# Simplicial Vector Bundles and Representations up to Homotopy

Giorgio Trentinaglia

CAMGSD Instituto Superior Técnico Lisbon

TQFT Club Seminar
19 January 2022

# Simplicial objects of a category $\mathcal{C}$

## The simplex category $\mathbf{\Delta}$

**Objects** The finite ordinals $[n] = \{0 < 1 < \cdots < n\} \quad (n \geq 0)$

**Morphisms** The *poset* (= order preserving) maps $[m] \to [n]$

$\delta_i : [n-1] \to [n]$ the poset injection skipping $i = 0, \ldots, n$

$\upsilon_i : [n+1] \to [n]$ the poset surjection sending $i, i+1 \mapsto i$

## Definition

The contravariant functors $X, Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ of $\mathbf{\Delta}$ into $\mathcal{C}$ are called *simplicial objects* of $\mathcal{C}$. A *simplicial morphism* is a natural transformation $X \to Y$ between two such functors.

## Presentation in terms of faces and degeneracies

$X_n = X([n]) \qquad d_i = X(\delta_i) : X_n \to X_{n-1} \qquad u_i = X(\upsilon_i) : X_n \to X_{n+1}$

*(i-th face morphism)* *(i-th degeneracy morphism)*

$$X = \left\{ \cdots \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix} X_2 \underset{u_i}{\begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix}} X_1 \overset{d_i}{\begin{smallmatrix}\rightarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix}} X_0 \quad \begin{array}{c}\text{satisfying the}\\ \textit{simplicial identites}\\ d_i d_j = d_{j-1} d_i \ \textit{etc...}\end{array} \right\}$$

## Simplicial sets and the nerve construction

- When $\mathcal{C} = \{\textbf{Sets}\}$, we speak of *simplicial sets/maps*.
- Elements of $X_0$ are called *vertices*. Elements of $X_1$ are called *edges*. Elements of $X_n$ are called *n-simplices*.

### Example — Nerve of a small category $G_1 \rightrightarrows G_0$

$G_0$ = Set of *objects* $x$

$G_1$ = Set of *arrows* $tg \xleftarrow{g} sg$

$$\begin{cases} d_0(g) = tg \\ d_1(g) = sg \end{cases} \qquad u_0(x) = x \xleftarrow{1x} x$$

$G_n$ = Set of length $n \geq 2$ *strings of arrows*

$$x_n \xleftarrow{g_n} x_{n-1} \xleftarrow{g_{n-1}} \cdots \xleftarrow{g_{i+1}} x_i \xleftarrow{g_i} \cdots \xleftarrow{g_2} x_1 \xleftarrow{g_1} x_0$$

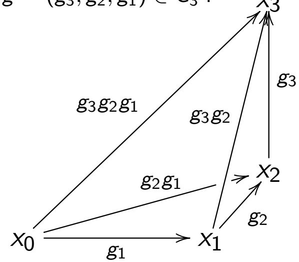
$$\begin{cases} d_0(g_n, \ldots, g_1) = (g_n, \ldots, g_2) & \text{for } n \geq 2 \\ d_i(g_n, \ldots, g_1) = (g_n, \ldots, g_{i+1}g_i, \ldots, g_1) & \text{for } 0 < i < n \\ d_n(g_n, \ldots, g_1) = (g_{n-1}, \ldots, g_1) & \text{for } n \geq 2 \\ u_i(g_n, \ldots, g_1) = (g_n, \ldots, g_{i+1}, 1x_i, g_i, \ldots, g_1) & \text{for } n \geq 1 \end{cases}$$

# Simplicial sets and the nerve construction

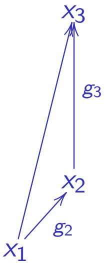
## Some related concepts

- $s_k = d_{k+1} \cdots d_n$ *back k-face* $s_k(g_n, \ldots, g_1) = (g_k, \ldots, g_1)$
- $t_k = (d_0)^{n-k}$ *front k-face* $t_k(g_n, \ldots, g_1) = (g_n, \ldots, g_{n-k+1})$
- $x_k = (d_0)^k d_{k+1} \cdots d_n$ *k-th vertex* $x_k(g_n, \ldots, g_1) = tg_k = x_k$
- $s = s_0$ *source* $s(g_n, \ldots, g_1) = sg_1 = x_0$
- $t = t_0$ *target* $t(g_n, \ldots, g_1) = tg_n = x_n$

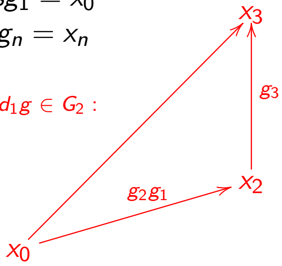
$g = (g_3, g_2, g_1) \in G_3$ :




