E-POLYNOMIALS AND GEOMETRY OF CHARACTER VARIETIES

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[1] joint work with Carlos Florentino (U. Lisboa, Portugal) and Azizeh Nozad (IPM Tehran, Iran)

Index

1. PRELIMINARIES
   - Mixed Hodge Structures and E-polynomials
   - Character Varieties
2. ARITHMETIC-GEOMETRIC METHODS FOR GL_n-CHARACTER VARIETIES
   - Stratifications by polystability type
   - Generating functions of E-polynomials
   - Explicit combinatorial formulae
3. EXPLICIT COMPUTATIONS FOR GL_n-CHARACTER VARIETIES
4. SL_n AND PGL_n-CHARACTER VARIETIES
   - Conjecture for Langlands dual groups SL_n and PGL_n
   - Computations for SL_n and PGL_n-character varieties of the free group

Index

Clasical Hodge Theory

- A (pure) Hodge structure (of weight k) on a $\mathbb{Z}$-module $V_{\mathbb{Z}}$ is

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}, \quad V^{p,q} = \overline{V^{q,p}}$$

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- Alternatively, define Hodge filtration:

$$V_{\mathbb{C}} \supset \cdots \supset F^p(V) \supset F^{p+1}(V) \supset \cdots$$

with $F^p(V) \cap \overline{F^q(V)} = V^{p,q}$ and $F^p(V) = \bigoplus_{i \geq p} V^{i,k-i}$.

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Hodge decomposition

X is a compact Kähler variety, k^{th}-cohomology carries (pure) Hodge structure of weight k:

$$H^k_{DR}(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{p,q}(X) = \overline{H^{q,p}(X)}$$

where $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$ are the Hodge numbers.

Hodge diamond

Using $H^{p,q}(X) \simeq H^q(X, \Omega^p)$, Serre duality ($h^{p,q} = h^{n-p,n-q}$), Hodge symmetry ($h^{p,q} = h^{q,p}$), Hodge numbers of a compact Kähler variety of dimension n are displaced in the Hodge diamond with symmetries:

$$\begin{array}{ccccccccc} & & & & h^{n,n} & & & & \\ & & & h^{n,n-1} & & h^{n-1,n} & & & \\ & & \iddots & & & & \ddots & & \\ h^{n,0} & h^{n-1,1} & & & \cdots & & & h^{1,n-1} & h^{0,n} \\ & & \ddots & & & & \iddots & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & & & h^{0,0} & & & & \end{array}$$

- Gives Betti numbers as sum of rows:

$$b_k = \dim H^k(X, \mathbb{C}) = \sum_{p+q=k} h^{p,q}$$

Examples of Hodge structures

(Smooth $\mathbb{C}$-projective genus g curve or compact Riemann surf.)

$$\begin{array}{ccc} & h^{1,1} = 1 & \\ h^{1,0} = g & & h^{0,1} = g \\ & h^{0,0} = 1 & \end{array}$$

with Betti numbers $b_0 = 1$, $b_1 = 2g$, $b_2 = 1$.

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$$

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(Complex projective space $\mathbb{P}^n_{\mathbb{C}}$)

$$
\begin{array}{ccccc}
 & & h^{n,n}=1 & & \\
 & h^{n-1,1}=0 & & h^{n,n-1}=0 & \\
 & \iddots & h^{n-1,n-1}=1 & \ddots & \\
h^{i,0}=0 & & \vdots & & h^{0,i}=0 \\
 & \ddots & h^{1,1}=1 & \iddots & \\
 & h^{1,0}=0 & & h^{0,1}=0 & \\
 & & h^{0,0}=1 & &
\end{array}
$$

with Betti numbers $b_{2k} = 1$, $k = 0, 1, \ldots, n$.

Mixed Hodge Structures

- On $V_{\mathbb{C}}$ with decreasing Hodge filtration $F^{\bullet}(V)$, add weight filtration

$$0 \subset \cdots W_{k-1} \subset W_k \subset \cdots \subset V$$

  such that $F^{\bullet}(V)$ induces weight k Hodge structure on $Gr_k^W(V) := W_k/W_{k-1}$, defining $V^{p,q} := Gr_F^p Gr_{p+1}^W(V)$
- $h^{k,p,q} = \dim_{\mathbb{C}} V^{p,q}(X)$ are the mixed Hodge numbers with $h^{k,p,q} = h^{k,q,p}$
- k-weights are (p,q) with $h^{k,p,q} \neq 0$ (it can be $p+q \neq k$)

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Mixed Hodge structures on cohomology by Deligne

X quasi-projective algebraic variety (not necessarily smooth nor complete nor irreducible).
Singular compactly supported cohomology $H_c^k(X)$ carry mixed Hodge structures.
Yield compactly supported Betti numbers $\dim H_c^k(X) = \sum_{p,q} h^{k,p,q}$.
Also give usual Betti numbers by Poincaré duality in the smooth case.

Topological and geometrical invariants from MHS

MIXED HODGE POLYNOMIAL

$\mu(X; t, u, v) = \sum_{k,p,q} h^{k,p,q} t^k u^p v^q$

Topological and geometrical invariants from MHS

Mixed Hodge polynomial

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Poincaré polynomial

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Hodge-Deligne-Serre or E-polynomial

$E(X;u,v) = \mu(X;-1,u,v) = \sum_{k,p,q} h^{k,p,q} (-1)^k u^p v^q \in \mathbb{Z}[u,v]$

- Hodge-Tate or balanced varieties: MHS only weights (p,p), then $E(X;u,v) = E(X;uv) = E(X;x)$ polynomial in 1 variable. Converse unknown!

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POINCARÉ POLYNOMIAL

$P^c(X;t) = \mu(X;t,1,1) = \sum_k \dim H^k_c(X) t^k$

HODGE-DELIGNE-SERRE OR E-POLYNOMIAL

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EQUIVARIANT E-POLYNOMIAL

If W finite group acting on X algebraically: $E^W(X;u,v) = \sum_{k,p,q} [H^{k,p,q}(X)]_W (-1)^k u^p v^q \in R(W)[u,v]$, coefficients in the representation ring of W.

Topological and geometrical invariants from MHS

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Compacty supported Euler characteristic

$\chi^c(X) = E(X;1,1) = \mu(X;-1,1,1) = P^c(X;-1) = \sum_k (-1)^k \dim H_c^k(X)$
Coincides with $\chi(X)$ for X quasi-projective.

