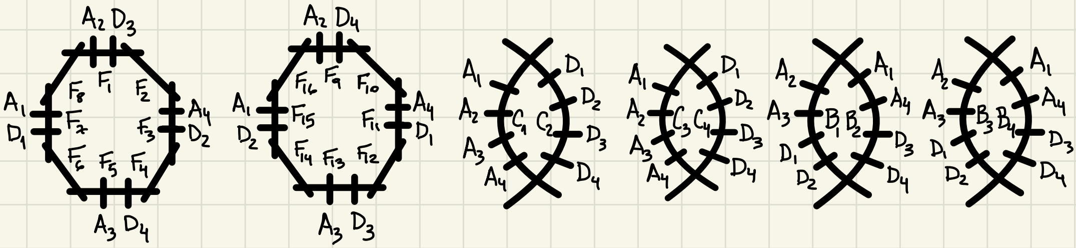
- We have an algorithm to do computer searches (J. Reyes), and we have found thousands of examples with several particular properties.
- In papers, there is a jungle of examples of $\mathbb{CP}^2 \# m \overline{\mathbb{CP}^2}$ and $3\mathbb{CP}^2 \# m \overline{\mathbb{CP}^2}$ with exotic structures, but many not from Rat. blow-downs and still some low m's are missing.

Theorem : (Javier Reyes, — 2021) (in the arxiv)

$\exists W$ with $K_W^2 = 1,2,3,4,5,6,7,8,9$ which produce $(20-2K_W^2)$- dimensional families of nonsingular projective surfaces of general type with ample canonical class, $p_g = 1$ and $\pi_1 = 1$. In particular they produce $3\mathbb{CP}^2 \# (19-K^2)\overline{\mathbb{CP}^2}$ with complex structure.

Observations.

- For $K^2 = 7, 9$ there were NO examples.
- For $K^2 = 8$ there was NO such family.
- Case $K^2 = 9$ shows a moduli space which is a surface!
- We got dramatically too many examples, difficult to analyze all of them!
- Frecuently we see that one configuration produce more than 1 W with same invariants.
- We close the unobstructed range!
- For this theorem, all configurations we use come from one configuration with 32 (-2)-curves.



«Theorem»: (Javier Reyes, __ 2021) (not available yet)

$\exists\, W$ with $K_W^2 = 10, 11, 12$ and K_W ample

(producing $\underbrace{\text{exotic } 3\mathbb{CP}^2 \# 9,8,7\, \overline{\mathbb{CP}^2}}_{\text{To be checked}}$)

observations:

- we have found very few in comparison (say between 10 and 20 examples for each!)
- Again, some configurations produce many. In $K^2 = 12$, we have one arrangement which produces 4 distinct examples!

$n = 267721$
$\ell = 27$

- For SNC we have the bound 13 but impossible so far. More general arrangements produce no BMY bound for K^2.
- $\exists\, 3\mathbb{CP}^2 \# m\, \overline{\mathbb{CP}^2}$ for $m = 6, 5, 4$ (Akhmedov, Park (2010) (2008); Baykur-Korkmaz (2017) $m=6$) no not blow-down here. Not known: $m = 3, 2, 1$. ($K^2 = 16, 17, 18$)

very few!

Apart: For $\mathbb{CP}^2 \# m\, \overline{\mathbb{CP}^2}$ we have $m=4$ via process above. Again, $m = 3, 2$ are also known by works around 2008 - 2011 (very few!). $m=1$ is still open.

Show ART.