$d_0 g \in G_2$ :




$d_1 g \in G_2$ :




$\ldots$

$d_3 g \in G_2$ :

# Dold–Kan correspondence

When $\mathcal{C} = \{\textbf{Vector Spaces}\}$, we speak of *simplicial vector spaces*.

$$V = \left\{ \begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & V_2 \arrow[l, bend left] \arrow[l, bend left=40] \arrow[l, bend left=50] \arrow[r, shift left=2, "d_i"] \arrow[r] \arrow[r, shift right=2] & V_1 \arrow[l, bend left, "u_j"] \arrow[l, bend left=40] \arrow[r, shift left] \arrow[r, shift right] & V_0 \arrow[l, bend left] \end{tikzcd} \right\}$$

## Definition

The *Moore complex* $V^\bullet, d^\bullet$ has $V^{-n} = V_n$ (zero for $n < 0$) and

$$d^{-n} = (-1)^{n-1} \sum_{j=0}^{n} (-1)^j d_j : V^{-n} \to V^{1-n} \quad n \geq 1.$$

The *Dold–Kan complex* $\hat{V}^\bullet, \hat{d}^\bullet$ is the subcomplex given by

$$\hat{V}^{-n} = \bigcap_{i=1}^{n} \ker\left(d_i : V_n \to V_{n-1}\right)$$
$$\hat{d}^{-n} = (-1)^{n-1} d_0 : \hat{V}^{-n} \to \hat{V}^{1-n}.$$

$$\cdots \xrightarrow{\hat{d}^{-3}} \hat{V}^{-2} \xrightarrow{\hat{d}^{-2}} \hat{V}^{-1} \xrightarrow{\hat{d}^{-1}} \hat{V}^{0} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

# Dold–Kan correspondence

## Theorem (Dold–Kan)

*The functor $V \mapsto (\hat{V}^\bullet, \hat{d}^\bullet)$ from simplicial vector spaces to cochain complexes of vector spaces vanishing in positive degrees is an equivalence of categories.*

## Proof.

This functor has a canonical quasi-inverse $(E, R) \mapsto E_\bullet$ given by

$$E_n = \bigoplus_{\nu:[n]\twoheadrightarrow[r]} E^{-r}$$

$$\underbrace{\left(E_{\theta:[m]\to[n]}\right)^{\mu:[m]\twoheadrightarrow[q]}_{\nu:[n]\twoheadrightarrow[r]}}_{\text{($\mu$-th row, $\nu$-th column)}} = \begin{cases} \mathrm{id} & \text{if } \nu\theta = \mu \\ (-1)^q R & \text{if } \nu\theta = \delta_0\mu \\ 0 & \text{otherwise.} \end{cases}$$

□

# Alternative formula for $(E,R) \mapsto E_\bullet$

Setting $\bar\nu(j) = \min \nu^{-1}(j)$ defines a bijection

$$\left\{ \begin{matrix} \text{poset surjections} \\ \nu : [n] \twoheadrightarrow [r] \end{matrix} \right\} \underset{\bar\rho \leftarrow\!\shortmid\, \rho}{\overset{\nu \mapsto \bar\nu}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \left\{ \begin{matrix} \text{poset injections} \\ \text{sending } 0 \mapsto 0 \\ \rho : [r] \underset{0}{\rightarrowtail} [n] \end{matrix} \right\}$$

We may thus just as well use the latter set to label the direct sum. However we do not do it the obvious way! Instead we mix up some of the components by considering the following linear bijection:

$$\bigoplus_{\nu:[n]\twoheadrightarrow[r]} E^{-r} \xrightarrow[\sim]{\Phi_n} \bigoplus_{\rho:[r]\underset{0}{\rightarrowtail}[n]} E^{-r} \qquad (\Phi_n)^\rho_\nu = \begin{cases} \mathrm{id} & \text{if } \nu\rho = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$$

By declaring the $\Phi_n$ $n \geq 0$ to be an isomorphism of simplicial vector spaces we get another valid formula for $(E,R) \mapsto E_\bullet$:

# Alternative formula for $(E, R) \mapsto E_\bullet$

1. For each $n \geq 0$ the vector space of $n$-simplices of $E_\bullet$ is

$$E_n = \bigoplus_{\rho:[r] \underset{0}{\rightarrowtail} [n]} E^{-r}$$

2. For each poset map $\theta : [m] \to [n]$ such that $\theta(0) = 0$ the matrix of the linear map $E_\theta : E_n \to E_m$ is

$$\left(E_{\theta:[m]\to[n]}\right)^{\rho:[r]\underset{0}{\rightarrowtail}[m]}_{\sigma:[s]\underset{0}{\rightarrowtail}[n]} = \begin{cases} \mathrm{id} & \text{if } \theta\rho = \sigma \\ 0 & \text{otherwise} \end{cases}$$