Properties of the E-polynomial

MULTIPLICATIVITY (KÜNNETH)

$$E(X \times Y) = E(X) \cdot E(Y)$$

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Additivity

$X = Z \sqcup (X \backslash Z)$ locally closed $\Rightarrow E(X) = E(Z) + E(X \backslash Z)$

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MULTIPLICATIVITY (KÜNNETH)

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ADDITIVITY

$X = Z \sqcup (X \backslash Z)$ locally closed $\Rightarrow E(X) = E(Z) + E(X \backslash Z)$

PROPOSITION (DIMCA-LEHRER ('97), LOGARES-MUÑOZ-NEWSTEAD ('13), FLORENTINO-NOZAD-Z. ('19))

Let $F \to^{W \curvearrowright} X \to B$ be a fibration with group W preserving fibers $\pi^{-1}(b)$ and verifying any of

A) *Locally Zariski trivial (LZT)*

B) *Smooth, locally analytic trivial and $\pi_1(B) \curvearrowright H_c^*(F)$ trivially*

C) *X, B smooth and F complex connected Lie group*

D) *F is special (F special if all principal F-bundles are LZT)*

E) *$X = G$ reductive, $F = Z(G)$ connected center, $B = PG = G/Z$ adjoint group*

$$\text{then,} \quad E^W(X) = E^W(F) \cdot E(B) .$$

Moreover, if W is trivial, $E(X) = E(F) \cdot E(B)$.

Examples of E-polynomials

(Riemann surface Σ_g carry pure not Hodge-Tate structure)

$$\mu(\Sigma_g; t, u, v) = 1 + gt(u+v) + t^2uv \quad P(\Sigma_g; t) = 1 + 2gt + t^2$$

$$E(\Sigma_g; u, v) = 1 - g(u+v) + uv \quad \chi(\Sigma_g) = 2 - 2g$$

Examples of *E*-polynomials

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(Projective space carry pure and Hodge-Tate structure)

$$\mu(\mathbb{P}^n_{\mathbb{C}}; t, u, v) = 1 + t^2uv + t^4u^2v^2 + \cdots + t^{2n}u^nv^n$$

$$P(\mathbb{P}^n_{\mathbb{C}}; t) = 1 + t^2 + t^4 + \cdots + t^{2n}$$

$$E(\mathbb{P}^n_{\mathbb{C}}; u, v) = 1 + uv + u^2v^2 + \cdots + u^nv^n = 1 + x + x^2 + \cdots + x^n$$

$$\chi(\mathbb{P}^n_{\mathbb{C}}) = n + 1$$

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(Locally closed decomposition $\mathbb{C} = \mathbb{C}^* \sqcup \{pt\}$)

$$E(\mathbb{C}; u, v) = E(\mathbb{C}^*; u, v) + E(\{pt\}; u, v) = (uv - 1) + 1 = (x - 1) + 1 = x$$

Examples of E-polynomials

(LZT fibration)

$$\mathrm{GL}_2(\mathbb{C}) \twoheadrightarrow \mathbb{C}^2 \setminus \{(0,0)\}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a,c)$$

fiber is $\simeq \mathbb{C}^2 \setminus \mathbb{C}$ *=vectors* (b,d) *linearly independent with* (a,c).

$$\textit{Then} \quad E(\mathrm{GL}_2(\mathbb{C}); u, v) =$$

$$E(\mathbb{C}^2 \setminus \mathbb{C}; u, v) \cdot E(\mathbb{C}^2 \setminus \{(0,0)\}; u, v) = (u^2v^2 - uv) \cdot (u^2v^2 - 1) =$$

$$(x^2 - x) \cdot (x^2 - 1) = x^4 - x^3 - x^2 + x$$

Examples of E-polynomials

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$$(x^2 - x) \cdot (x^2 - 1) = x^4 - x^3 - x^2 + x$$

(Fiber needs to be connected)

$$\mathbb{Z}_2 \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}})$$

$$(1 + uv)^2 \neq 2 \cdot (1 + uv + u^2v^2)$$

Character varieties

- Γ finitely presented group, $\Gamma = \langle \gamma_1, \gamma_2, \dots, \gamma_r : r_1, r_2, \dots, r_s \rangle$
- G complex reductive affine algebraic group (for this talk $G = \mathrm{GL}_n(\mathbb{C}), \mathrm{PGL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$)

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Representation variety

$\mathcal{R}_\Gamma G := \mathrm{Hom}(\Gamma, G) = \left\{ \rho(\gamma) = \big(\rho(\gamma_1), \ldots, \rho(\gamma_r)\big) : r_j(\rho) = 1, j = 1, \ldots, s \right\}$ is an affine algebraic variety

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Action of G on $\mathcal{R}_\Gamma G$ by conjugation

For $\rho \in \mathcal{R}_\Gamma G$, $g \in G$, $\gamma \in \Gamma$:

$$(g \cdot \rho)(\gamma) := g\rho(\gamma)g^{-1} = \left(g\rho(\gamma_1)g^{-1}, \ldots, g\rho(\gamma_r)g^{-1}\right)$$

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G-Character variety of Γ is affine GIT quotient

$$\mathcal{X}_\Gamma G := \mathcal{R}_\Gamma G /\!/ G = \mathrm{Spec}\,\mathbb{C}[\mathcal{R}_\Gamma G]^G = \mathcal{R}_\Gamma^{ps} G / G$$

Some finitely presented groups Γ

Surface groups

Fundamental group of Σ_g compact orientable Riemann surface

$$\Gamma = \pi_1(\Sigma_g) = \left\langle a_1, b_1 \ldots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

(more generally, a central extension of $\pi_1(\Sigma_g)$).

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FREE GROUPS

$\Gamma = F_r$ free group of rank r. The character variety is $\mathcal{X}_r G := \mathcal{X}_{F_r} G \simeq G^r /\!/ G$.

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Abelian groups

$\Gamma = \mathbb{Z}^r$, free abelian group of rank r. Also abelian groups with torsion.

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OTHER GROUPS

Twisted surface groups, torus knot groups, non-orientable surface groups.

Example: surface group GL_2-character variety

For $\Gamma = \pi_1(\Sigma_g)$, $G = \mathrm{GL}_2$, *representation variety is*

$$\mathcal{R}_\Gamma G = \left\{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} : A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = I\right\}$$

and character variety $\mathcal{X}_\Gamma G$ *is the GIT quotient of* $\mathcal{R}_\Gamma G$ *by action given by simultaneous conjugation:*

$$(A_1, B_1, \ldots, A_g, B_g) \sim (CA_1C^{-1}, CB_1C^{-1}, \ldots, CA_gC^{-1}, CB_gC^{-1}), \quad C \in G$$

- *Reducible representations* $\mathcal{X}_\Gamma^{red} G$ *are those simultaneously conjugated to*

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_{2g} & 0 \\ 0 & \mu_{2g} \end{pmatrix}\right)$$

$$\in (\mathbb{C}^*)^{4g} / (\lambda_1, \mu_1, \ldots, \lambda_{2g}, \mu_{2g}) \sim (\mu_1, \lambda_1, \ldots, \mu_{2g}, \lambda_{2g})$$

- *Irreducible representations form smooth locus* $\mathcal{X}_\Gamma^{irr} G := \mathcal{R}_\Gamma^{irr} G / G$.

