Extremal metrics on blowups

Lars Martin Sektnan

University of Gothenburg

Geometria em Lisboa Seminar, January 11, 2022

This work is funded by a Marie Skłodowska-Curie Individual Fellowship, funded from the European Union's Horizon2020 research and innovation programme under grant agreement No 10102804.

Extremal metrics on blowups

A central theme in complex geometry is the search for canonical metrics, such as constant scalar curvature metrics. In this talk, I will discuss a construction of certain canonical Kähler metrics on the blowup of a manifold in a point. The problem has a long history of study, and we will survey previous results, before discussing the core new ideas that allows one to deduce stronger results. This is joint work with Ruadhaí Dervan.

Canonical Kähler metrics

Some Kähler geometry

Let X be a (compact) complex manifold. Then the holomorphic structure determines an endomorphism $J : TX \to TX$ of the tangent bundle, satisfying

$$J^2 = -\operatorname{Id}.$$

This is the associated *almost complex structure*.
A Riemannian metric g on X is *Hermitian* if J is an isometry, i.e. if

$$J^*g := g(J(\cdot), J(\cdot)) = g(\cdot, \cdot).$$

Some Kähler geometry

$$J^*g = g.$$

If g is Hermitian, then

$$\omega(\cdot,\cdot) = g(J(\cdot),\cdot)$$

is a 2-form, as $J^2 = -\mathrm{Id}$:

$$\omega(u,v) = g(Ju,v) = J^*g(Ju,v) = g(J^2u,Jv) = -g(Jv,u) = -\omega(v,u).$$

The metric g is *Kähler* if

$$d\omega = 0.$$

The Kähler condition

By the equation

$$d\omega = 0.$$

if g is Kähler, there is then an associated class

$$\Omega = [\omega] \in H^2(X, \mathbb{R}),$$

the Kähler class of ω. By the $i\partial\bar{\partial}$ Lemma, if $\omega' \in \Omega$ is another Kähler form in the same class, then there exists $\phi : X \to \mathbb{R}$ such that

$$\omega' = \omega + i\partial\bar{\partial}\phi =: \omega_\phi.$$

Canonical metrics

The set

$$\{\phi : \omega_\phi \text{ is a Kähler form}\} \subseteq_{\text{open}} C^\infty(X)$$

is an open subset in the set of smooth functions on X that parametrises Kähler metrics in the class Ω. In particular, in any Kähler class, there is an infinite dimensional set of Kähler metrics. The following question is therefore very natural:
Question: Is there a canonical representative $\omega_\phi \in \Omega$?

Canonical metrics

To begin to answer this, we first need to ask ourselves: what is a good notion of canonical metric?

$\dim_{\mathbb{C}} X = 1$: The uniformisation theorem gives a unique metric of constant curvature. This gives a good canonical choice in complex dimension 1.

$\dim_{\mathbb{C}} X > 1$: Should also be some curvature property, but there are many curvature notions, leading to different notions of canonical metrics!

Scalar curvature

The two important notions in this talk are the following. The metric Kähler metric has an associated scalar curvature $S(\omega)$. We can then seek a *constant scalar curvature (cscK)* metric in Ω:

$$S(\omega_\phi) = \text{constant}.$$

Extremal Kähler metrics

There is a generalisation of this, due to Calabi, called *extremal* Kähler metrics. Let $\mathcal{D}_\omega$ be defined as follows:

$$\mathcal{D}_\omega(f) = \bar{\partial}(\nabla^{1,0}_\omega(f)).$$

Definition (Calabi)

A Kähler metric ω is *extremal* if

$$\mathcal{D}_\omega(S(\omega)) = 0.$$

This says that $S(\omega)$ is the potential for a holomorphic vector field on X.

The YTD conjecture

A central conjecture in the field is the Yau–Tian–Donaldson (YTD) conjecture. This says that the existence of cscK/extremal metrics should be related to algebro-geometric conditions of stability. This notion is called K-stability.

Conjecture (Yau–Tian–Donaldson)

A polarised projective manifold (X, L) *admits a cscK/extremal metric in* $c_1(L)$ *if and only if it is K-stable/relatively K-stable.*

The linearisation of the scalar curvature operator

The problem we consider is a perturbation problem. The linearisation of the equation is therefore key. The *Lichnerowicz operator* L_ω is given by

$$L_\omega(f) = \mathcal{D}_\omega^* \mathcal{D}_\omega(f)$$

where we recall

$$\mathcal{D}_\omega(f) = \bar{\partial}(\nabla_\omega^{1,0}(f)).$$

The kernel of L_ω are precisely the *holomorphy potentials*, i.e. the potentials for holomorphic vector fields. The linearisation of the scalar curvature is essentially this operator.

