Higher Berry classes for many-body quantum lattice systems

joint work with

Nikita Sopenko
Plan

• Review of “Berry phase” and difficulties with extending to many-body systems

• Local Noether theorem for quantum lattice systems

• Application to “higher Berry phase”.
Review of "Berry phase".

Let $M$ be a compact manifold ("parameter space").

Let $V$ be a fixed f.d. Hilbert space.

Let $H_\mu : V \to V$, $\mu \in M$

be a smooth family of Hamiltonians
(self-adjoint operators) s.t.

lowest eigenvalue is "non-degenerate"
(eigenspace has dimension 1 $\forall \mu \in M$)

$L \to M$

- $L$ is a subbundle of a trivial bundle with fiber $V$.
- There is a canonical connection $\nabla^\mu$ on $L$
  ("Berry connection")
\[ F = \nabla^2_B \in \Omega^2(M, i\mathbb{R}) \quad ("\text{Berry curvature}") \]

Purely imaginary b.c. \( \nabla_B \) is unitary.

\( \left[ \frac{F}{2\pi i} \right] \in H^2(M, \mathbb{R}) \) is an integral class (lie in the image of \( H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \))

\( \left[ \frac{F}{2\pi i} \right] \) depends only on \( \mathcal{L} \), not on

the Hamiltonian \( \mathcal{H}(\rho) \).

\( \mathcal{L} + \left[ \frac{F}{2\pi i} \right] \neq 0 \), the family of

ground states is "topologically non-trivial."

Can we generalize this to the case of a many-body quantum system?
Quantum lattice systems

Typical setup: \( \mathbb{Z}^d \subset \mathbb{R}^d \)

Hilbert space is not specified at the outset.

Instead, specify an algebra of observables, \( \mathcal{A} = \bigotimes_{p \in \mathbb{Z}^d} \mathcal{A}_p \), \( \mathcal{A}_p \subset \text{Mat}(n_p, \mathbb{C}) \)

Define somehow a Hamiltonian on \( \mathcal{A} \) which generates the time evolution.

A ground state is now a positive linear function \( \sigma : \mathcal{A} \to \mathbb{C} \) invariant under evolution.

\( \sigma \) defines the Hilbert space via GNS construction.

Problem: as the Hamiltonian varies, so will \( \sigma \). The Hilbert spaces for different \( \sigma \) are not naturally isomorphic.
The same problem from a different angle

Suppose the parameters are themselves physical degrees of freedom...

\[ \phi^x \rightarrow \phi^x(\vec{x}, t) \]

local coord. \[ \vec{x} = (x_1', \ldots, x_d') \]

\[ d = 0 \]

\[ S_{\text{eff}} = \int A^B_\phi(\phi(t)) \frac{d\phi^x}{dt} \, dt = \int \phi^* A_B \quad (t, \ldots) \]

Here \[ A^B = A^B_\phi \frac{d\phi^x}{dt} \] is the "connection 1-form".

\[ D_B = d + A^B \]

\[ A^B \] is defined only locally.

\[ \begin{align*}
A^{(1)} - A^{(2)} &= df^{(12)} \\
f^{(12)} : U^{(1)} \rightarrow i \times \mathbb{R}^2 / 2\pi\mathbb{Z} \\
F = dA^B & \text{ is an honest 2-form}
\end{align*} \]
\[ S_{\text{eff}} = \int \phi^* C + \ldots \]

\[ C = \frac{1}{(d+1)!} C_{\alpha_1 \ldots \alpha_{d+1}} d\phi^{\alpha_1} \ldots d\phi^{\alpha_{d+1}} \]

Is a \((d+1)\)-form (locally)

\[ C^{(1)} - C^{(2)} = d\lambda^{(12)} \]

\[ \lambda^{(12)} \in \Omega^d (U^{(12)}, i\mathbb{R}) \]

\[ \lambda^{(12)} + \lambda^{(23)} + \lambda^{(31)} = d\rho^{(123)} \]

\[ \rho^{(123)} \in \Omega^{d-1} (U^{(123)}, i\mathbb{R}) \]

\[ F = dC \in \Omega^{d+2} (M, i\mathbb{R}) \] is an honest \((d+2)\)-form

\[(C, \lambda, \rho, \ldots) \] is a Beilinson-Deligne cocycle

\((d+1)\)-form gauge field

Alternative description:

Cheeger-Simons differential character
Summary

• For \( d > 0 \) expect a "higher Berry phase" described by a \((d+1)\)-form gauge field.