3. The matrix of the linear map $d_0 = E_{\delta_0:[n-1]\to[n]}$ is

$$(d_0)^{\rho:[r]\underset{0}{\rightarrowtail}[n-1]}_{\sigma:[s]\underset{0}{\rightarrowtail}[n]} = \begin{cases} (-1)^{i-1}\mathrm{id} & \text{if } \rho^+\delta_i = \sigma \ \exists i \leq r \\ (-1)^r R & \text{if } \rho^+ = \sigma \\ 0 & \text{otherwise} \end{cases}$$

where $\rho^+ : [r+1] \to [n]$ is the only poset map satisfying $\rho^+(0) = 0$ and $\rho^+\delta_0 = \delta_0\rho : [r] \to [n]$

# A first generalization

The Dold–Kan correspondence extends to simplicial vector bundles over a given manifold $M$ i.e. simplicial objects in the category $\mathcal{C} = \mathbf{VB}(M)$ with only one "subtlety": you need to prove that $\hat{V}^{-n}$ is a smooth subbundle of $V^{-n} = V_n$.

## Proof.

$\hat{V}^{-n}$ is the image of the smooth involutive endomorphism of $V_n$ given by $v \mapsto \mathrm{nor}(v) = (\mathrm{id} - u_0 d_1) \cdots (\mathrm{id} - u_{n-1} d_n) v$. (By the way the kernel of $\mathrm{nor}$ is the linear subspace of $V_n$ generated by all degenerate $n$-simplices.) $\square$

## Theorem

*There is a canonical equivalence of categories (canonical pair of mutually quasi-inverse functors)*

$$\left\{ \begin{matrix} \textbf{Simplicial vector} \\ \textbf{bundles over } M \end{matrix} \right\} \underset{E_\bullet \leftarrow\!\shortmid (E, R)}{\overset{V \mapsto (\hat{V}^\bullet, \hat{d}^\bullet)}{\rightleftarrows}} \left\{ \begin{matrix} \textbf{(Co)chain complexes} \\ \textbf{of vector bundles} \\ \textbf{over } M \textbf{ vanishing} \\ \textbf{in positive degrees} \end{matrix} \right\}$$

# Generalizing to other simplicial vector bundles?

There are however good reasons not to require our vector bundles to have a fixed manifold $M$ as base. Applications demand that we consider more general simplicial objects of $\mathcal{C} = \mathbf{VB} =$ the category of all vector bundles (covering arbitrary bases or base maps):

$$\begin{tikzcd} V \arrow[d, "p"] \\ G \end{tikzcd} \quad = \quad \left\{ \begin{tikzcd}[column sep=large] \cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & V_2 \arrow[l, shift left=4] \arrow[l, shift left=6] \arrow[l, shift left=8] \arrow[r, shift left=2, "d_i"] \arrow[r] \arrow[r, shift right=2] \arrow[d, "p_2"] & V_1 \arrow[l, shift left=4] \arrow[l, shift left=6, "u_j"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, "p_1"] & V_0 \arrow[l, shift left=3] \arrow[d, "p_0"] \\ \cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G_2 \arrow[l, shift left=4] \arrow[l, shift left=6] \arrow[l, shift left=8] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G_1 \arrow[l, shift left=4] \arrow[l, shift left=6] \arrow[r, shift left=1] \arrow[r, shift right=1] & G_0 \arrow[l, shift left=3] \end{tikzcd} \right\}$$

The base of the *simplicial vector bundle* $p : V \to G$ is the *simplicial manifold* $G$—a simplicial object of $\mathcal{C} = \{\textbf{Manifolds}\}$.

## Example

The *tangent bundle* $TG \to G$ of the simplicial manifold $G$:

$$T(G)_n = T(G_n) \qquad {}^{TG}d_i = T({}^{G}d_i) \qquad {}^{TG}u_j = T({}^{G}u_j)$$

# Motivation: representations of Lie groupoids

A small category $G_1 \rightrightarrows G_0$ is called a *groupoid* if all its arrows are invertible. A *Lie groupoid* is a groupoid whose nerve is given the structure of a "nice" simplicial manifold:

- $G_1 \xrightarrow{s} G_0$ is a submersion of smooth manifolds (it follows that every $G_n$ is a smooth manifold)
- the maps $G_0 \xrightarrow{1} G_1$ and $G_2 \xrightarrow{d_1} G_1$ are smooth (it follows that all $d_i$ and $u_j$ are) and so is the map $G_1 \to G_1$, $g \mapsto g^{-1}$.

