Looking at the Euler flows through a contact mirror

Seminar 'Geometria em Lisboa'

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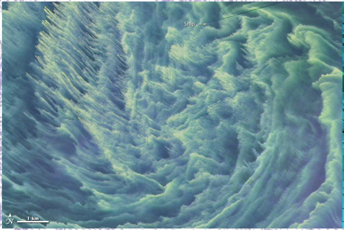
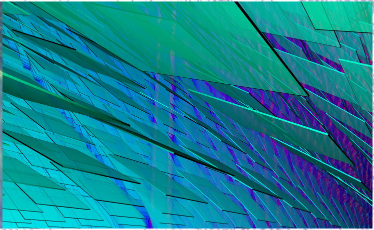
UPC-CRM



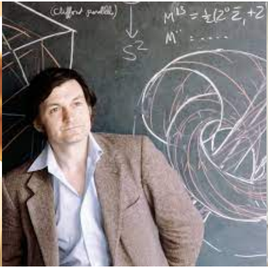
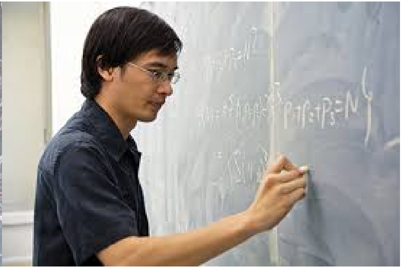


Computational complexity and Fluid Dynamics

In Nature fluids (seas or volcano's lava) often rebel against what is expected....

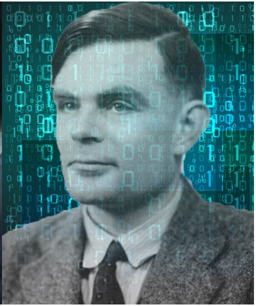
Fluid Computers?

Are fluids *"complicated"* enough to create a Fluid Computer?

Levels of complexity and Alan Turing

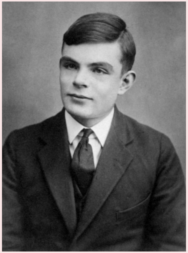
Is the complexity of fluids enough to simulate any Turing machine?

Turing machines and the halting problem

In computability theory, the halting problem is the problem of determining, from a description of an arbitrary computer program and an input, whether the program will finish running (halting state), or continue to run forever.

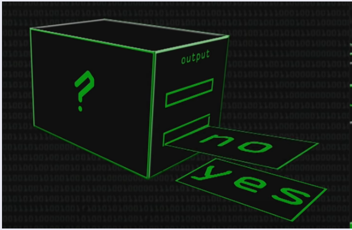
Turing, 1936: The halting problem is **undecidable**.

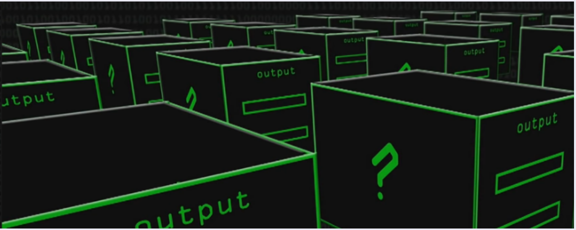


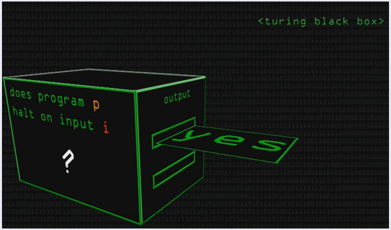
Alan Turing proved in 1936 that a general algorithm to solve the halting problem for all possible program-input pairs cannot exist.

Proof by picture

Proof.







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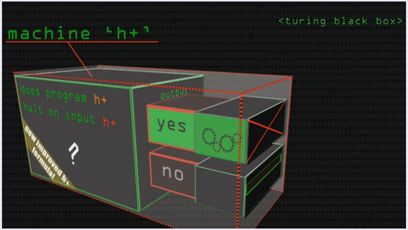
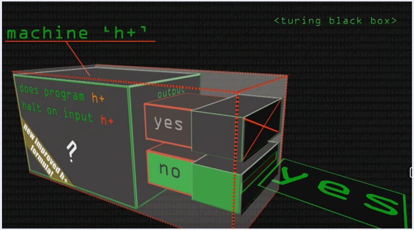
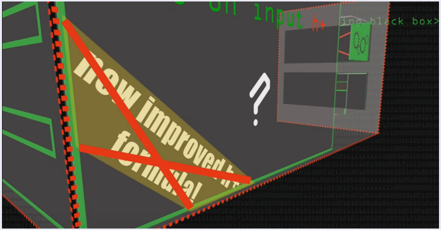
Proof by picture

Proof.