• \( \left[ \frac{F}{2\pi i} \right] \in H^{d+2}(M, \mathbb{R}) \) should be possible to define without knowing the details of the Hamiltonian.

• \( \left[ \frac{F}{2\pi i} \right] \) is possibly quantized - (perhaps under some additional conditions)
• To understand lattice systems, it is natural to use Quantum Statistical Mechanics.

• Mathematical apparatus:
  
  Operator algebras (analysis)

• This is not as hard as it seems (some nice algebra emerges)
Symmetry $\Rightarrow$ current $j^\mu$

- "global"
- acts locally

Ambiguity:

\[ j^\mu \mapsto j^\mu + \partial_\nu h^{\mu \nu}, \quad h^{\mu \nu} = - h^{\nu \mu} \]

Equivalently:

\[ j^0 = p \mapsto p + \partial_i h^{0i}, \quad i = 1, 2, 3 \]

\[ \left( Q = \int p \, d^3x \text{ is unchanged} \right) \]

\[ j^k \mapsto j^k - \partial_0 h^{0k} + \partial_i h^{ki} \]

\[ \uparrow \text{additional ambiguity} \]

All these ambiguities are physically harmless*

* I think.

I think.
Currents and conserved quantities on a lattice.

\[ \Lambda \subset \mathbb{R}^d \]  
(e.g. \( \Lambda = \mathbb{Z}^d \))

Algebra of observables:  
\[ A = \bigotimes_{p \in \Lambda} A_p \]  
\[ A_p = \text{Mat}(n_p, \mathbb{C}) \]

Hamiltonian:
\[ A \rightarrow \left[ H, A \right] = \delta_H(A) \]

derivation of \( A \)
\[ \delta(AB) = \delta(A)B + A \cdot \delta(B) \]
\[ A, B \in A. \]

- \( \delta_H \) is NOT bounded.
- \( \delta_H \) is not defined everywhere on \( A \).
- Unbounded derivations do not form a Lie algebra.
Physically relevant derivations:
\[ \delta_\phi : A \mapsto \sum_{\Gamma \in \mathcal{P}(\Lambda)} [\phi(\Gamma), A], \quad \phi(\Gamma) \in \mathcal{A} \]

Properties of \( \phi(\Gamma) \)

- \( \phi(\Gamma)^* = - \phi(\Gamma) \).
- \( \phi(\Gamma) \to 0 \) as \( \text{diam}(\Gamma) \to \infty \).
- \( \text{Tr} (\phi(\Gamma)) = 0 \).

N.B.
- \( \phi(\Gamma) \) are not uniquely defined by \( \delta_\phi \).
- No notion of "density of energy".

Alternative:
\[ \delta_F : A \mapsto \sum_{\rho \in \Lambda} [F_\rho, A], \quad F_\rho \in \mathcal{A} \]

- \( F_\rho^* = - F_\rho \).
- \( F_\rho \) is approximately localized near \( \rho \).
- \( \text{Tr} (F_\rho) = 0 \).
Q1. How do we describe the ambiguity in $F_p$ for a given $\delta F$?

Conservation equation:

$$\delta_H F_j = - \sum_{k \in \Lambda} J_{kj}^F$$

"current from $j$ to $k$"

Conserved quantity at $j$:

- $J_{kj}^F = -(J_{kj}^F)^*$
- $J_{kj}^F = -J_{jk}^F$
- $J_{kj}^F \to 0$ as $|j-k| \to \infty$
- $J_{kj}^F$ is approximately localized near $j,k$.

Q2. Does such a $J^F$ exist for any "Hamiltonian" $F$?

Q3. How does one describe the ambiguity in $J^F$ for a given $F_p$, $\rho \Lambda$?
Finite-range (UL) chains

\[ a : \Lambda \times \Lambda \times \ldots \times \Lambda \to A \]

9+1 times

q-chain

* shear-symmetric

* traceless

* localized on a ball of radius \( R \) centered at \( j_h \), \( h \in \{0, \ldots, q\} \)

* Bounded

Let \( C_{UL}^q \) be the space of q-chains.

\[(\partial a)_{v_1 \ldots j_q} = \sum_{i_0} a_{i_0 j_0 j_1 \ldots j_q} \]

\[ \Theta : C_{UL}^q \to C_{UL}^{q-1} \] is a differential:

\[ \Theta^2 = 0. \]
Let $J_{j\mu}$ be a 1-chain.

\[(\partial J^F)_k = \sum_j J_{j\mu} \cdot \]

Conservation equation takes the form

\[\delta_H(F) = -\partial J^F.\]

Ambiguity?

\[J \mapsto J + \partial M, \quad M \in C_2^u.\]

No other ambiguities f. r.