Motivation

- For surface groups $\Gamma := \pi_1(\Sigma_g)$, Σ_g a Riemann surface, character varieties are related to moduli spaces of Higgs bundles through non-abelian Hodge correspondence (Hitchin, Donaldson, Corlette, Simpson):

$$\mathcal{X}_\Gamma G = \mathcal{R}_\Gamma G /\!/ G \approx \text{moduli space of } G\text{-Higgs bundles over } \Sigma_g$$

- QFT interpretation of geometric Langlands program in mirror symmetry (Simpson, Kapustin-Witten).
- In SYZ mirror symmetry, hyperkähler nature of Hitchin systems allows topological criterion for mirror symmetry: same/mirror Hodge numbers for G and ${}^L G$.
- (Hausel-Thaddeus, Groechenig-Wyss-Ziegler) establish topological mirror simmetry for SL_n and PGL_n (smooth/orbifold case, pure HS).

For other Γ, character varieties more singular, Hodge structure not pure, we expect other topological mirror symmetries.

Results on topological invariants of character varieties

For Γ surface group (related with smooth varieties):

- Poincaré polynomials for surface groups: Hitchin ('87), Gothen ('94) for $G = \mathrm{SL}_2, \mathrm{SL}_3$, García-Prada-Heinloth-Schmitt ('13, '14), Schiffman ('16), Mellit ('17) for $G = \mathrm{SL}_n, \mathrm{PGL}_n$.
- Mixed Hodge polynomials with arithmetic methods: Hausel-Rodriguez-Villegas ('08): for $G = \mathrm{GL}_n$, Mereb ('10): $G = \mathrm{SL}_n$.

For other Γ (singular character varieties) computations of E-polynomials are harder.
Geometric approach:

- Logares, Muñoz, Newstead, Martínez ('13,'14,'17), surface groups for $G = \mathrm{SL}_2, \mathrm{PGL}_2$.
- Cavazos, Lawton, Muñoz, Porti ('14,'15,'17): free groups for $G = \mathrm{SL}_2, \mathrm{SL}_3$ and torus knot groups for $G = \mathrm{GL}_3, \mathrm{SL}_3, \mathrm{PGL}_3$.
- Florentino-Lawton-Casimiro-Oliveira ('09,'15), free group retraction of G-character variety to K-character variety (K maximal compact).

Arithmetic approach:

- Mozgovoy-Reineke: ('15) compute points of $\mathcal{X}_r \mathrm{GL}_n$ over $\mathbb{F}_q$ and Baraglia-Hekmati ('17) E-polynomials for $G = \mathrm{SL}_2, \mathrm{GL}_2$.
- Florentino-Silva ('18): Combined methods for abelian character varieties $\mathcal{X}_{\mathbb{Z}^r} G$, $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_n$.

Index

Geometric methods

GEOMETRIC METHODS

Based on decomposing the character variety into strata with different stabilizers and use additivity (stratifications) and multiplicativity (fibrations with stabilizer as the fiber) to compute E-polynomials.

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- $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_r : r_1, \ldots, r_s \rangle$, a finitely presented group and G complex reductive algebraic group.
- Locally closed stratification by stabilizer dimension

$$\mathcal{X}_\Gamma G = \bigsqcup_{m \geq m_0} \mathcal{X}_\Gamma^m G$$

where $m_0 = \dim \bigcap_{\rho \in \mathcal{R}_\Gamma G} \mathrm{Stab}(\rho)$, center of the action of G on $\mathcal{R}_\Gamma G$.

Geometric methods

Geometric methods

Based on decomposing the character variety into strata with different stabilizers and use additivity (stratifications) and multiplicativity (fibrations with stabilizer as the fiber) to compute E-polynomials.

- $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_r : r_1, \ldots, r_s \rangle$, a finitely presented group and G complex reductive algebraic group.
- Locally closed stratification by stabilizer dimension

$$\mathcal{X}_\Gamma G = \bigsqcup_{m \geq m_0} \mathcal{X}_\Gamma^m G$$

where $m_0 = \dim \bigcap_{\rho \in \mathcal{R}_\Gamma G} \mathrm{Stab}(\rho)$, center of the action of G on $\mathcal{R}_\Gamma G$.

In the linear case $G = \mathrm{GL}_n$, can perform a refinement by polystability type, connected to affine GIT and representation theory of symmetric group (=Weyl of GL_n).

Stratification by polystability type for $G = \mathrm{GL}_n$

Partition

$[k] = [1^{k_1} 2^{k_2} \cdots n^{k_n}] \in \mathcal{P}_n$, $\sum_{j=1}^{n} j \cdot k_j = n$, with length (number of blocks) $|[k]| = \sum_{j=1}^{n} k_j$.

- For example $[1^2\ 2\ 4] \in \mathcal{P}_8$ with length 4.

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$[k]$-strata

$\rho \in \mathcal{R}_\Gamma \mathrm{GL}_n$ is $[k]$-polystable if $\mathcal{R}_\Gamma^{[k]} \mathrm{GL}_n \ni \rho \sim_{conj} \bigoplus_{j=1}^{n} \rho_j$, where $\rho_j \in \mathcal{R}_\Gamma^{irr}(\mathrm{GL}_j^{\oplus k_j})$.

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Stratification by polystability type for $G = GL_n$

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Define $\mathcal{X}_\Gamma^{[k]} GL_n := \mathcal{R}_\Gamma^{[k]} GL_n /\!/ GL_n$, the $[k]$-stratum of $\mathcal{X}_\Gamma GL_n$. Particular cases:

- Abelian stratum: $\mathcal{X}_\Gamma^{[1^n]} GL_n \simeq \mathcal{X}_{\Gamma_{Ab}} GL_n$ (of maximal length n).
- Irreducible stratum: $\mathcal{X}_\Gamma^{[n]} GL_n = \mathcal{X}_\Gamma^{irr} GL_n$ (of minimal length 1), equals smooth locus for GL_n.