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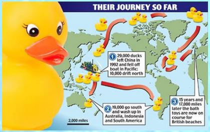
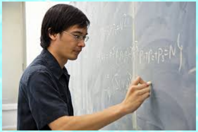
Chronology



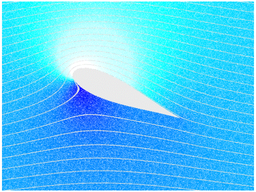


- 1991, Moore: Is hydrodynamics capable of performing computations?
- January 10, 1992: 29000 rubber ducks were lost in the ocean.
- July 2007: One rubber duck show ups in Scotland.
- July 2017: Tao asks about universality of Euler flows.
- November 2019: (Cardona--M.--Peralta-Salas --Presas) Steady Euler flows are universal.
- December 2020: (Cardona--M.--Peralta-Salas-- Presas) There exist stationary Turing complete Euler flows in dimension 3.
- April 2021: (Cardona--M.--Peralta-Salas) There exist time-dependent Euler flows which are Turing complete in high dimension.
- November 2021:(Cardona--M.--Peralta-Salas) Euclidean case.

Joint work with Cardona, Peralta-Salas and Presas

- R. Cardona, E. Miranda, D. Peralta-Salas, F. Presas. *Universality of Euler flows and flexibility of Reeb embeddings.* arXiv:1911.01963.
- R. Cardona, E. Miranda, D. Peralta-Salas, F. Presas. *Constructing Turing complete Euler flows in dimension* 3. PNAS, Proceedings of the National Academy of Sciences, 2021, 118 (19) e2026818118.
- R. Cardona, E. Miranda, D. Peralta-Salas, *Turing universality of the incompressible Euler equations and a conjecture of Moore.* International Mathematics Research Notices, 2021, rnab233, https://doi.org/10.1093/imrn/rnab233.
- R. Cardona, E. Miranda, D. Peralta-Salas, *Computability and Beltrami fields in Euclidean space.* arXiv:2111.03559.

Incompressible fluids on Riemannian manifolds



Classical Euler equations on $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial X}{\partial t} + (X \cdot \nabla)X = -\nabla P \\ \operatorname{div} X = 0 \end{cases}$$

The evolution of an **inviscid and incompressible fluid flow** on a Riemannian n-dimensional manifold (M, g) is described by the Euler equations:

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla P\,, \qquad \operatorname{div} X = 0$$

- X is the velocity field of the fluid: a non-autonomous vector field on M.
- P is the inner pressure of the fluid: a time-dependent scalar function on M.

Incompressible fluids on Riemannian manifolds

If X does not depend on time, it is a steady or stationary Euler flow: it models a fluid flow in equilibrium. The equations can be written as:

$$\nabla_X X = -\nabla P\,, \qquad \operatorname{div} X = 0\,,$$

$$\iff \iota_X d\alpha = -dB\,, \qquad d\iota_X \mu = 0\,, \qquad \alpha(\cdot) := g(X, \cdot)$$

where $B := P + \frac{1}{2}||X||^2$ is the Bernoulli function.

Beltrami fields:

$\operatorname{curl} X = fX,$ with $f \in C^\infty(M) \qquad \operatorname{div} X = 0.$

Example (Hopf fields on S^3 and ABC fields on T^3)

- The Hopf fields $u_1 = (-y, x, \xi, -z)$ and $u_2 = (-y, x, -\xi, z)$ are Beltrami fields on S^3.
- The ABC flows
  $(\dot{x}, \dot{y}, \dot{z}) = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$,
  $((x, y, z) \in (\mathbb{R}/2\pi\mathbb{Z})^3)$ are Beltrami.

The Millennium problem list

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Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

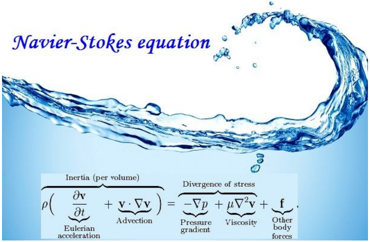
Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

The Navier-Stokes problem: Existence of global smooth solutions



The *Navier–Stokes* equations are then given by

$$(1)\qquad \frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu\Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \geq 0),$$

$$(2)\qquad \operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \geq 0)$$

with initial conditions

$$(3)\qquad u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n).$$

Here, $u^\circ(x)$ is a given, C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given, externally applied force (e.g. gravity), ν is a positive coefficient (the viscosity), and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables. The *Euler equations* are equations (1), (2), (3) with ν set equal to zero.

$$(4)\qquad |\partial_x^\alpha u^\circ(x)| \leq C_{\alpha K}(1+|x|)^{-K} \quad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K$$

and

$$(5)\qquad |\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^n \times [0,\infty), \text{ for any } \alpha, m, K$$