**Theorem**

Homology of $(C^u, \partial)$ is trivial for $q > 0$.

\[H_0(C^u) \cong \text{finite-range derivations}\]

**Def.** UL Noether complex is

\[
\begin{align*}
\partial^u & \colon C^u & \partial & \colon C^u & \partial & \colon C^u & \delta & \colon C^u \\
\cdots & \to C_2 & \to C_1 & \to C_0 & \to C_{-1} & \\
& & & & & \text{"nice" derivations}
\end{align*}
\]

UL Noether complex has trivial homology.
Rapidly decaying (UAL) chains

Uniformly Almost Local chains.

\( a : \Lambda \times \ldots \times \Lambda \rightarrow A \)

- \( q+1 \) times
- skew-symmetric
- traceless
- \( a_{j_0 \ldots j_q} \) approximately localized near any of \( j_0, \ldots, j_q \).

\( \Theta : C_q^{\text{UAL}} \rightarrow C_{q-1}^{\text{UAL}} \), \( \Theta^2 = 0 \).

UAL derivations:

\( a : A \rightarrow \sum_j [a_j, A] \). \( A \in A_{\text{al}} \)

\[ \text{Theorem} \]

Homology of \( \text{UAL} \)

is trivial

\[ \ldots \rightarrow C_1^{\text{UAL}} \rightarrow C_0^{\text{UAL}} \rightarrow C_{-1}^{\text{UAL}} \]

\[ \text{UAL derivations} \]
Important technical point:

one can set up the definitions so that

\( \mathfrak{A} \), \( C^q \) are "nice" topological vector

spaces (Fréchet spaces) \( \forall q \geq -1 \)

and \( \varphi \) is a continuous map \( \forall q \geq 0 \)
Applications

• any "local" symmetry of a "local" Hamiltonian gives a "local" current

• currents are determined up to exact 1-chains.

• conserved quantity determines its density up to exact 0-chains

Example: energy current.

\[ J_{jk}^E = [H_k, H_j]. \]

More generally, for any two 0-chains ("densities") \( F_j, G_j \) can define a 1-chain:

\[ [F, G]_{jk} = [F_j, G_k] - [F_k, G_j]. \]

How does this binary operation fit into our story?
There is a bracket of degree +1 on \( C^* \) & \( C^* \):

\[
[a, b]_{i_0 \ldots i_p, q+1} = \frac{1}{p! q!} [a_{i_0 \ldots i_p}, b_{i_{p+1} \ldots i_{p+q+1}}] \]

\( p \)-chain \( q \)-chain

\( \frac{1}{p! q!} \) permutations

\[
(\mathfrak{c}^{ul}, \mathfrak{c}^{ul}, \partial, [\cdot, \cdot]) \text{ is a (1-shifted) DG Lie algebra (DGLA)}
\]

- graded Jacobi for \([\cdot, \cdot]\)
- graded Leibniz for \((\mathfrak{c}, \cdot, \partial)\)

\( C_{-1} \times C_{-1} \rightarrow C_{-1} \) - Lie bracket of derivations

\( C_0 \times C_0 \rightarrow C_1 \) - symmetric operation on 0-chains

C. f. in QFT:

\[
\left[ J^a (\hat{x}, 0), J^b (\overline{\hat{x}}, 0) \right] = \oint J^c (\hat{x}) \delta (\hat{x} - \overline{\hat{x}})
\]

\( \text{symmetric in } a, b \)

+ Schwinger terms
<table>
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<th>QFT in $D+1$ space-time dimensions</th>
<th>Lattice models in $D$ spatial dimensions</th>
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<td>$D$-form</td>
<td>$0$-chain</td>
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<tr>
<td>$(D-1)$-form $= \text{current}$</td>
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<td>$p$-form</td>
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<td>de Rham $d$</td>
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<tr>
<td>Submanifold of dimension $p$</td>
<td>$(D-p)$-cochain (to be discussed)</td>
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What is this good for?

Not sure ... but there is an analogous structure which is VERY useful.