Stratification by polystability type for $G = \mathrm{GL}_n$

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Theorem (Florentino-Nozad-Z. ('19))

There exists a locally closed stratification by partition type:

$$\mathcal{X}_\Gamma \mathrm{GL}_n = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_\Gamma^{[k]} \mathrm{GL}_n .$$

Arithmetic methods

POLYNOMIAL COUNT

X is of polynomial type if there is a counting polynomial $C_X(t) \in \mathbb{Z}[t]$ such that $|X/\mathbb{F}_q| = C_X(q)$, for almost every prime p, with $|\mathbb{F}_q| = p^m$.

- (Katz ('08)) If X is of polynomial type then $E(X; u, v) = C_X(uv)$.

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Plethystic operators

Let $f(x,y,z) = \sum_n f_n(x,y)z^n \in \mathbb{Q}[x,y][[z]]$ be a formal power series.
Define plethystic exponential $\mathrm{PExp}(f) = e^{\Psi(f)}$, where $\Psi(x^i y^j z^k) = \sum_l \frac{x^{li} y^{lj} z^{lk}}{l}$ is the Adams operator.

- Particularly, $\mathrm{PExp}\left(\left(\sum_{p,q\geq 0} a_{p,q} u^p v^q\right) y\right) = \prod_{p,q\geq 0} (1 - u^p v^q y)^{-a_{p,q}}$.

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Theorem (Mozgovoy-Reineke ('15))

For $\Gamma = F_r$, GL_n-character varieties are of polynomial type and

$$\sum_{n\geq 0} A_n^r(q) z^n = \mathrm{PExp}\left(\sum_{n\geq 1} B_n^r(q) z^n\right), \quad \textit{where}$$

$A_n^r(q) := |\mathcal{X}_r \mathrm{GL}_n / \mathbb{F}_q| = E(\mathcal{X}_r \mathrm{GL}_n; q)$ *and* $B_n^r(q) := |\mathcal{X}_r^{irr} \mathrm{GL}_n / \mathbb{F}_q| = E(\mathcal{X}_r^{irr} \mathrm{GL}_n; q)$.

Generalization to Γ finitely presented (even if X is not of polynomial type)

$[k]$-Levy and $[k]$-symmetric group

$$L_{[k]} := \mathrm{GL}_1^{k_1} \times \mathrm{GL}_2^{k_2} \times \cdots \times \mathrm{GL}_n^{k_n} \subset \mathrm{GL}_n, \qquad S_{[k]} := S_{k_1} \times S_{k_2} \times \cdots \times S_{k_n} \subset S_n$$

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Proposition (Florentino-Nozad-Z. ('19))

A) $\mathcal{X}_\Gamma^{[k]} \mathrm{GL}_n \simeq \left(\mathcal{R}_\Gamma^{[k]} \mathrm{GL}_n /\!/ L_{[k]}\right)/S_{[k]} \simeq \times_{j=1}^n \mathrm{Sym}^{k_j}(\mathcal{X}_\Gamma^{irr} \mathrm{GL}_j)$.

B) $\sum_{n\geq 0} E(\mathrm{Sym}^n(X); u, v) y^n = \mathrm{PExp}(E(X; u, v) y)$.

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Theorem (Florentino-Nozad-Z. ('19))

If Γ is finitely presented,

$$\sum_{n\geq 0} A_n^\Gamma(u,v) t^n = \mathrm{PExp}\left(\sum_{n\geq 1} B_n^\Gamma(u,v) t^n\right)$$

$A_n^\Gamma(u,v) = E(\mathcal{X}_\Gamma \mathrm{GL}_n; u, v)$ *and* $B_n^\Gamma(u,v) = E(\mathcal{X}_\Gamma^{irr} \mathrm{GL}_n; u, v)$.

Rectangular partitions

- Further combinatorial analysis allows to relate A_n^Γ with B_l^Γ, $l \leq n$.

Rectangular partitions

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Rectangular partition

Idea: For a partition $[k] = [1^{k_1} \cdots n^{k_n}]$ choose a partition $[l] \in \mathcal{P}_{k_j}$ for each k_j.

$$[[k]] = [(1\times 1)^{k_{1,1}} (1\times 2)^{k_{1,2}} \cdots (1\times n)^{k_{1,n}} \cdots (n\times n)^{k_{n,n}}] \in \mathcal{RP}_n$$

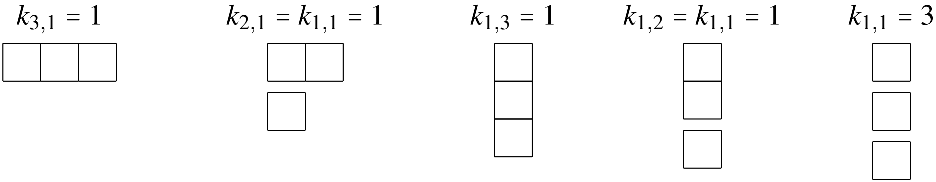
satisfying $n = \sum_{l,h=1}^n l\, h\, k_{l,h}$.

Gluing map

$$\pi : \mathcal{RP}_n \to \mathcal{P}_n$$
$$[[k]] \mapsto [m] = [1^{m_1} \cdots n^{m_n}]$$

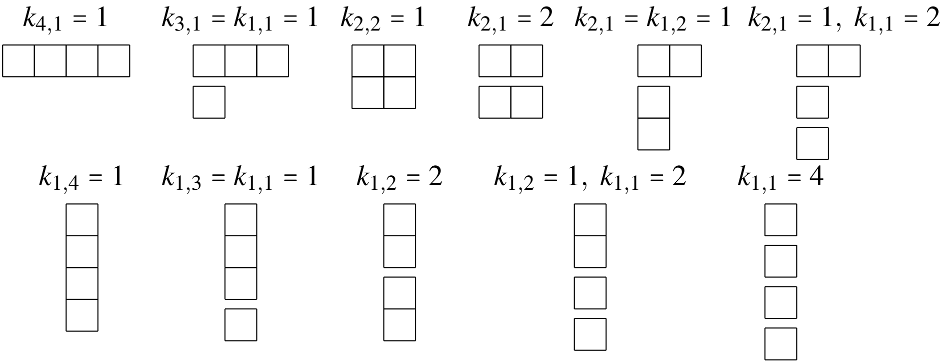
defined by $m_l := \sum_{h=1}^n h \cdot k_{l,h}$

Example $n = 3$



5 *rectangular partitions of* $n = 3$*. Gluing map* π *takes the first one to the Young diagram of the partition* $[3]$*, the second one to* $[1\,2]$ *and the last three to* $[1^3]$.