We accept a solution of (1), (2), (3) as physically reasonable only if it satisfies

$$(6)\qquad p, u \in C^\infty(\mathbb{R}^n \times [0,\infty))$$

and

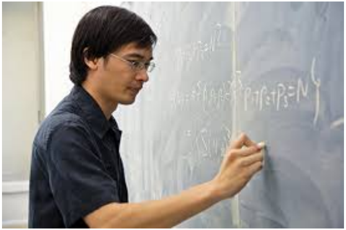
$$(7)\qquad \int_{\mathbb{R}^n} |u(x,t)|^2 dx < C \quad \text{for all } t \geq 0 \quad \text{(bounded energy)}.$$

The Navier-Stokes problem

(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

Tao's program for universality of Euler flows



Can any incompressible vector field be embedded as an Euler flow by increasing the dimension of the manifold?

Tao's program and blow-up

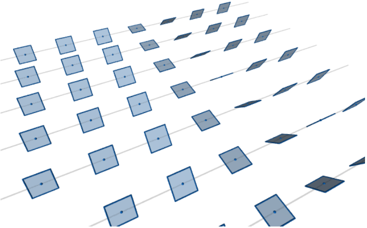
Is it possible to construct Turing complete fluid flows? Motivation: to create an initial datum that is "programmed" to evolve to a rescaled version of itself (as a Von Neumann self-replicating machine). Can this be applied to prove blow-up of Navier-Stokes?

Can you spot 7 differences?

Figure: Barcelona and Paris in December

Maybe yes!.... but if you are far enough you will not find them....

All dynamics represented by an Euler flow

Symplectic	**Contact**
$\dim M = 2n$	$\dim M = 2n + 1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Darboux theorem $\omega = \sum_{i=1}^{2n} dx_i \wedge dy_i$	$\alpha = dx_0 - \sum_{i=1}^{2n} x_i dy_i$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

An example

The kernel of a 1-form α on M^{2n+1} is a contact structure whenever $\alpha \wedge (d\alpha)^n$ is a volume form $\Leftrightarrow d\alpha|_\xi$ is non-degenerate.

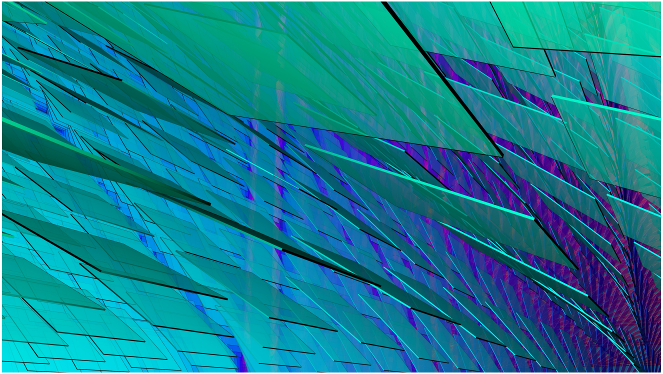


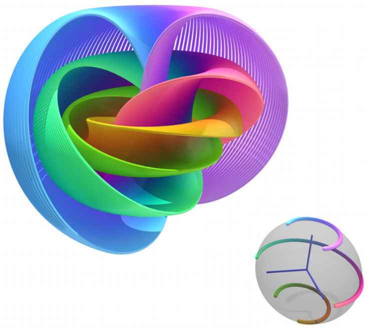
Figure: Standard contact structure on $\mathbb{R}^3$

$$\alpha = dz - y dx \quad \xi = \ker \alpha = \left\langle \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle \; d\alpha = -dy \wedge dx = dx \wedge dy$$

$$\Rightarrow \; \alpha \wedge d\alpha \; = \; dx \wedge dy \wedge dz$$

The Hopf fibration revisited

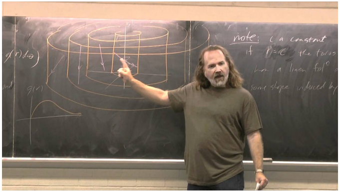
- $S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \alpha = \frac{1}{2}(ud\overline{u} - \overline{u}du + vd\overline{v} - \overline{v}dv)$.
  The orbits of the Reeb vector field form the Hopf fibration!
  $$R_\alpha = iu\frac{\partial}{\partial u} - i\overline{u}\frac{\partial}{\partial \overline{u}} + iv\frac{\partial}{\partial v} - i\overline{v}\frac{\partial}{\partial \overline{v}}$$
- $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ can be endowed with Hopf coordinates $(z_1, z_2) = (\cos s \exp i\phi_1, \sin s \exp i\phi_2)$, $s \in [0, \pi/2]$, $\phi_{1,2} \in [0, 2\pi)$. The Hopf field $R := \partial_{\phi_1} + \partial_{\phi_2}$ is a steady Euler flow (Beltrami) with respect to the round metric.