A *representation* of $G$ on a vector bundle $E \in \mathbf{VB}(G_0)$ is an isomorphism $R : s^*E \xrightarrow{\sim} t^*E \in \mathbf{VB}(G_1)$ such that the linear maps $R(g) : E_{sg} \to E_{tg}$ satisfy $R(g_2 g_1) = R(g_2)R(g_1)$ $R(1x) = \mathrm{id}$.

## Representations as 1-strict simplicial vector bundles

$$E_n = s^*E \in \mathbf{VB}(G_n) \qquad d_0(g_n, \dots, g_1; e) = (g_n, \dots, g_2; R(g_1)e)$$

$$d_i(g; e) = (d_i g; e) \qquad i > 0 \qquad u_j(g; e) = (u_j g; e)$$

*$V \xrightarrow{p} G$ is in the essential image of $(E, R) \mapsto E_\bullet$ iff $V_n \xrightarrow[\sim]{(p_n, s)} s^* V_0$ for every $n$ (1-strictness property).*

# The adjoint representation

## Problem

What is the adjoint representation of a Lie groupoid?

*A little reflection shows this cannot be a representation in the traditional sense!*

## Remarks

The tangent bundle of the nerve of a Lie groupoid $G$ is itself the nerve of a Lie groupoid $TG_1 \rightrightarrows TG_0$, the *tangent groupoid* of $G$. When $G$ is a group, the adjoint representation of $G$ can be expressed in terms of the tangent group operations: for all group elements $g \in G$ and Lie algebra vectors $X \in \mathfrak{g} = T_1 G$,

$$\mathrm{Ad}(g)X = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g \exp(tX) g^{-1} = 0_g \cdot X \cdot {0_g}^{-1}.$$

*Now, for general $G$, on what vector bundle $\mathfrak{g}$ over $G_0$ should the adjoint representation operate?*

# The adjoint representation

## Underlying vector bundle?

- $\mathfrak{g}_x = T_{1x}G(x)$, where $G(x) = G(x,x)$? This does not work: not a smooth vector bundle!
- $\mathfrak{g}_x = \ker T_{1x}s$ in other words $\mathfrak{g} = 1^*(\ker Ts)$? This gives a smooth vector bundle, but does not work either since the formula for Ad is no longer meaningful when the target of $X$ is nonzero.

## The RUTH viewpoint — basic ideas

- Avoid smoothness issues by working with complexes of vector bundles rather than with vector bundles.
- Relax the homomorphism condition $R(g_2g_1) = R(g_2)R(g_1)$ by allowing it to be only true "up to homotopy".
- Give up expectations that Ad should be definable in a canonical way.

# Representations up to homotopy

## Definition

A *representation up to homotopy* of a Lie groupoid $G$ consists of

- $E = \bigoplus E^{-n}$ a vector bundle over $G_0$ with grading over the additive group of all integers $n$
- for each integer $m \geq 0$ a degree $1 - m$ morphism $R_m : s^*E \to t^*E$ of smooth vector bundles over $G_m$

such that for every $g \in G_m$ the following equation is satisfied where $R_m(g) = R_m^{-n}(g) : E_{sg}^{-n} \to E_{tg}^{1-m-n}$ denotes the linear map that $R_m$ induces between the fibers of $E$ at the source and at the target of $g$.

$$\boxed{\sum_{k=1}^{m-1}(-1)^k R_{m-1}(d_k g) = \sum_{k=0}^{m}(-1)^k R_{m-k}(t_{m-k} g) R_k(s_k g)}$$

# Representations up to homotopy

## Remarks

- For $m = 0$ the equation says that the pair $E_x, d_x = R_0(x)$ is a (co)chain complex of vector spaces for every $x \in G_0$.
- For $m = 1$ the equation says that $R(g) = R_1(g)$ is a chain map of $E_{sg}, d_{sg}$ into $E_{tg}, d_{tg}$.
- For $m = 2$ the equation says that $R_2(g_2, g_1)$ is a chain homotopy between the chain maps $R(g_2 g_1)$ and $R(g_2)R(g_1)$.

## Further requirements

We demand that our representations up to homotopy be *unital* in the following sense:

- $R_1(1x)$ is the identity on $E_x$ for every $x \in G_0$.
- $R_{k+1}(u_j g) = 0$ for all $k \geq 1$, $0 \leq j \leq k$, and $g \in G_k$.