Rectangular partitions for $n = 4$



11 *rectangular partitions of* $n = 4$. *Gluing map* π *takes the first one to the Young diagram of the partition* $[4]$, *the second one to* $[1\,3]$, *the third and fourth ones to* $[2^2]$, *the fifth and sixth to* $[1^2\,2]$ *and the last five to* $[1^4]$.

E-polynomials of CVs in terms of irreducible lower dimensional strata

- Use plethystic exponential relations + rectangular partitions:

Theorem (Florentino-Nozad-Z. ('19))

Let Γ be a finitely presented group. Then,

$$E(\mathcal{X}_\Gamma \,\mathrm{GL}_n; u, v) = \sum_{[[k]] \in \mathcal{RP}_n} \prod_{l,h=1}^{n} \frac{B_l^\Gamma(u^h, v^h)^{k_{l,h}}}{k_{l,h}!\, h^{k_{l,h}}}$$

Moreover, for a given $[m] \in \mathcal{P}_n$, the E-polynomial of the corresponding stratum is:

$$E(\mathcal{X}_\Gamma^{[m]} \,\mathrm{GL}_n; u, v) = \sum_{[[k]] \in \pi^{-1}[m]} \prod_{l,h=1}^{n} \frac{B_l^\Gamma(u^h, v^h)^{k_{l,h}}}{k_{l,h}!\, h^{k_{l,h}}}$$

where $B_l^\Gamma(u, v) := E(\mathcal{X}_\Gamma^{irr} \,\mathrm{GL}_l; u, v)$.

Example for $n = 4$

- $A_4^\Gamma(u,v) = E(\mathcal{X}_\Gamma \mathrm{GL}_4; u,v)$ is the sum of these 5 strata comprising the 11 terms coming from the rectangular partitions in the previous figure.

$$
\begin{aligned}
E(\mathcal{X}_\Gamma^{[4]}\,\mathrm{GL}_4; u,v) &= B_4^\Gamma(u,v) \\
E(\mathcal{X}_\Gamma^{[1\;3]}\,\mathrm{GL}_4; u,v) &= B_3^\Gamma(u,v) B_1^\Gamma(u,v) \\
E(\mathcal{X}_\Gamma^{[2^2]}\,\mathrm{GL}_4); u,v &= \frac{B_2^\Gamma(u,v)^2}{2} + \frac{B_2^\Gamma(u^2v^2)}{2} \\
E(\mathcal{X}_\Gamma^{[1^2 2]}\,\mathrm{GL}_4; u,v) &= \frac{B_2^\Gamma(u,v) B_1^\Gamma(u^2v^2)}{2} + \frac{B_2^\Gamma(u,v) B_1^\Gamma(u,v)^2}{2} \\
E(\mathcal{X}_\Gamma^{[1^4]}\,\mathrm{GL}_4; u,v) &= \frac{B_1^\Gamma(u^4v^4)}{4} + \frac{B_1^\Gamma(u^3v^3) B_1^\Gamma(u,v)}{3} + \frac{B_1^\Gamma(u^2v^2)^2}{8} \\
&+ \frac{B_1^\Gamma(u^2v^2) B_1^\Gamma(u,v)^2}{4} + \frac{B_1^\Gamma(u,v)^4}{24}
\end{aligned}
$$

Explicit computations in the free group case

- Furthermore, for $\Gamma = F_r$, GL_n-character varieties are of polynomial type, then use Katz-Mozgovoy-Reineke to get combinatorial formulae for irreducible polynomials $B_n^r(u,v) = B_n^r(x) = E(\mathcal{X}_r^{irr} GL_n; x)$.

Proposition (Mozgovoy-Reineke ('15), Florentino-Nozad-Z. ('19))

For $r, n \geq 2$, we have $E(\mathcal{X}_r^{irr} GL_n; x) =$

$$(x-1)\sum_{d|n}\frac{\mu(n/d)}{n/d}\sum_{[k]\in\mathcal{P}_d}\frac{(-1)^{|[k]|}}{|[k]|}\binom{|[k]|}{k_1,\cdots,k_d}\prod_{j=1}^{d} b_j(x^{n/d})^{k_j} x^{\frac{n(r-1)k_j}{d}\binom{j}{2}},$$

where μ is the Möbius function, and the $b_j(x)$ are polynomials defined by

$$\left(1+\sum_{n\geq 1} b_n(x)\, t^n\right)\left(1+\sum_{n\geq 1}\left((x-1)(x^2-1)\dots(x^n-1)\right)^{r-1} t^n\right) = 1.$$

Explicit expressions for $B_n^r(x) = E(\mathcal{X}_r^{irr}\,\mathrm{GL}_n; x)$, $n \leq 4$, $(s = r-1)$

$$\frac{B_1^r(x)}{x-1} = (x-1)^s,$$

$$\frac{B_2^r(x)}{x-1} = (x-1)^s\Big((x-1)^s x^s((x+1)^s - 1) + \frac{1}{2}(x-1)^s - \frac{1}{2}(x+1)^s\Big),$$

$$\frac{B_3^r(x)}{x-1} = (x-1)^s\Big(-\frac{1}{3}(x^2+x+1)^s + (x-1)^{2s}(\frac{1}{3} - x^s + x^s(x+1)^s,$$
$$+x^{3s} + x^{3s}(x+1)^s(x^2+x+1)^s - 2x^{3s}(x+1)^s)\Big)$$

$$\frac{B_4^r(x)}{x-1} = (x-1)^{2s}\Big(\frac{1}{4}(x-1)^{2s} - \frac{1}{4}(x+1)^{2s} + (x^2-1)^s x^s(1-(x+1)^s),$$
$$+\frac{1}{2}(x+1)^{2s}x^{2s}(1-(x^2+1)^s) + \frac{1}{2}(x-1)^{2s}x^{2s}(1-(x+1)^s)^2$$
$$-(x-1)^{2s}x^{3s}(-(x+1)^s(x^2+x+1)^s + 2(x+1)^s - 1)$$
$$-(x-1)^{2s}x^{6s}(-(x+1)^s(x^2+x+1)^s(x^3+x^2+x+1)^s$$
$$+2(x+1)^s(x^2+x+1)^s + (x+1)^{2s} - 3(x+1)^s + 1)\Big).$$

Index

Consequences

This analysis allows to:

- Recover arithmetic computations of $E(\mathcal{X}_r \mathrm{GL}_n; x)$.
- Use locally closed stratification by partition type

$$\mathcal{X}_\Gamma \mathrm{GL}_n = \bigsqcup_{[k]\in\mathcal{P}_n} \mathcal{X}_\Gamma^{[k]} \mathrm{GL}_n$$

  to relate computations and geometry of $E(\mathcal{X}_\Gamma \mathrm{GL}_n; x)$ to each stratum $E(\mathcal{X}_\Gamma^{[k]} \mathrm{GL}_n; x)$.
- In particular, compute $E(\mathcal{X}_\Gamma^{[1^n]} \mathrm{GL}_n; x)$ (abelian stratum, abelian character varieties) and $E(\mathcal{X}_\Gamma^{[n]} \mathrm{GL}_n; x)$ (irreducible stratum, smooth).