The magic mirror

In terms of $\alpha = \iota_X g$ and μ (volume form) the **stationary Euler equations** read

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$



- Etnyre-Ghrist:
  {Rotational non singular Beltrami v.f.} $\rightleftarrows$ {Reeb v.f. reparametrized}
- With Cardona and Peralta-Salas we have extended this picture to manifolds with cylindrical ends to get singular contact structures.
- CMPP: The Beltrami/contact correspondence works in higher dimensions.

Let's prove it!

- The Beltrami equation $\leftrightsquigarrow$ $d\alpha = f\iota_X\mu$. Since $f > 0$ and X does not vanish $\rightsquigarrow$ $\alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$.
- X satisfies $\iota_X(d\alpha) = f\iota_X\iota_X\mu = 0$ so $X \in \ker d\alpha$ $\leftrightsquigarrow$ it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.

A magic mirror

- Weinstein conjecture for Reeb vector fields $\rightsquigarrow$ **periodic orbits for Beltrami vector fields**
- h-principle $\rightsquigarrow$ Reeb embeddings $\rightsquigarrow$ **universality of Euler flows** (Cardona–M–Peralta-Salas–Presas)
- Reeb suspension of area preserving diffeomorphism of the disc $\rightsquigarrow$ **Construction of universal 3D Turing machine** (Cardona–M–Peralta-Salas–Presas)
- Uhlenbeck's genericity properties of eigenfunctions of Laplacian $\rightsquigarrow$ **existence of singular periodic orbits** (M–Oms–Peralta-Salas)

Geometrical approach to Tao's question

(Tao) Can any incompressible vector field be embedded as an Euler flow in higher dimensions?

When the Euler vector field is Beltrami we can use our magic mirror

Beltrami vector field  **Reeb vector field**

(CMPP) Can we realize a vector field on a manifold N as a Reeb vector field on a bigger compact contact manifold?

- **Necessary condition**: X geodesible in N $\Longleftrightarrow$ preserves transverse hyperplane distribution.

  A geodesible vector field is a vector field for which there is a Riemannian metric g on M such that the orbits of X are geodesics of unit length. $\leftrightsquigarrow$ there exists a 1-form α such that $\alpha(X) = 1$ and $\iota_X d\alpha = 0$.
- **Question 1**: Is this condition sufficient?
- **Question 2**: How hard is to get a geodesible vector field?

Geodesible vector fields

Let us answer question 2.

Lemma

The suspension of a t-periodic vector field $X(p,t)$ is geodesible.

Consider $N \times S^1$, the vector field

$$Y(p,\theta) = (X(p,\theta), \frac{\partial}{\partial \theta}).$$

is geodesible (Take $\alpha = d\theta$. It satisfies $\iota_Y \alpha = 1$ and $\iota_Y d\alpha = 0$.)

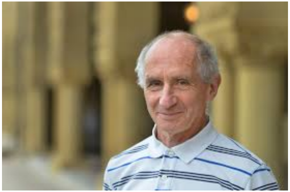
Variation in tactics: Flexibility

Inspirational: All 3-dimensional manifolds are contact (Martinet-Lutz) and in higher dimensions:

Theorem (Borman-Eliashberg-Murphy)

Any almost contact closed manifold is contact.

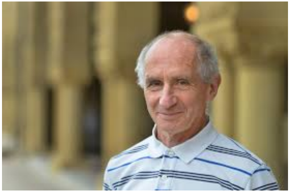
The almost contact condition is a formal condition and h-principle is the key ingredient of the proof.

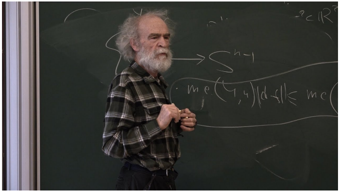


The h-principle

The philosophy of the h-principle:

- Goal: Solve an equation (PDE, partial differential relation...).
- Semigoal: Solve a linearized/formal equation.
- Miracle: Prove that there exist an homotopy that allows to deform a formal solution to a honest solution (one needs to be lucky for that: the system is undetermined or has high codimension).





Our h-principle scenario

- Goal: AN EMBEDDING $f : (N, X, \eta = \ker \beta) \to (M, \xi)$ IS AN ISO-REEB EMBEDDING IF $f_*(\eta) = \xi$.

(Inaba) Let $i : N \subset (M, \xi)$ denote the inclusion and η be the restriction $i^*\xi$. A non-vanishing vector field X on N can be extended to a Reeb field on $M \Leftrightarrow X \pitchfork \eta$ and the flow of X preserves η.