## From VB-groupoids to RUTHs

The *adjoint representation* can be defined for any Lie groupoid as a RUTH whose underlying complex of vector bundles has only two nonzero terms namely in degrees $-1$ and $0$ (*two-term* RUTH):

| | |
|---|---|
| degree $-1$: | $1^*(\ker Ts)$ |
| degree $0$: | $T(G_0)$ |
| differential: | $1^*(\ker Ts) \xrightarrow{1^* \mathrm{d}t} 1^* t^* T(G_0) = T(G_0)$ |

These data are canonical, but the construction of the chain map and chain homotopy is not. It is a special case of a more general construction which takes as input a simplicial vector bundle $p : V \to G$ over the nerve of $G$ such that $V$ itself is the nerve of a Lie groupoid $V_1 \rightrightarrows V_0$. Such simplicial vector bundles are known as *VB-groupoids.* The example of $V = TG$ the tangent bundle/groupoid of $G$ should come to mind.

# The Dold–Kan representation $\hat{V}, \{\hat{R}_m\}$

## Construction — Step 0: defining the chain complex $\hat{V}, \hat{R}_0$

- The maps $1_n = u_{n-1} \cdots u_0 : G_0 \to G_n$ provide a *smooth simplicial map* in other words a morphism of simplicial manifolds $1 : G_0 \to G$ along which $p : V \to G$ can be pulled back.
- The pullback $1^*p : 1^*V \to G_0$ is a simplicial vector bundle over the (constant simplicial) manifold $M = G_0$ to which we may apply the Dold–Kan construction described earlier.
- The result is a (co)chain complex $\hat{V}, \hat{R}_0 = \hat{V}, \hat{d}$ of smooth vector bundles over $G_0$ called the *Dold–Kan complex* of $p : V \to G$. Explicitly

$$\hat{V}^{-n}_x = \{v \in V_n : p_n v = 1_n x,\ d_i v = 0\ \forall i > 0\}$$
$$\hat{R}^{-n}_0(x) v = (-1)^{n-1} d_0 v.$$

# The Dold–Kan representation $\hat{V}, \{\hat{R}_m\}$

## Construction — Step 1: defining $\hat{R}_1 = \hat{R}_1^{-1} \oplus \hat{R}_1^0$

- Since $V_1 \rightrightarrows V_0$ is a Lie groupoid, the vector bundle morphism $V_1 \xrightarrow{(p,s)} G_1 \times_{G_0} V_0 = s^* V_0 \in \mathbf{VB}(G_1)$ is onto.
- Let $c : s^* V_0 \to V_1 \in \mathbf{VB}(G_1)$ be any section $g, v \mapsto c(g, v)$, smooth and linear in $v$. We call $c$ a *splitting* or *cleavage.*
- Since $1(G_0) \subset G_1$ is a closed submanifold, we can always adjust $c$ so that $c(1x, v) = 1v$ (*normality* property).
- Given $g \in G_1$, $v \in \hat{V}^0_{sg}$, and $w \in \hat{V}^{-1}_{sg}$, making use of the groupoid operations of $V_1 \rightrightarrows V_0$, we define

$$\hat{R}^0_1(g)v = t(c(g, v)) \in \hat{V}^0_{tg}$$

$$\begin{aligned} \hat{R}^{-1}_1(g)w &= c(g, tw) \cdot w \cdot c(g, sw)^{-1} \\ &= c(g, tw) \cdot w \cdot {0_g}^{-1} \in \hat{V}^{-1}_{tg}. \end{aligned}$$

# The Dold–Kan representation $\hat{V}, \{\hat{R}_m\}$

## Construction — Step 2: defining $\hat{R}_2 = \hat{R}_2^0$

- Given $g = (g_2, g_1) \in G_2$ and $v \in \hat{V}^0_{sg}$, again in terms of the groupoid operations of $V_1 \rightrightarrows V_0$, we define

$$R_2(g)v = c(g_2, \hat{R}_1(g_1)v) \cdot c(g_1, v) \cdot c(g_2 g_1, v)^{-1} \in V^{-1}_{tg_2}$$

and then "normalize" this to get

$$\begin{aligned} \hat{R}_2^0(g)v &= (\mathrm{id} - u_0 d_1) R_2^0(g)v \\ &= \mathrm{nor}(R_2^0(g)v) \in \hat{V}^{-1}_{tg}. \end{aligned}$$

## Conclusion

$\hat{V} = \hat{V}^{-1} \oplus \hat{V}^0$, $\hat{R}_0$, $\hat{R}_1$, $\hat{R}_2$, is a two-term unital RUTH of $G$, the *Dold–Kan representation* associated with $V$ and $c$.
The *adjoint representation* is the special case where $V = TG$.