The Blowup Problem

The blowup in a point

Let X be a compact Kähler manifold, and let $p \in X$ be a point. We can then define the blowup

$$\pi : \mathrm{Bl}_p X \to X$$

of X in p. This is a manifold satisfying:

- It is isomorphic to $X \setminus \{p\}$ outside the preimage of p;
- The preimage of p via the blowdown map π is a copy E of $\mathbb{P}^{n-1}$, called the exceptional divisor.

The local model

The local model is the blowup $\mathrm{Bl}_0\,\mathbb{C}^n$ of $\mathbb{C}^n$ in the origin. This can be seen as the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$. In other words,

$$\begin{aligned}\mathrm{Bl}_0\,\mathbb{C}^n =& \{([z], v) : v = \lambda z \text{ for some } \lambda \in \mathbb{C}\} \\ \subset& \mathbb{P}^{n-1} \times \mathbb{C}^n\end{aligned}$$

The map to the second factor is an biholomorphism away from the origin. At the origin, the fibre is $\mathbb{P}^{n-1}$.

The Kähler cone of the blowup

First note that $H^2(\mathrm{Bl}_p X) \cong H^2(X) \oplus \langle [E] \rangle$. Moreover, if Ω is a Kähler class, then $\pi^*(\Omega)$ is on the boundary of the Kähler cone of $\mathrm{Bl}_p X$ – all subvarieties have non-negative volume, but the exceptional divisor has volume 0. So the class cannot be Kähler. However, if we allow the exceptional divisor to get some positive volume, we move into the Kähler cone of $\mathrm{Bl}_p X$. That is, for all $\varepsilon > 0$ sufficiently small, the class

$$\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$$

is a Kähler class on $\mathrm{Bl}_p X$.

The question we want to answer

The question we want to answer is the following: under what conditions on (X, Ω) and p does $(\mathrm{Bl}_p X, \Omega_\varepsilon)$ admit a cscK or extremal metric for all $\varepsilon > 0$ sufficiently small? Note, we are not trying to understand what happens for every value of ε such that Ω_ε is Kähler. We are only trying to understand what happens when ε is very small, i.e. when the volume of the exceptional divisor is very small.

Previous results

This question has a rich history of study, through:

- Works of Arezzo–Pacard, Arezzo–Pacard–Singer and Székelyhidi giving sufficient conditions for when the blowup does admit an extremal metric in these classes.
- Stoppa and Stoppa–Székelyhidi, on the other hand, investigated the algebro-geometric counterpart and provided necessary conditions.
- Seyyedali–Székelyhidi have also considered the case of blowing up higher dimensional subvarieties (see also work of Hashimoto).

In particular, Székelyhidi completely settled the question in the cscK case, in dimension at least 3.

First case – discrete automorphism group

The initial case considered by Arezzo–Pacard is:

- X admits a cscK metric in Ω;
- The automorphism group is discrete.

In this case, any point will do in the construction.

Theorem (Arezzo–Pacard)

Suppose X is as above. Let $p \in X$ be any point. Then there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{Bl}_p X$ admits a cscK metric in the class

$$\Omega_\varepsilon = \pi^* \Omega - \varepsilon [E].$$

What if the automorphism group is non-trivial?

The situation quickly gets more complicated in the presence of automorphisms, however. The simplest case in the presence of automorphisms is when blowing up a fixed point of a maximal torus in the reduced automorphism group of X. Here there is still no obstruction, provided one allows extremal metrics. The key is that all the relevant holomorphic vector fields lift and the extremal equation is unobstructed in this case.

What if the automorphism group is non-trivial?

Theorem (Arezzo–Pacard–Singer/Székelyhidi)

Suppose X admits an extremal metric in Ω. Let $p \in X$ be a point fixed under the action of a maximal torus in the reduced automorphism group of X. Then there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{Bl}_p X$ admits an extremal metric in the class

$$\Omega_\varepsilon = \pi^* \Omega - \varepsilon [E].$$

What if the automorphism group is non-trivial?

The issue comes when we want to blow up a point not fixed by a maximal torus. This is when we see start to see obstructions to solving the extremal equation. The core reason can be understood from the linear theory.