- Semigoal: AN EMBEDDING $f : (N, X, \eta) \to (M, \xi)$ IS A FORMAL ISO-REEB EMBEDDING IF THERE EXISTS A HOMOTOPY OF MONOMORPHISMS $F_t : TN \longrightarrow TM$, SUCH THAT F_t COVERS f, $F_0 = df$, $h_1 \alpha \circ F_1 = \beta$ AND $d\beta|_\eta = h_2 d\alpha \circ F_1|_\eta$ FOR STRICTLY POSITIVE h_1 AND h_2 ON N. ($\alpha \circ F_1$ for $\alpha(F_1(\cdot))$ and $d\alpha \circ F_1$ for $d\alpha(F_1(\cdot), F_1(\cdot))$)
- Miracle: In the overtwisted case: h-principle for codimension 0 iso-contact embeddings into overtwisted targets (BEM).

(CMPP) Let $e : (N, X, \eta) \to (M, \xi)$ be an embedding such that there is an homotopy of monomorphisms $F_t : TN \to TM$ covering e satisfying $F_0 = de$ and $F_1(\eta)$ is an isotropic subbundle of ξ. Then e is a formal iso-Reeb embedding.

Vanishing of obstructions: Diagram chasing.

Short comprehensive dictionary

- A volume-preserving (autonomous) vector field u on M is Eulerisable if there exists a Riemannian metric g on M compatible with the volume form, such that u satisfies the stationary Euler equation on (M, g)

$$\nabla_u u = -\nabla P\,, \qquad \operatorname{div} u = 0\,. \tag{1}$$

- A non-autonomous time-periodic vector field $u_0(x,t)$ on a compact manifold M is Euler-extendible if there exists an embedding $e : M \times \mathbb{S}^1 \to \mathbb{S}^n$ for some (high-enough) dimension n and an Eulerisable flow u on $\mathbb{S}^n$, such that $e(M \times \mathbb{S}^1)$ is an invariant submanifold of u and $u = e_*(u_0(x,\theta) + \partial_\theta)$, $\theta \in \mathbb{S}^1$.

Theorem 1: Universality

Theorem 1 (Cardona, M., Peralta-Salas & Presas)

The Euler flows are universal. The dimension of the ambient manifold $\mathbb{S}^n$ or $\mathbb{R}^n$ is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$.
In the time-periodic case, the extended field u is a steady Euler flow with a metric $g = g_0 + g_P$, where g_0 is the canonical metric on $\mathbb{S}^n$ and g_P is supported in a ball that contains the invariant submanifold $e(N \times \mathbb{S}^1)$.

Proof:

- **Step 1:** Using the correspondence between Beltrami flows and Reeb vector fields we reduce the problem to studying the universality of high-dimensional Reeb flows.

- **Step 2:** The Reeb flows are geodesible. The converse also holds: Any geodesible flow is Reeb-extendible.

Key steps in the proof II

Theorem (Cardona,M.,Peralta-Salas,Presas)

Let N be a compact manifold and X a geodesible field. Then there is a smooth embedding $e : N \to \mathbb{S}^n$ n the smallest odd integer $n \in \{3\,\text{dim}\ N + 2, 3\,\text{dim}\ N + 3\}$ and a 1-form α defining the standard contact structure ξ_{std} on $\mathbb{S}^n$ such that $e(N)$ is an invariant submanifold of the Reeb field R and $e_*X = R$. Moreover α equals α_{std} in the complement of a ball containing $e(N)$.

- Embed (N, X) on an open contact manifold of dim $2n - 1 \rightsquigarrow$ (Gromov) on contact $(\mathbb{S}^{4\,\text{dim}\ N-1}, \xi_{st})$.
- Use more sophisticated techniques in contact topology to prove an h-principle for Reeb embeddings and sharpen the dimension:

Theorem (Reeb embeddings)

Let $e : (N, X) \to (M, \xi)$ be an embedding of N into a contact manifold (M, ξ),

- *If $\dim M \geqslant 3 \dim N + 2$, then e is isotopic to a Reeb embedding $\tilde{e}$, and $\tilde{e}$ can be taken C^0-close to e.*
- *If $\dim M \geqslant 3 \dim N$ and M is overtwisted, then e is isotopic to a Reeb embedding.*

Final step

To end the proof we use that suspensions are always geodesible.

Proof.

- By the previous theorem applied to an $n+1$-dimensional manifold, there exist a Reeb embedding $e : N \to \mathbb{S}^n$ extending Y with n the smallest odd integer $n \in \{3\,\text{dim } N + 5, 3\,\text{dim } N + 6\}$.
- By the contact-Beltrami correspondence theorem, there are a metric and a volume making Y a steady solution of the Euler equations.

□

Theorem 2: Turing complete Euler flows

Turing completeness

A vector field on M is Turing complete if it can simulate any Turing machine $\leftrightsquigarrow$ The halting of any Turing machine with a given input is equivalent to a certain bounded trajectory of the field entering a certain open set of M (precise definition later).