Let \( w: A \to C \) be a state.

\[ \text{Def. } \text{An observable } A \text{ does not excite } w \text{ if } \langle [A, B] \rangle = 0 \forall B \in A. \]

This seems rare.

\[ \text{Def. } \text{A derivation } \delta_f \text{ does not excite } w \text{ if } \langle \delta_f(B) \rangle = 0 \forall B \in A \]

\[ \text{Example: Derivations associated to unbroken symmetries} \]

\[ \text{Def. } \text{A } q \text{-chain } \alpha \text{ does not excite } w \text{ if } \langle [\alpha_{j_0 \ldots j_q}, B] \rangle = 0 \forall B \in A \]

Also seems rare.
Let $C_{w}^{q}$ be the space of chains which do not excite $\omega$, $q \geq -1$

**Theorem:** If $\omega$ is a gapped ground state of a Hamiltonian $H = \sum_{j} H_{j}$ arising from a 0-chain $H_{j}$, then $(C_{w}^{q}, \partial)$ has trivial homology.

**Application #1**

$\delta_{H} : \mathcal{A} \rightarrow \sum_{j} \left[ H_{j}, \mathcal{A} \right]$ does not excite the ground state $\omega$ of $H$

If $H$ is gapped, then $\exists \tilde{H}_{j}$ s.t.

- $\tilde{H}_{j}$ is a 0-chain
- $\delta_{H} = \delta_{\tilde{H}}$ (i.e. $H$ & $\tilde{H}$ generate the same dynamics)
- $\langle [\tilde{H}_{j}, B] \rangle = 0 \quad \forall \quad B \in \mathcal{A}$

i.e. $|0\rangle$ is the eigenstate of each $\tilde{H}_{j}$. $\leftarrow$ due to Kitaev
Application #2: Higher Berry class

Def. A family of states \( \omega: \mathcal{A} \rightarrow \mathcal{C} \) is called smooth if \( \mu \in \mathcal{M} \) exists such that \( G \in \mathcal{L}^2(M, \mathcal{D}_ad) \) is the space of UAL derivations such that \( (M, \omega) \) is "covariantly constant" w.r.t. \( \mathcal{D} = d + G \).

\[
\frac{d}{d\langle A \rangle}_\mu = \left< G(A) \right>_\mu \quad \forall \, A \in \mathcal{A}
\]

Or, if \( A: M \rightarrow \mathcal{A} \) is a smooth function:
\[
\frac{d}{d\langle A \rangle} = \left< dA + G(A) \right> = \left< DA \right>
\]

Note:
\[
0 = \frac{d^2}{d\langle A \rangle} = \left< D^2(A) \right> = \left< F(A) \right>
\]
where \( F = dG + \frac{1}{2} \{G, G\} \).

Thus, \( F \in \mathcal{L}^2(M, \mathcal{D}_ad) \)
also, \( DF = 0 \).
Th. Let $H: M \to D$ be a smooth family of "Hamiltonians" with unique ground states $\omega_\mu$, $\mu \in M$ such that $\langle A \rangle_\mu \in C^\infty(M, C)$ for $A \in A$, then such a $G$ exists.

( follows from a theorem of Y. Ogata and A. Moon ).

This provides a justification for studying smooth families of states.

We will further assume that for some $\mu_0 \in M$ the state $\omega_{\mu_0}$ is a unique ground state of a gapped Ul "Hamiltonian". Then

$$\cdots \to \Omega^0(M, C^\omega) \xrightarrow{\partial} \Omega^1(M, C^\omega) \xrightarrow{\partial} \Omega^2(M, \mathcal{D}_d^\omega)$$

has trivial homology $\forall \rho \geq 0$. 
Now let's try to define the Berry curvature \( F \in \mathcal{M}^{d+2}(M, i\mathbb{R}) \).

\[
d = 0
\]

\[
F = \langle F \rangle
\]

(makes sense b.c. \( \mathcal{D}_\text{al} = \mathcal{D} = \{ \text{traceless anti-self-adjoint elements of } \mathcal{A} \} \)).

Check that if is closed:

\[
dF = \langle DF \rangle = 0.
\]

\[
d > 0
\]

\[
F = dG + \frac{1}{2} [G, G] \in \mathcal{M}^{2}(M, \mathcal{D}^{\text{al}})
\]

But \( \langle F \rangle \) now does not make sense.