The reduced automorphism group of $\mathrm{Bl}_p X$ is the subgroup of the reduced automorphism group of X generated by the vector fields that vanish at the blown up point. When p is not fixed by the maximal torus, there are then real holomorphic vector fields on X that do not lift to $\mathrm{Bl}_p X$. Hence there will be a *discrepancy* between the mapping properties of the linearised operator on the blowup, and the initial extremal metric. Analytically, this is the core source of the obstruction to solve the extremal equation.

What if the automorphism group is non-trivial?

The goal is then to obtain a more geometric understanding of this obstruction. Székelyhidi's approach allowed him to relate the conditions on the point to K-stability. In the cscK case, when X has dimension at least 3, he showed that the blowup admits a cscK metric if and only if the manifold is K-polystable. Moreover he gave a finite dimensional GIT condition that captures precisely what is needed to check K-polystability in this case. The main result of the work with Dervan is an extension of this to the non-cscK case, and dimension 2, as well as to certain manifolds that themselves initially do *not* admit an extremal metric.

The main result

Ultimately, we prove the following.

Theorem (Dervan–S. '21)

Suppose X is analytically relatively K-semistable. Let $p \in X$. Let $\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$. Then the following are equivalent:

1. $\mathrm{Bl}_p X$ *admits an extremal metric in* Ω_ϵ *for all* $0 < \varepsilon \ll 1$*;*
2. $(\mathrm{Bl}_p X, \Omega_\epsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$*.*

Moreover, the relative K-stability criterion is an explicit finite dimensional condition.

An important consequence

Conjecture (Donaldson)

Let X be any Kähler manifold. Then there exists a collection of points $p_1, \ldots, p_k \in X$ such that $\mathrm{Bl}_{p_1,\ldots p_k} X$ admits an extremal metric.

We show that for analytically semistable manifolds, one can always find a point in whose blowup we get an extremal metric. The fact that we can deal with the strictly semistable case means that we solve the above conjecture for those manifolds, by picking one carefully chosen point.

Székelyhidi's approach

Székelyhidi's proof

We will now outline Székelyhidi's approach. We begin with the case of blowing up a fixed point of the action of a maximal torus, which is unobstructed. We then explain briefly why the general case is obstructed.

Székelyhidi's proof

Consider $\mathrm{Bl}_p X$ as the union of two non-compact manifolds:

- $X \setminus \{p\}$;
- the preimage $\pi^{-1}(D)$ of a large disk $D \subset \mathbb{C}^n$ via the blowdown map $\mathrm{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$.

We will equip

- $X \setminus \{p\}$ with an extremal metric ω from X invariant under the corresponding real maximal torus;
- $\mathrm{Bl}_0 \mathbb{C}^n$ with a certain asymptotically flat, scalar-flat metric η, the *Burns–Simanca metric*.

Over an annular region (whose size depends on ε), we will then interpolate between the two metrics at the level of Kähler potentials to create an initial approximate solution to the equation.

Székelyhidi's proof

This gluing is performed as follows.

- pick holomorphic normal coordinates $(z_1, \ldots, z_n)$ at p in which the torus action becomes a usual linear action;
- Let $(w_1, \ldots, w_n)$ be coordinates on $\mathrm{Bl}_0 \mathbb{C}^n$ away from the exceptional divisor;
- Glue the two coordinate systems via the coordinate change

$$w = \varepsilon^{-1} z.$$

Székelyhidi's proof

Around p,

$$\omega = i\partial\bar{\partial}\left(|z|^2 + \phi(|z|)\right),$$

for a function $\phi = O(|z|^4)$. Similarly, the Burns–Simanca metric η satisfies that

$$\varepsilon^2\eta = i\partial\bar{\partial}\left(|z|^2 + \varepsilon^2\psi(\varepsilon^{-1}z)\right)$$

under the coordinate change $w = \varepsilon^{-1}z$.

Székelyhidi's proof

We then interpolate between the two metrics on the level of potentials over an annular region

$$D_{2r_\varepsilon} \setminus D_{r_\varepsilon} = \{z : r_\varepsilon < |z| \leq 2r_\varepsilon\},$$

Then we define ω_ε to be

- ω on $X \setminus D_{2r_\varepsilon}$;
- $\varepsilon^2\eta$ on $\pi^{-1}(D_{r_\varepsilon})$;
- $i\partial\bar\partial \left(|z|^2 + \chi(z)\phi(z) + \varepsilon^2(1-\chi)(z)\psi(\varepsilon^{-1}z)\right)$ on $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$.