Recovering arithmetic computations of $E(\mathcal{X}_r\,\mathrm{GL}_n;x)$

- For $s \geq 0$, the E-polynomial of the GL_3-character variety of the free group $\Gamma = F_{s+1}$ is:

$$\frac{E(\mathcal{X}_{s+1}\,\mathrm{GL}_3;x)}{(x-1)^{s+1}} = \tfrac{1}{2}(x-1)^{s+1}(x+1)^s x + \tfrac{1}{3}(x^2+x+1)^s x(x+1) + \\ + (x-1)^{2s}\Big((x+1)^s[x^{3s}(x^2+x+1)^s + x^{s+1} - 2x^{3s}] \\ + x^{3s} - x^{s+1} + \tfrac{x}{6}(x+1)\Big).$$

- Every $[k]$-polystable stratum $\mathcal{X}_r^{[k]}\,\mathrm{GL}_n$ is irreducible and has zero Euler characteristic:

$$\chi(\mathcal{X}_r^{[k]}\,\mathrm{GL}_n) = 0, \quad \chi(\mathcal{X}_r\,\mathrm{GL}_n) = 0.$$

Computations for GL_2

Stratification for $n = 2$

$$\mathcal{X}_\Gamma \mathrm{GL}_2 = \mathcal{X}_\Gamma^{[2]} \mathrm{GL}_2 \sqcup \mathcal{X}_\Gamma^{[1^2]} \mathrm{GL}_2 \cong \mathcal{X}_\Gamma^{irr} \mathrm{GL}_2 \sqcup \mathcal{X}_{\Gamma_{Ab}} \mathrm{GL}_2$$

where use $\mathcal{X}_\Gamma^{[1^n]} \mathrm{GL}_n \simeq \mathcal{X}_{\Gamma_{Ab}} \mathrm{GL}_n$.

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Use

- (Baraglia-Heckmati ('17)) use arithmetic arguments to compute $E(\mathcal{X}_\Gamma \mathrm{GL}_2; x)$ for various Γ.
- (Florentino-Silva ('19)) compute E-polynomials of abelian character varieties $E(\mathcal{X}_{\mathbb{Z}^r} \mathrm{GL}_n; x)$ through symmetric functions.

to calculate:

$$E(\mathcal{X}_\Gamma^{irr} \mathrm{GL}_2; x) = E(\mathcal{X}_\Gamma \mathrm{GL}_2; x) - E(\mathcal{X}_{\Gamma_{Ab}} \mathrm{GL}_2; x).$$

E-polynomials of irreducible loci of GL_2-character varieties

Theorem (Florentino-Nozad-Z. ('19))

1. *For free groups F_{s+1}:*

$$\frac{E(\mathcal{X}^{irr}_{s+1}\,\mathrm{GL}_2; x)}{(x-1)^{s+1}} = (x-1)^s x^s((x+1)^s - 1) - \frac{1}{2}(x+1)^s + \frac{1}{2}(x-1)^s.$$

2. *For surface goups $\Gamma_g = \pi_1(\Sigma_g)$, with $c = 2g-2$,*

$$\frac{E(\mathcal{X}^{irr}_{\Gamma_g}\,\mathrm{GL}_2; x)}{(x-1)^{c+2}} = (x^2-1)^c(x^c+1) + \frac{(x^{c+1}-x-1)}{2}(x+1)^c - \frac{(x^{c+1}-x+1)}{2}(x-1)^c - x^c.$$

3. *For non-orientable surface groups $\hat{\Gamma}_k = \pi_1(\hat{\Sigma}_g)$, with $h = k-2$,*

$$\frac{E(\mathcal{X}^{irr}_{\Gamma_k}\,\mathrm{GL}_2; x)}{(x-1)^{h+1}} = 2(x^h+1)(x^2-1)^h + x^h(x-1)\frac{(x-1)^h+(x+1)^h}{2} + (2-4x^h)(x-1)^h - (x+1)^h - 2x^h.$$

4. *For torus knot groups $\Gamma_{a,b}$ we have:*

$$\frac{E(\mathcal{X}^{irr}_{\Gamma_{a,b}}\,\mathrm{GL}_2; x)}{x-1} = \begin{cases} \frac{1}{4}(a-1)(b-1)(x-2), & a, b \text{ both odd} \\ \frac{1}{4}(b-1)(ax-3a+4), & a \text{ even, } b \text{ odd.} \end{cases}$$

E-polynomial of irreducible GL_3-character variety of surface group

STRATIFICATION FOR $n = 3$

$$\mathcal{X}_\Gamma \mathrm{GL}_3 = \mathcal{X}_\Gamma^{[3]} \mathrm{GL}_3 \sqcup \mathcal{X}_\Gamma^{[1\,2]} \mathrm{GL}_3 \sqcup \mathcal{X}_\Gamma^{[1^3]} \mathrm{GL}_3$$

$$E(\mathcal{X}_{\Gamma_g} \mathrm{GL}_3; x) = E(\mathcal{X}_{\Gamma_g}^{irr} \mathrm{GL}_3; x) + E(\mathcal{X}_{\Gamma_g}^{irr} \mathrm{GL}_1; x) \cdot E(\mathcal{X}_{\Gamma_g}^{irr} \mathrm{GL}_2; x) + E(\mathcal{X}_{\Gamma_g^{ab}} \mathrm{GL}_3; x)$$

THEOREM (FLORENTINO-NOZAD-Z. ('19))

The E-polynomial of the irreducible GL_3*-character variety of* Γ_g*, setting* $c = 2g-2$*, is*

$$\begin{aligned}
\frac{E(\mathcal{X}_{\Gamma_g}^{irr} \mathrm{GL}_3; x)}{(x-1)^{c+2}} &= (x-1)^{2c+2}\Big[x^{3c} - \frac{x^{c+1}}{2} - (x+1)^c(x^c+1) + \frac{1}{3}\Big] \\
&+ (x-1)^{2c+1}(x-2x^{2c})\Big[\frac{x^c(x-2)}{2} + (x+1)^c(x^c+1)\Big] \\
&+ (x-1)^{2c}(x^2+x+1)^c\big[(x+1)^c(x^{3c}+1) + x^{2c}\big] \\
&+ (x-1)^{2c}(x-2)x^{2c}\Big[(x+1)^c(x^c+1) + \frac{x^c(x-3)}{6}\Big] + \frac{(x-1)^{c+1}(x+1)^c}{2}\big[x^{c+1} - x^{3c+1}\big] \\
&+ (x-1)^c(x^c-1)\big[x^{c-2} + x^{c+1} - 2\big] + (x-1)^{c+2}\big[x^{2c-2} - x^{c-2}\big] \\
&+ \frac{(x^2+x+1)^c}{3}\big[x^{3c+1}(x+1) - (x^2+x+1)\big] - x^{3c}.
\end{aligned}$$