# Converse: From two-term RUTHs to VB-groupoids

Grothendieck construction

Given: $E = E^{-1} \oplus E^0, \quad R_0 = R_0^{-1}, \quad R_1 = R_1^{-1} \oplus R_1^0, \quad R_2 = R_2^0$

an arbitrary two-term unital RUTH

Define:

- $V_0 = E^0$ a smooth vector bundle over $G_0$
- $V_1 = s^*E^0 \oplus t^*E^{-1}$ a smooth vector bundle over $G_1$
- source, target, composition, and inversion operations

$$s(g; e, f) = e$$
$$t(g; e, f) = R_0(tg)f + R_1(g)e$$
$$(g_2; e_2, f_2) \cdot (g_1; e_1, f_1) = (g_2 g_1; e_1, f_2 + R_1(g_2)f_1 + R_2(g_2, g_1)e_1)$$
$$(g; e, f)^{-1} = (g^{-1}; t(g; e, f), -R_1(g^{-1})f - R_2(g^{-1}, g)e).$$

Then: $V_1 \rightrightarrows V_0$ thus defined is a VB-groupoid.

There is a canonical normal cleavage $c(g, e) = (g, e, 0)$.

$E, \{R_m\}$ can be recovered from $V$ and $c$.

# Recapitulation

### Theorem (Dold–Kan correspondence for vector bundles)

*There is a categorical equivalence between simplicial vector bundles over a manifold $M$ and (unital) representations up to homotopy of $M$ (viewed as a groupoid with only unit arrows).*

### Theorem (Grothendieck construction for VB-groupoids)

*There is a categorical equivalence between VB-groupoids over $G$ and two-term unital representations up to homotopy of $G$.*
*This restricts to an equivalence between 1-strict simplicial vector bundles over $G$ and ordinary representations (= one-term unital representations up to homotopy) of $G$.*

* * * * *

### Natural question

Is there a general theorem connecting these two results, that is to say, containing them as special cases?

# Vector fibrations

## Definition

Let $X$ be a simplicial set. For any $0 \leq k \leq n$ let $X_{n,k}$ denote the set of all length $n$ sequences $x_0, \ldots, \widehat{x_k}, \ldots, x_n$ of $n-1$-simplices $x_i \in X_{n-1}$ $(i \neq k)$ that satisfy $d_i x_j = d_{j-1} x_i$ for all $i < j$. The elements of $X_{n,k}$ are called *$n, k$-horns.*

Any element $x$ of $X_n$ defines a corresponding $n, k$-horn $d_{n,k} x = (d_i x)_{i \neq k}$. Not every $n, k$-horn arises in this way. Whenever it does, we say it *can be filled* (by at least one and in principle more than one $x$).

## Kan lifting problem

Given a simplicial vector bundle $p : V \to G$, an $n$-simplex $g \in G_n$, and an $n, k$-horn $(v_i)_{i \neq k} \in V_{n,k}$ with $(p_{n-1} v_i)_{i \neq k} = d_{n,k} g$, can you lift $g$ to an $n$-simplex $v \in V_n$ so that $d_{n,k} v = (v_i)_{i \neq k}$?

## Definition

Whenever the Kan lifting problem can be solved for all $0 \leq k \leq n$ and all $g, (v_i)$ as specified, we call $p : V \to G$ a *vector fibration.*

# General cleavages of a vector fibration

It is an easily proven fact, although perhaps at first a bit surprising, that for any vector fibration each projection $G_n \times_{G_{n,k}} V_{n,k} \to G_n$ can be turned uniquely into a smooth vector bundle over $G_n$ so that

$$(p_n, d_{n,k}) : V_n \longrightarrow G_n \times_{G_{n,k}} V_{n,k}$$

becomes an epimorphism of smooth vector bundles over $G_n$.

### Definition

By an *$n, k$-cleavage* for $0 \leq k \leq n$ we mean an arbitrary morphism

$$c_{n,k} : G_n \times_{G_{n,k}} V_{n,k} \longrightarrow V_n \in \mathbf{VB}(G_n)$$

that is a cross-section of the above epimorphism.

The idea is that the $n, k$-cleavage $c_{n,k}$ picks out a solution $c_{n,k}(g; v_0, \ldots, \widehat{v_k}, \ldots, v_n)$ to the Kan lifting problem which depends both smoothly and linearly on the givens of the problem.