Here χ is a cutoff function, depending on ε, supported in $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$.

Székelyhidi's proof

ω_ε is Kähler when ε is sufficiently small. The final steps are then:

- Obtain a uniform bound for the scalar curvature, showing it is approximately extremal on the blowup;
- Establish a uniform bound for the (right) inverse to the linearised operator;
- Apply the contraction mapping theorem to perturb to an actual solution.

This produces extremal metrics in the case of blowing up a fixed point of the torus action.

Blowing up a point that is not fixed

When the point is not a fixed point of the maximal torus, not all vector fields will lift. Székelyhidi defines a lift

$$l : \overline{\mathfrak{h}} \to C^{\infty}(\mathrm{Bl}_p X)$$

of the corresponding holomorphy potentials by using cut-off functions. These are no longer holomorphy potentials on the blowup. The difference then comes in the second step. This only gives a uniform bound orthogonally to $l(\overline{\mathfrak{h}})$. Since not all these functions are holomorphy potentials, this means that we are solving a more general equation than the extremal equation, and the construction is therefore obstructed.

Understanding the obstruction

Székelyhidi then proceeds to understand this obstruction in terms of K-stability. To get precise information on the relation to K-stability and a finite-dimensional GIT condition, a better approximate solution with higher matching with the Burns–Simanca metric is required. The need to have this higher order matching is why Székelyhidi's results are restricted to the cscK case and dimension at least 3.

The new approach

The new approach

We now explain the core new idea in the work of Dervan and the author. We work with a fixed *symplectic* manifold in the gluing process, and let the complex structure vary instead. When blowing up a complex manifold, the complex structure will depend on the point chosen. On the other hand, in the symplectic category, we have no dependence on the point. As a smooth manifold, the blowup $\mathrm{Bl}_p M$ is $X \# \overline{\mathbb{CP}^n}$. We take the symplectic form to only depend on ε, not the point. The *almost complex structure* will be what is changing when we do this.

The new approach

In order to this, we construct a symplectomorphism $f : M \to M$ sending a fixed point p to the point q we want to blowup. This uses Moser's trick. We let $J_q = f^*J$. The blowup of (M, J_q) in p will then be isomorphic to the blowup of X in q. Since f is a symplectomorphism, we even have that

$$(M, J_q, \omega) \cong (X, \omega) = (M, J, \omega)$$

as Kähler manifolds.

The new approach

To achieve this, we consider the local model of blowing up the origin in $\mathbb{C}^n$ with its Euclidean symplectic form. The flow generated by the real vector field ν corresponding to

$$\sum_{j=1}^{r} \lambda_j \frac{\partial}{\partial z^j}$$

sends the origin to $q = (\lambda_1, \ldots, \lambda_n)$. This vector field is dual via the Euclidean symplectic form to

$$d(\sum_{j=1}^{r} \lambda_j |z^j|^2).$$

Using cut-off functions, we can globalise this to an exact form β generating a Hamiltonian symplectomorphism $f : M \to M$.

The new approach

The advantage is that in the construction we can now take the point of view that we are blowing up a fixed symplectic manifold (M, ω) in one given point p. The almost complex structure is what changes, not the point.

The new approach

We can also let the complex structure vary even before changing the point, and this allows us to tackle also certain strictly *semistable* manifolds. That is, we can allow X to be *analytically relatively K-semistable*. This means it admits a degeneration, invariant with respect to the reduced automorphism group, to an extremal central fibre.

The new approach

Thus we have a map

$$\Psi : B \to \mathcal{J}(M, \omega),$$

the space of ω-compatible AC structures, where B is an open ball in some vector space. This parametrises the isomorphism classes of complex structures near the extremal central fibre X_0 in the Kuranishi model and all nearby points to a point p fixed by a maximal torus in the reduced automorphism group of the central fibre.

The new approach

$$\Psi : B \to \mathcal{J}(M, \omega),$$

By work of Székelyhidi and Brönnle, we can ensure that the scalar curvature lands in the space $\overline{\mathfrak{h}}$ of holomorphy potentials on the central fibres. I.e. we have

$$S(J_b, \omega) \in \overline{\mathfrak{h}}$$

for all $b \in B$, where $J_b = \Psi(b)$.

Dividing up the problem

Next, we divide the problem up into two more manageable bits.