Index

PGL_n and SL_n fibrations for free group $\Gamma = F_r$

- From $\mathbb{C}^* \to \mathrm{GL}_n \to \mathrm{PGL}_n$, get $E(\mathrm{GL}_n; u, v) = (1 - uv) \cdot E(\mathrm{PGL}_n; u, v)$.

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Proposition (Florentino-Nozad-Z. ('19))

The fibration

$$\mathcal{R}_r\mathbb{C}^* \to \mathcal{X}_r^{[k]} \mathrm{GL}_n \to \mathcal{X}_r^{[k]} \mathrm{PGL}_n$$

is special, therefore

$$E(\mathcal{X}_r^{[k]} \mathrm{GL}_n; x) = (x-1)^r E(\mathcal{X}_r^{[k]} \mathrm{PGL}_n; x)$$

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$$E(\mathcal{X}_r \mathrm{GL}_n; x) = (x-1)^r E(\mathcal{X}_r \mathrm{PGL}_n; x).$$

- Also $E(\mathrm{GL}_n; u, v) = (1 - uv) \cdot E(\mathrm{SL}_n; u, v) \Rightarrow E(\mathrm{SL}_n; x) = E(\mathrm{PGL}_n; x)$ but hard to prove $E(\mathcal{X}_r \mathrm{GL}_n; x) = (x-1)^r E(\mathcal{X}_r \mathrm{SL}_n; x)$.

Solution of conjecture for Langlands dual groups PGL_n and SL_n

Theorem (Florentino-Nozad-Z. ('19))

For $\Gamma = F_r$, $E(\mathcal{X}_r \mathrm{SL}_n; x) = E(\mathcal{X}_r \mathrm{PGL}_n; x)$.

Solution of conjecture for Langlands dual groups PGL_n and SL_n

THEOREM (FLORENTINO-NOZAD-Z. ('19))

For $\Gamma = F_r$, $E(\mathcal{X}_r \mathrm{SL}_n; x) = E(\mathcal{X}_r \mathrm{PGL}_n; x)$.

- Try to imitate the PGL_n-fibration:

$$\mathbb{Z}_n^r \to \mathcal{X}_r \mathrm{SL}_n \to \mathcal{X}_r \mathrm{PGL}_n$$

but the fiber is not connected, then we cannot directly apply multiplicative property for E-polynomials!

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  but the fiber is not connected, then we cannot directly apply multiplicative property for E-polynomials!

- We prove the Theorem by distinguishing between partitions of two or more blocks ($[k]$ of length > 1, reducible) and the irreducible case ($[k] = [n]$ of length $= 1$).

PROPOSITION (FLORENTINO-NOZAD-Z. ('19))

If length $[k] \in \mathcal{P}_n$ *is* > 1, *then* $E(\mathcal{X}_r^{[k]} \mathrm{GL}_n; x) = (x-1)^r E(\mathcal{X}_r^{[k]} \mathrm{SL}_n; x)$.
Therefore $E(\mathcal{X}_r^{[k]} \mathrm{SL}_n; x) = E(\mathcal{X}_r^{[k]} \mathrm{PGL}_n; x)$.

Sketch of the proof for $|[k]| > 1$

- Let $[k]$ a partition whose blocks have size $n_1, n_2, \ldots, n_s$, $s > 1$, and $g.c.d(n_1, \ldots, n_s) = d$.
- Let $m(\sigma_1, \ldots, \sigma_s) \mapsto \sigma_1^{m_1}, \ldots, \sigma_s^{m_s}$, with $m_i = n_i/d$

$$\begin{array}{ccccc}
H := \ker m & \subset & J := (\mathcal{R}_r\mathbb{C}^*)^s & \xrightarrow{m} & \mathcal{R}_r\mathbb{C}^* \\
\downarrow & & \downarrow & & \\
S\mathsf{X}_r^{\mathbf{n}} := \{x \in \mathsf{X}_r^{\mathbf{n}} | \det(x) = 1\} & \subset & \mathsf{X}_r^{\mathbf{n}} := \times_{i=1}^{s} \mathcal{X}_r^{irr} GL_{n_i} & & \\
\downarrow & & \downarrow & & \\
S\mathsf{X}_r^{\mathbf{n}}/H & = & \mathsf{X}_r^{\mathbf{n}}/J, & &
\end{array}$$

- H is abelian, connected and reductive, then $H \simeq (\mathbb{C}^*)^{r(s-1)}$.
- Fibrations are special and enjoy multiplicative property for E-polynomials.
- Take quotients by actions of symetric groups $S_{[k]}$ (permuting blocks) and take invariant parts of the equivariant E-polynomials.

Main theorem: Irreducible locus $[n]$

Theorem (Florentino-Nozad-Z. ('19))

The quotient map

$$\mathcal{X}_r^{irr}\,\mathrm{SL}_n \to \mathcal{X}_r^{irr}\,\mathrm{PGL}_n$$

given by the central action of $\mathbb{Z}_n^r$ *on* $\mathcal{X}_r^{irr}\,\mathrm{SL}_n$ *induces an isomorphism of mixed Hodge structures*

$$H_c^*(\mathcal{X}_r^{irr}\,\mathrm{SL}_n) \cong H_c^*(\mathcal{X}_r^{irr}\,\mathrm{PGL}_n).$$

Therefore,

$$E(\mathcal{X}_r^{irr}\,\mathrm{SL}_n; x) = E(\mathcal{X}_r^{irr}\,\mathrm{PGL}_n; x).$$

Sketch of the proof for $[n]$

- Define $C_{r,n} = \mathrm{Hom}(F_r, \mathbb{Z}_n)$.
- Define $\mathrm{U}^*_{r,n} = \mathrm{Hom}^{irr}(F_r, \mathrm{U}(n)) \subset \mathrm{U}_{r,n} = \mathrm{Hom}(F_r, \mathrm{U}(n))$ and similarly $\mathrm{SU}^*_{r,n}$, $\mathrm{SU}_{r,n}$, $\mathrm{PU}^*_{r,n}$, $\mathrm{PU}_{r,n}$ for representations into SL_n and PGL_n.