\( \mathcal{D}^{\text{al}} \) now consists of formal sums

\[
\mathcal{B} = \sum_{j \in \Lambda} B_{j}, \quad B_{j} \in \mathcal{D}_{\text{al}} \quad \text{is}
\]

"approximately localized at } j \)."

\[
\sum_{j} \langle B_{j} \rangle \quad \text{is divergent} \ldots
\]
Maurer - Cartan element

Recall that given a DG Lie algebra $(g^*, d, [], J)$, an MC element is a degree-1 element $G$ satisfying

$$dG + \frac{1}{2} [G, G] = 0.$$ 

$G \in \Omega^1(M, \mathcal{D}e)$ is a degree-1 element in

$$g^* = \bigoplus_{p, q} \Omega^p(M, \mathcal{T}^q)$$

where $\mathcal{T}^q$ is the complex

$$\cdots \to C^{w, 2} \to C^w \to C_0 \to \mathcal{D}e$$

deg: $-2, -1, 0$

$$d = d + \partial$$

$[\cdot, \cdot]$ is a combination of $[\cdot, \cdot]$ and wedge product of forms.
G is not an MC element b.c. 
\[ F = dG + \frac{1}{2} [G, G] \neq 0. \]

But can use G as a "seed":

\[ G = G + \sum_{p=2}^{\infty} g^p, \quad g^p \in \mathcal{N}^p(M, T_{p-1}) \]

Can solve the MC equation recursively:

\[ F + \partial g^{(2)} = 0 \quad \checkmark \]
\[ Dg^{(2)} - \partial g^{(3)} = 0 \quad \checkmark \]
\[ Dg^{(3)} + \frac{1}{2} \{ g^{(2)}, g^{(2)} \} + \partial g^{(4)} = 0 \quad \checkmark \]

What do we do now?

Need to extract an observable out of all these form-valued chains...
Integrating chains

\[ d = 1 \]

How to integrate a 1-chain (= current):

\[ \int h = \sum_{p \in A_0}^{q \in A_1} h_{pq} = \text{"flux of } h^{(1)} \text{ through a section"} \]

Main property

\[ \int_{A_0A_1} \Theta h' = 0 \text{ for a 2-chain } h'. \]

Consider \[ F^{(3)} = \int_{A_0A_1} \langle g^{(3)} \rangle \in \mathbb{S}^3(M, i\mathbb{R}) \]

\[ dF^{(3)} = \int_{A_0A_1} \langle Dg^{(3)} \rangle = -\int_{A_0A_1} \langle \Theta g^{(4)} \rangle = 0 \]

Can also check that \[ [F^{(3)}] \in H^3(M, i\mathbb{R}) \] does not depend on the section.
**General d**: Integrating a d-chain

- Choose d+1 conical regions
  \[ A_0, \ldots, A_d \subset \mathbb{R}^d \text{ s.t. } \bigcup_{i=0}^{d} A_i = \mathbb{R}^d \]

- \[ \int h = \sum_{i_0 \in A_0} h_{i_0} \ldots i_d \text{ for } A_0, \ldots, A_d \]

Hence we let

\[ F^{(d+2)} = \int \left< g^{(d+2)} \right>_{A_0 \ldots A_d} \]

\[ dF^{(d+2)} = \int \left< Dg^{(d+2)} \right>_{A_0 \ldots A_d} = \pm \int \left< \partial g^{(d+3)} \right>_{A_0 \ldots A_d} \]

Then, \( [F^{d+2}] \in H^{d+2}(M, i\mathbb{R}) \)

does not depend on the choice of \( A_0 \ldots A_d \), or the MC element \( G \)

\[ \Rightarrow \text{ it is a topological invariant of the family.} \]
Concluding remarks

- \( [g \cdot Cdt] \) is not expected to be quantized, in general.
- But can be shown to be quantized for \( d = 1 \).

- \( F_{(d+2)} \) depends on various choices \( \Rightarrow \) not physical.
- Suppose \((M, \omega)\) is a G-equivariant family (G = a compact Lie group) can attach to it an element of \( H^*_G(M, i\mathbb{R}) \).
- This is interesting even for \( M = \xi \times Y \) get topological invariants of G-invariant states taking values in G-invariant polynomials on the Lie algebra of G \( \Rightarrow \) Chern-Simons forms!
This work originated with a proposal by A. Kitaev, see his talk at Dan Freed's 60th birthday conference.


The talk is based on work with Nikita Sopenko (to appear soon)