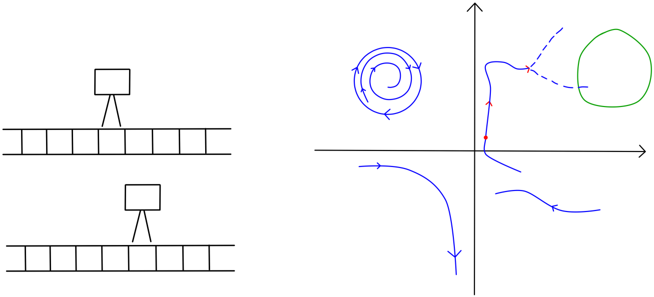
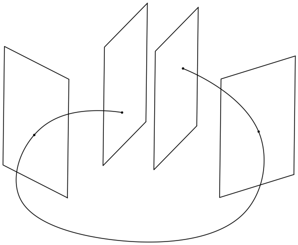


Figure: Turing machine and Turing complete vector field associated to a point and an open set.

Turing completeness of the Euler flows

Theorem 2 (Cardona, M., Peralta-Salas, Presas)

There exists a Turing complete Eulerisable flow on $\mathbb{S}^{17}$. This flow is Beltrami with constant proportionality factor.



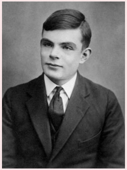
Key ingredients of the proof: There exists an orientation-preserving diffeomorphism ϕ of $\mathbb{T}^4$ encoding a universal Turing machine (Tao) and the h-principle given by holonomic approximation is algorithmic.
Remark: Using an overtwisted disk and the h-principle in BEM we can even reduce to $\mathbb{S}^{15}$.

Can we do it better?

Theorem 3 (Cardona, M., Peralta-Salas & Presas)

There exists an Eulerisable flow X in $\mathbb{S}^3$ that is Turing complete. The metric g that makes X a stationary solution of the Euler equations can be assumed to be the round metric in the complement of an embedded solid torus.

Turing, 1936: The halting problem is **undecidable**.



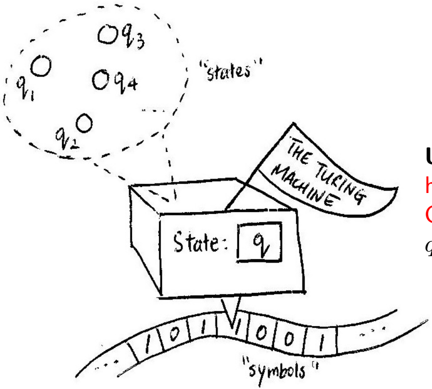
Corollary

There exist undecidable fluid particle paths: there is no algorithm to decide whether a trajectory will enter an open set or not in finite time.

Turing machines

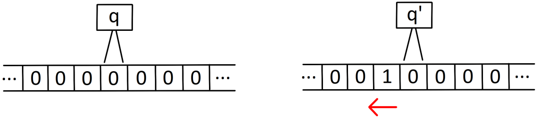
Turing machine

A Turing machine is defined as $T = (Q, q_0, q_{halt}, \Sigma, \delta)$, where Q is a finite set of states, including an initial state q_0 and a halting state q_{halt}, Σ is the alphabet, and $\delta : (Q \times \Sigma) \longrightarrow (Q \times \Sigma \times \{-1, 0, 1\})$ is the transition function. The input of a Turing machine is the current state $q \in Q$ and the current tape $t = (t_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.



User's guide: If the current state is q_{halt} then halt the algorithm and return t as output. Otherwise compute $\delta(q, t_0) = (q', t_0', \varepsilon)$, replace q with q', t_0 with t_0' and t by the ε-shifted tape.

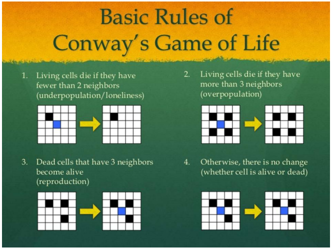
Example



Example: $\delta(q, 0) = (q', 1, +1)$, we replace 0 by 1, the new state is q' and we shift the tape to the left.

See also the video:

Conway’s game of life



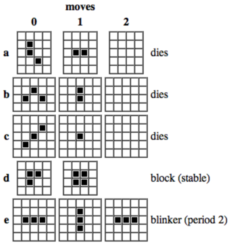
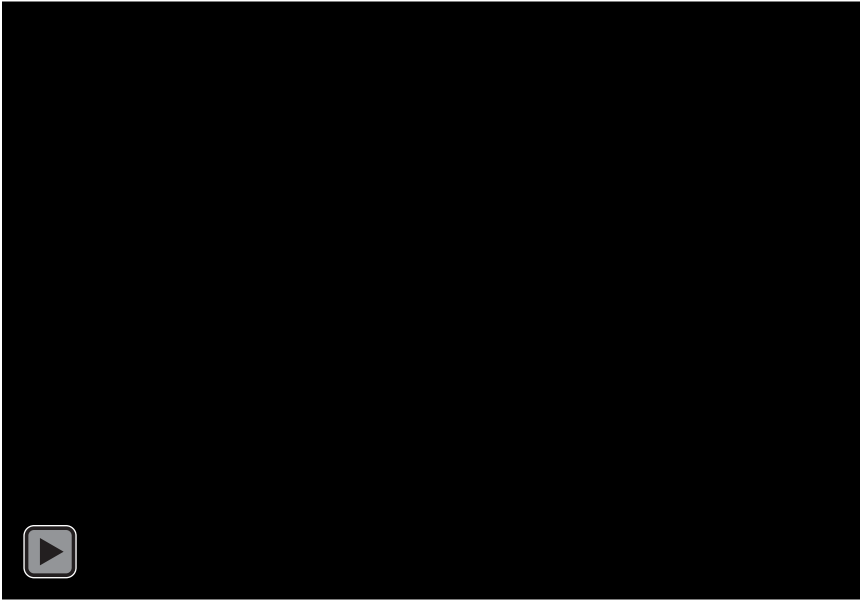


Figure: John von Neumann: every Turing machine has a cellular automaton which simulates it.

Conway’s game of life on a torus:

A simple dynamical system: The cantor set



Dynamical systems simulating Turing machines

A vector field X on $\mathbb{S}^3$ is Turing complete if

for any integer $k \geqslant 0$, given a Turing machine T, an input tape t, and a finite string $(t^*_{-k}, ..., t^*_k)$ of symbols of the alphabet, there exist an explicitly constructible point $p \in \mathbb{S}^3$ and an open set $U \subset \mathbb{S}^3$ such that the orbit of X through p intersects U if and only if T halts with an output tape whose positions $-k, ..., k$ correspond to the symbols $t^*_{-k}, ..., t^*_k$.

- Moore generalized the notion of shift to simulate any Turing machine (Generalized shifts).
- Generalized shifts are conjugated to maps of the square Cantor set $C^2 \subset I^2$.

A symbolic dynamics tool: generalized shifts

Generalized shifts

Let A be an alphabet. A generalized shift $\phi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is specified by two maps F and G. Denote by $D_F = \{i, ..., i + r - 1\}$ and $D_G = \{j, ..., j + \ell - 1\}$ the sets of positions on which F and G depend. The function G modifies the sequence only at the positions indicated by D_G:

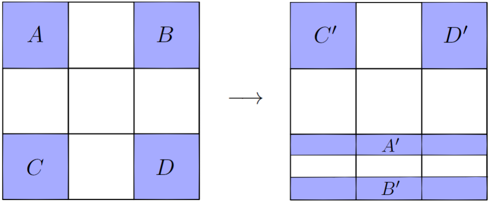
$$G : A^\ell \longrightarrow A^\ell$$
$$(s_j ... s_{j+\ell-1}) \longmapsto (s'_j ... s'_{j+\ell-1})$$

On the other hand, the function F assigns to the finite subsequence $(s_i, ..., s_{i+r-1})$ of the infinite sequence $S \in A^{\mathbb{Z}}$ an integer $F : A^r \longrightarrow \mathbb{Z}$. ϕ is defined as:

- Compute $F(S)$ and $G(S)$.
- Modify S changing the positions in D_G by the function $G(S)$, obtaining a new sequence S'.
- Shift S' by $F(S)$ positions.

Dynamical systems simulating Turing machines

- **Point assignment**: Take $\Sigma = \{0,1\}$. Given $s = (...s_{-1}.s_0 s_1...) \in \Sigma^{\mathbb{Z}}$, we can associate to it an explicitly constructible point in C^2. Express the coordinates of the point in base 3: The coordinate x corresponds to the expansion $(x_1, x_2, ...)$ in base 3 where $x_i = 0$ if $s_{-i} = 0$ and $x_i = 2$ if $s_{-i} = 1$. The same for y.
- Each linear component is the composition of a translation and a power of the horseshoe map.

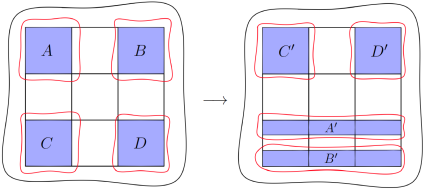


A Turing complete diffeomorphism of the disk

Proposition (Cardona, Miranda, Peralta-Salas & Presas)

For each bijective generalized shift and its associated map of the square Cantor set ϕ, there exists an area-preserving diffeomorphism of the disk $\varphi : D \to D$ which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ.