# Strictness and normality

- *In the case of VB-groupoids considered above, our cleavage $c$ was what now goes under the name of "$1, 0$-cleavage $c_{1,0}$."*
- *For VB-groupoids, the case $n = 1$ is the only one which allows some freedom of choice for the $n, k$-cleavages. Indeed the VB-groupoids are precisely the 2-strict vector fibrations:*

### Definition

A vector fibration for which every $(p_n, d_{n,k}) : V_n \to G_n \times_{G_{n,k}} V_{n,k}$ is an isomorphism whenever $n \geq m$ is called *m-strict.*

- *In the case of VB-groupoids our cleavage $c$ was also normal:*

### Definition

An $n, k$-cleavage $c_{n,k} : G_n \times_{G_{n,k}} V_{n,k} \to V_n$ is *normal* whenever its image contains all degenerate $n$-simplices $\bigcup_{j=0}^{n} u_j(V_{n-1})$.

### Proposition

Any vector fibration admits normal $n, k$-cleavages for all $0 \leq k \leq n$.

# From vector fibrations to RUTHs

### Definition

Let $p : V \to G$ be an arbitrary vector fibration. By our proposition we know that there is a normal $n, k$-cleavage $c_{n,k}$ for each pair of integers $n \geq k \geq 0$. Let us select one such cleavage for each pair $n > k \geq 0$. We refer to the whole collection $c = \{c_{n,k}\}_{n>k\geq 0}$ as a *normal cleavage* of the vector fibration.

### The general Dold–Kan representation $\hat{V}, \{\hat{R}_m\}$

- It is a unital RUTH constructed canonically out of the vector fibration $p : V \to G$ and its normal cleavage $c$.
- When the vector fibration $p : V \to G$ is $m$-strict, $\hat{V}, \{\hat{R}_m\}$ is an *$m$-term* RUTH in the sense that its only nonzero homogeneous components are $\hat{V}^{1-m}, \ldots, \hat{V}^0$.
- When the vector fibration $p : V \to G$ is 2-strict, i.e. a VB-groupoid, $\hat{V}, \{\hat{R}_m\}$ is a two-term RUTH and is precisely the one we constructed earlier.

# The general Dold–Kan representation $\hat{V}, \{\hat{R}_m\}$

## Outline of the construction

- The underlying (co)chain complex of vector bundles $\hat{V}, \hat{R}_0$ is the Dold–Kan complex of $p : V \to G$ defined earlier.
- We construct a (non-unital) RUTH $V^\bullet, \{R_m\}$ on the Moore complex $V^\bullet, d^\bullet$ by solving a certain recursive lifting problem involving all of the cleavages $c_{n,k}$.
  We call $V^\bullet, \{R_m\}$ the *Moore representation* associated with $p : V \to G$ and $c$.
- We apply the *normalization map*

$$\mathrm{nor}^{-n} = (\mathrm{id} - u_0 d_1) \cdots (\mathrm{id} - u_{n-1} d_n) : V^{-n} \to \hat{V}^{-n}$$

  to the Moore representation in order to obtain the desired (unital) Dold–Kan representation $\hat{V}, \{\hat{R}_m\}$:

$$\hat{R}_m^{-n}(g)v = \mathrm{nor}_{tg}^{1-m-n}(R_m^{-n}(g)v).$$

# Converse: From RUTHs to vector fibrations

The semidirect product construction $(E, \{R_m\}) \mapsto E_\bullet$

Given: $E, \{R_m\}$ a unital RUTH of $G$ with $E^n = 0$ for $n > 0$

Define:

- $E_n = \bigoplus_{\rho:[r] \underset{0}{\rightarrowtail} [n]} {x_{\rho(r)}}^* E^{-r}$ a smooth vector bundle over $G_n$
- For each poset map $\theta : [m] \to [n]$ such that $\theta(0) = 0$

$$(E_\theta : E_n \to E_m)_{\sigma:[s] \underset{0}{\rightarrowtail} [n]}^{\rho:[r] \underset{0}{\rightarrowtail} [m]} = \begin{cases} \mathrm{id} & \text{if } \theta\rho = \sigma \\ 0 & \text{otherwise} \end{cases}$$

- $(d_0 : E_n \to E_{n-1})_{\sigma:[s] \underset{0}{\rightarrowtail} [n]}^{\rho:[r] \underset{0}{\rightarrowtail} [n-1]}$ at $g \in G_n$ equals

$$\begin{cases} (-1)^{i-1}\mathrm{id} & \text{if } \rho^+\delta_i = \sigma \ \exists i \leq r \\ (-1)^r R^{-s}_{r+1-s}({d_0}^s G_{\rho^+} g) & \text{if } \rho^+ \mid [s] = \sigma \\ 0 & \text{otherwise} \end{cases}$$