Get stratifications

$$U_{r,n} = \bigsqcup_{[k]\in\mathcal{P}_n} U_r^{[k]}, \qquad SU_{r,n} = \bigsqcup_{[k]\in\mathcal{P}_n} SU_r^{[k]}, \qquad PU_{r,n} = \bigsqcup_{[k]\in\mathcal{P}_n} PU_r^{[k]}.$$

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Step 1

For the stratification $\mathrm{SU}_{r,n} = \bigsqcup_{[k]\in\mathcal{P}_n} \mathrm{SU}_r^{[k]}$:

- If length $[k] > 1$, action $C_{r,n} \curvearrowright H^*(\mathrm{SU}_{r,n}^{[k]})$ is trivial (construct homotopies to the identity).
- Given that $C_{r,n} \subset \mathrm{SU}_{r,n}$ connected, action $C_{r,n} \curvearrowright H^*(SU_{r,n})$ is trivial.
- 5-lemma on stratification to conclude action $C_{r,n} \curvearrowright H^*(SU^*_{r,n})$ is trivial.

Sketch of the proof for $[n]$

STEP 2

$\pi : \mathrm{SU}^*_{r,n} \to \mathrm{PU}^*_{r,n} = \mathrm{SU}^*_{r,n}/C_{r,n}$ is $\mathrm{PU}(n)$-equivariant, then

$$H^*(\mathrm{Hom}^{irr}(F_r, \mathrm{SL}_n)) \overset{ret}{\simeq} H^*(\mathrm{SU}^*_{r,n}) \simeq H^*(\mathrm{PU}^*_{r,n}) \overset{ret}{\simeq} H^*(\mathrm{Hom}^{irr}(F_r, \mathrm{PGL}_n))$$

Sketch of the proof for $[n]$

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STEP 3

$$\mathrm{Hom}^{irr}(F_r, \mathrm{SL}_n) \to \mathrm{Hom}^{irr}(F_r, \mathrm{PGL}_n) = \mathrm{Hom}^{irr}(F_r, \mathrm{SL}_n)/C_{r,n}$$

is $\mathrm{PGL}(n)$-equivariant between orbifolds, then (similar to Step 2):

$$H^*_c(\mathcal{X}^{irr}_r \mathrm{SL}_n) \cong H^*_c(\mathcal{X}^{irr}_r \mathrm{PGL}_n).$$

It is an algebraic map + inducing an isomorphism in cohomology $\Rightarrow$ Mixed Hodge structures coincide.

E-polynomial of the SL_4 and PGL_4-character varieties of the free group

Theorem (Florentino-Nozad-Z. ('19))

The E-polynomials of the SL_4 (and PGL_4)-character varietes of the free group $\Gamma = F_{s+1}$ are equal to

$$E(\mathcal{X}_{s+1}\,\mathrm{SL}_4; x) =$$

$$\frac{1}{24}(x-1)^{3s+3} + (x-1)^{3s+1}\Big[(x+1)^{2s}\frac{x^{2s}}{2} + (x+1)^s\Big(x^{3s}(x^2+x+1)^s - 2x^{3s} - x^{2s} + \frac{3x^s}{2}\Big)\Big]$$

$$+(x-1)^{3s+1}\Big[x^{3s} + \frac{x^{2s}}{2} - \frac{3x^s}{2} + \frac{11}{24}\Big] + (x-1)^{3s}(x+1)^{2s}\Big(-x^{6s} + \frac{x^{2s}}{2}\Big)$$

$$+(x-1)^{3s}(x+1)^s x^{6s}\Big[(x^2+x+1)^s(x^3+x^2+x+1)^s - 2(x^2+x+1)^s + 3\Big]$$

$$+(x-1)^{3s}(x+1)^s\Big[x^{3s}\Big((x^2+x+1)^s - 2\Big) - x^{2s} + \frac{x^s}{2}\Big] + (x-1)^{3s}\Big(-x^{6s} + x^{3s} + \frac{x^{2s}}{2} - \frac{x^s}{2} + \frac{1}{2}\Big)$$

$$+(x-1)^{2s+2}\frac{(x-1)^{s+1}}{4} + (x-1)^{2s+1}\frac{(x+1)^s}{2}\Big(-(x+1)^s x^s + x^s - \frac{1}{2}\Big) + (x-1)^{2s}(x+1)^s\frac{x^s}{2}\big(1-(x+1)^s\big)$$

$$+(x-1)^{s+1}\Big[(x+1)\frac{x}{3}(x^2+x+1)^s + \frac{(x+1)^{2s}}{8}(x^2+2x+2)\Big]$$

$$+(x-1)^s(x+1)^{2s}\Big[\frac{x^{2s+1}}{2}\big((x^2+1)^s - 1\big) + \frac{x-1}{4}\Big] - \frac{1}{4}(x+1)^{s+1}(x^2+1)^s + \frac{1}{4}(x^3+x^2+x+1)^{s+1}.$$

Euler characteristics of SL_4 and PGL_4-character varieties of the free group

Theorem (Florentino-Nozad-Z. ('19))

The Euler characteristics of the PGL_n and SL_n-character varieties of the free group are

$$\chi(\mathcal{X}_r \mathrm{PGL}_n) = \chi(\mathcal{X}_r \mathrm{SL}_n) = \phi(n) n^{r-2} ,$$

where $\phi(n)$ is the arithmetic Euler function. For $[d^{n/d}] \in \mathcal{P}_n$,

$$\chi(\mathcal{X}_r^{[d^{n/d}]} \mathrm{PGL}_n) = \chi(\mathcal{X}_r^{[d^{n/d}]} \mathrm{SL}_n) = \frac{\mu(d)}{d} n^{r-1} ,$$

$$\textit{otherwise} \quad \chi(\mathcal{X}_r^{[k]} \mathrm{PGL}_n) = \chi(\mathcal{X}_r^{[k]} \mathrm{SL}_n) = 0 ,$$

where $\mu(n)$ is the arithmetic Möebius function.

Further conjectures

Conjectures on topological mirror symmetry

For other pairs of Langlands dual groups G and ${}^LG$, and $\Gamma = F_r$, are

- $\mathcal{X}_r G$ and $\mathcal{X}_r {}^LG$ Hodge-Tate ?
- $\mathcal{X}_r G$ and $\mathcal{X}_r {}^LG$ polynomial type?
- $E(\mathcal{X}_r G; x) = E(\mathcal{X}_r {}^LG; x)$?

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