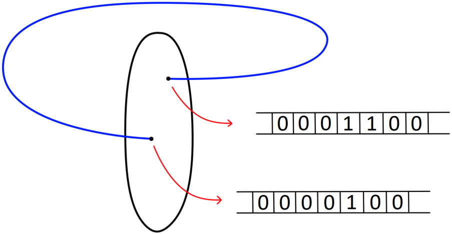
Idea of the proof: Extend a construction by Moore which is a piecewise linear map using disjoint blocks containing all the Cantor set:



The magic mirror again

Theorem (Cardona, M., Peralta-Salas & Presas)

Let (M, ξ) be a contact 3-manifold and $\varphi : D \to D$ an area-preserving diffeomorphism of the disk which is the identity (in a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ.



The contact form α can be fixed in the complement of a toroidal domain.
By means of the Reeb-Beltrami mirror this proves Theorem 3.

Does this give finite-time blow-up for Navier-Stokes?

Short answer: No

Long answer: On a Riemannian 3-manifold (M, g) the Navier-Stokes read as

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u - \nu \Delta u = -\nabla p \,, \\ \operatorname{div} u = 0 \,, \\ u(t=0) = u_0 \,, \end{cases} \tag{2}$$

where $\nu > 0$ is the viscosity.

- Δ is the Hodge Laplacian (whose action on a vector field is defined as $\Delta u := (\Delta u^\flat)^\sharp$).
- The vector field X is of Beltrami type (with constant factor 1). When considered as an initial datum of NS, we obtain:

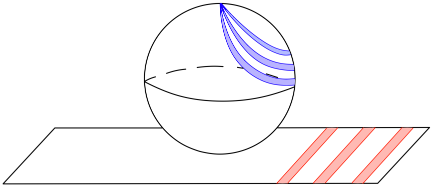
$$X(t) = X\, e^{-\nu t}$$

$\Longrightarrow$ it exists for all time.

- The exponential decay implies that it can simulate just a finite number of steps of any Turing machine.

What about time-dependent Euler flows?

The Euler equations on (M, g) are Turing complete if: for any integer $k \geqslant 0$, given a Turing machine T, an input tape t, and a finite string $(t^*_{-k}, ..., t^*_k)$ of symbols of the alphabet, there exist an explicitly constructible vector field $X_0 \in \mathfrak{X}^\infty_{vol}(M)$ and an open set $U \subset \mathfrak{X}^\infty_{vol}(M)$ such that the solution to the Euler equations with initial datum X_0 is defined for all time and intersects U if and only if T halts with an output tape whose positions $-k, ..., k$ correspond to the symbols $t^*_{-k}, ..., t^*_k$.



The manifold

The manifold M is diffeomorphic to $SO(N) \times \mathbb{T}^{N+1}$ and $\dim(M) \lesssim 10^{35}$.

Theorem 4 (Cardona, M., & Peralta-Salas)

There exists a smooth compact Riemannian manifold (M, g) such that the Euler equations on (M, g) are Turing complete. In particular, the problem of whether the solution to the Euler equations with an initial datum X_0 will reach a certain open set $U \subset \mathfrak{X}^\infty_{vol}(M)$ or not is undecidable.

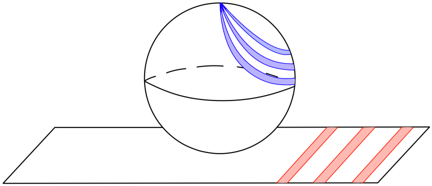
Main steps of the proof

- There exists a polynomial vector field on the sphere which is Turing complete. Idea: *Compactify* a construction by *Graça et al* of polynomial vector fields on $\mathbb{R}^n$ and *regularize* to get a global smooth vector field.
- Recall:

Theorem (Torres de Lizaur)

Let Y be a polynomial vector field on $\mathbb{S}^n$. Then there is a (constructible) compact Riemannian manifold (M, g) such that $(\mathbb{S}^n, Y)$ can be embedded into the Euler equations for (M, g).

- Apply the former result to our vector field to embed the vector field as a Turing complete t-dependent Euler flow.



The manifold

The manifold M is diffeomorphic to $SO(N) \times \mathbb{T}^{N+1}$ and dim$(M) \lesssim 10^{35}$.

What's next?

- The metric of our construction on S^3 is not Euclidean. Can this be improved? Yes in joint work with Cardona and Peralta-Salas.
- Topological complexity vs computational complexity. Entropy? Can a Turing complete system have zero entropy? (joint work with Bruera, Cardona and Peralta-Salas).
- Undecidability and periodic orbits: Weinstein conjecture and undecidability of periodic orbits. arXiv:2107.09471

Questions?

However, on the second time round, she came upon a low curtain she had not noticed before, and behind it was a little door about fifteen inches high: she tried the little golden key in the lock, and to her great delight it fitted!
Lewis Carrol, Alice in Wonderland