# Converse: From RUTHs to vector fibrations

The semidirect product construction $(E, \{R_m\}) \mapsto E_\bullet$

$$(g, \{e_\sigma\}) = \bigoplus_\sigma (g, e_\sigma) \in \bigoplus_{\sigma:[s] \underset{0}{\rightarrowtail} [n]} x_{\sigma(s)}{}^* E^{-s}$$

$$d_0(g, \{e_\sigma\}) = \Big(d_0 g, \Big\{ \sum_{i=1}^{r} (-1)^{i-1} e_{\rho^+ \delta_i} + (-1)^r \sum_{s=0}^{r+1} R^{-s}_{r+1-s}(d_0{}^s G_{\rho^+} g) e_{\rho^+|[s]} \Big\}\Big)$$

$$d_i(g, \{e_\sigma\}) = (d_i g, \{e_{\delta_i \rho}\}) \quad i > 0$$

$$u_j(g, \{e_\sigma\}) = \Big(u_j g, \begin{cases} e_{\upsilon_j \rho} & \text{if } \upsilon_j \rho \text{ is injective} \\ 0 & \text{otherwise} \end{cases}\Big\}\Big)$$

# The semidirect product construction $(E, \{R_m\}) \mapsto E_\bullet$

## Conclusions

- $E_\bullet \to G$ thus defined is a vector fibration over $G$, called the *semidirect product* vector fibration associated with the unital representation up to homotopy $E, \{R_m\}$ with $E^n = 0$ $n > 0$.
- When $E, \{R_m\}$ is a two-term representation, there is a canonical identification between $E_\bullet \to G$ and the nerve of the VB-groupoid $V_1 \rightrightarrows V_0$ defined earlier.
- The semidirect product fibration $E_\bullet \to G$ has a canonical normal $n, k$-cleavage for every pair of integers $n > k \geq 0$. It is the only such cleavage whose image is the subbundle $C_n \subset E_n$ consisting of all $\oplus_\sigma(g, e_\sigma)$ such that $e_{\mathrm{id}} = 0 \in E_{tg}^{-n}$.

## Definition

The cleavage $c = \{c_{n,k}\}_{n>k\geq 0}$ of the vector fibration $p : V \to G$ is *coherent* if for each $n$ the image of $c_{n,k}$ is independent of $k$ i.e. there is a subbundle $C_n \subset V_n$ such that $\mathrm{im}(c_{n,k}) = C_n$ for every $k$.

# Correspondence theorem — Part 1

## Hypotheses

- Let $E, \{R_m\}$ be an arbitrary unital representation up to homotopy of $G$ such that $E^n = 0$ for all $n > 0$.
- Let $\hat{E}, \{\hat{R}_m\}$ be the Dold–Kan representation associated with the semidirect product fibration $E_\bullet \to G$ and with its canonical coherent normal cleavage.

## Conclusions

- $\hat{E}_x^{-n} = \{(1_n x, \{e_\sigma\}) : e_\sigma = 0 \ \forall \sigma \neq \mathrm{id}\}$
- The following formula holds:

$$\hat{R}_m^{-n}(g)(1_n sg; 0, \ldots, 0, e_{\mathrm{id}}) = (1_{m-1+n} tg; 0, \ldots, 0, R_m^{-n}(g) e_{\mathrm{id}})$$

- In particular $\hat{E}, \{\hat{R}_m\}$ is canonically identified with (canonically *strictly isomorphic* to) the original representation $E, \{R_m\}$.

# Correspondence theorem — Part 2

## Hypotheses

- Let $V \to G$ be an arbitrary vector fibration and let $c = \{c_{n,k}\}_{n>k\geq 0}$ be an arbitrary normal cleavage of it.
- Let $(\hat{V}, \{\hat{R}_m\})_\bullet \to G$ be the semidirect product vector fibration arising from the Dold–Kan representation associated with $V \to G$ and $c$.

## Conclusions

- There is a canonical *homotopy equivalence* of vector fibrations over $G$

  $$\Theta : V \xrightarrow{\simeq} (\hat{V}, \{\hat{R}_m\})_\bullet$$

  between $V \to G$ and $(\hat{V}, \{\hat{R}_m\})_\bullet \to G$.

# Correspondence theorem — Part 3

Under additional hypotheses on the cleavage, we get stronger conclusions:

## Hypotheses

The same as for Part 2, plus

- The (normal) cleavage $c = \{c_{n,k}\}_{n>k\geq 0}$ is coherent.

## Conclusions

- $\Theta : V \xrightarrow{\sim} (\hat{V}, \{\hat{R}_m\})_\bullet$ is an isomorphism of simplicial vector bundles over $G$.

## Corollary

*The existence of a coherent normal cleavage characterizes the essential image (closure under isomorphism of the image) of the semidirect product construction among all vector fibrations.*

The results described in this talk were obtained in collaboration with Matias del Hoyo (UFF)

More details are available at
https://arxiv.org/pdf/2109.01062.pdf

THANKS FOR YOUR ATTENTION