- We first solve a more general equation on the blowup than the extremal equation;
- Then we analyse when this more general equation actually is a solution to the extremal equation.

This is a strategy going back to at least to ideas of Donaldson, which is a very useful general principle in obstructed perturbation problems.

The lifts

Working with the symplectic manifold $(\mathrm{Bl}_p M, \omega_\varepsilon)$, even vector fields that do not correspond to holomorphy potentials on the non-zero fibres have a natural lift now – we just use the lift on the central fibre. For any function $h \in \overline{\mathfrak{h}}$ of average 0 with respect to ω, we have a naturally defined lift

$$h_\varepsilon \in C^\infty(\mathrm{Bl}_p M),$$

the Hamiltonian of average 0 with respect to ω_ε of the lift of the corresponding vector field to $\mathrm{Bl}_p X_0$. In the non-zero fibres not all of these functions are holomorphy potentials – they are Hamiltonians on the blowup, but they do not all produce a holomorphic vector field.

The equation we solve

Let $\overline{\mathfrak{h}}_\varepsilon$ denote the space of such lifted potentials, with respect to ω_ε. **Goal:** Construct an AC structure $J_{\varepsilon,\boldsymbol{q}}$ such that

$$S(\omega_\varepsilon, J_{\varepsilon,\boldsymbol{q}}) \in \overline{\mathfrak{h}}_\varepsilon.$$

It is only after solving this equation that we analyse the relation to K-stability. To do so, we perform the same construction as on the central fibre, on the non-zero fibres.

The equation we solve

$$S(\omega_\varepsilon, J_{\varepsilon,q}) \in \overline{\mathfrak{h}}_\varepsilon.$$

In fact, on the non-zero fibres, the scalar curvature can be viewed as a perturbation of that of the central fibre. This allows us to rely on many of the bounds/mapping properties already established by Székelyhidi. Through a couple of iterations of Moser's trick, we can then follow Székelyhidi's strategy to produce solutions to the above equation.

Note: $\overline{\mathfrak{h}}_\varepsilon$ corresponds to holomorphy potentials on the *central fibre*. So we have not necessarily solved the extremal equation on the non-zero fibres yet. We turn to understanding when we have done this now.

Relating the construction to K-stability

Relating the construction to K-stability

We have parametrised nearby points to p and nearby complex structures to J by a ball B in a vector space. So far, we have solved

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon.$$

We now want to understand when we can have

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_{\varepsilon,b} \subseteq \overline{\mathfrak{h}}_\varepsilon,$$

the space of holomorphy potentials for $J_{\varepsilon,b}$.

Relating the construction to K-stability

The key to achieving this is that we can view this as looking for a zero, or a critical point, of a certain moment map. The reason we can do this is that we can view the map

$$b \mapsto J_{\varepsilon,b}$$

as giving a symplectic embedding into the space $\mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$ of ω_ε-compatible almost complex structures.

Proposition

The image of the map $B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$ given by

$$b \mapsto J_{\varepsilon,b}$$

is a symplectic submanifold.

Relating the construction to K-stability

The scalar curvature is a moment map for the action of the Hamiltonian symplectomorphism group on $\mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$. Hence its composition with the orthogonal projection to $\overline{\mathfrak{h}}_\varepsilon$ is a moment map for the restriction to T of this action. This then also holds on the symplectic submanifold given by the image of the embedding

$$B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon).$$

Thus we have managed to put ourselves in the position that solving the cscK/extremal equation becomes a finite dimensional moment map problem, which can then be related to a GIT notion of stability/relative stability.

Relating the construction to K-stability

- The functions in $\bar{\mathfrak{h}}$ induce test configurations for the blowup (Székelyhidi);
- Their Donaldson–Futaki invariant is precisely the value via the moment map above;
- Upshot: the value of the moment map can be realised as the algebro-geometric quantity we wish to relate the construction to;
- Analysing the Hamiltonian functions produces another point b' in the same orbit as b where the value of all the Hamiltonians orthogonal to those that are holomorphy potentials vanish (technique due to Dervan);
- Thus, assuming relative K-stability, we get the scalar curvature lies in the space of holomorphy potentials at the complex structure over b'.

Thus we have produced an extremal metric under the assumption of relative K-stability, the main result.

Relating the construction to K-stability

Note that we also obtain explicit expansions that give the exact criterion for stability in this case. This follows from an explicit expansion of the Donaldson–Futaki invariant. This uses similar ideas to the analogous statements in Székelyhidi's work on blowups.

Thank you!