Multiunitary matrices and their applications

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QM^3, Quantum Matter meets Math

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What is this talk about ?

we analyze

a) distinguished subsets of the set of unitary matrices $U(d^k)$ of a power dimension $N = d^k$

b) corresponding **discrete** structures in a finite **Hilbert space** $\mathcal{H}_{N^2}$ relevant for the standard **Quantum Theory**,

for instance:

- **Absolutely Maximally Entangled (AME) states** of $2k$ subsystems with d levels each

Why we do it ? Because we

a) do not fully understand these structures relevant for **quantum theory** !

b) wish to construct novel schemes of **generalized measurements**,

c) construct original models of quantum dynamics,

d) quantum error correction codes

Composed systems & **entangled states**

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states **not** of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Schmidt decomposition & Entanglement measures

Any pure state from $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle \; = \; \sum_{ij} G_{ij} |i\rangle \otimes |j\rangle \; = \; \sum_i \sqrt{\lambda_i} |i'\rangle \otimes |i''\rangle,$$

where $|\psi|^2 = \mathrm{Tr}\, GG^\dagger = 1$. The partial trace, $\sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi| = GG^\dagger$, has spectrum given by the **Schmidt vector** $\{\lambda_i\}$ = squared **singular values** of G. Linear entanglement entropy of $|\psi\rangle$ is equal to **linear entropy** of the reduced state σ, $E_L(|\psi\rangle) := 1 - \mathrm{Tr}\, \sigma^2 = 1 - \sum_i \lambda_i^2$.

The more **mixed** partial trace, the more **entangled** initial pure state...

Maximally entangled bi–partite quantum states

Bipartite systems $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B = \mathcal{H}_d \otimes \mathcal{H}_d$

generalized Bell state (for two qu*d*its),

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

distinguished by the fact that all **singular values** are equal, $\lambda_i = 1/d$,
hence the reduced state is **maximally mixed**,
$\rho_A = \mathrm{Tr}_B |\psi_d^+\rangle\langle\psi_d^+| = \mathbb{1}_d/d$.
This property holds for all locally equivalent states, $(U_A \otimes U_B)|\psi_d^+\rangle$.

Observations:

A) State $|\psi\rangle$ is **maximally entangled** if $\rho_A = GG^\dagger = \mathbb{1}_d/d$,
which is the case if the matrix $U = G/\sqrt{d}$ of size d is **unitary**,
(and all its **singular values** are equal to 1).

B) For a bipartite state the **singular values** of G characterize
entanglement of the state $|\psi\rangle = \sum_{i,j} G_{ij}|i,j\rangle$.

Bipartite quantum gates: unitary $U \in U(d^2)$

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **local gates** $U_{\rm loc} = V_A \otimes V_B$
- **non-local gates:** all unitaries **not** of the above **product** form.

Let $|m\rangle \otimes |\mu\rangle$ be a product basis in $\mathcal{H}_A \otimes \mathcal{H}_B$.
For any operator X with entries $X_{\substack{m\mu\\n\nu}} = \langle m\mu| X |n\nu\rangle$ define **reshuffled** matrix X^R with entries $X^R_{\substack{m\mu\\n\nu}} = X_{\substack{mn\\\mu\nu}} = \langle mn| X |\mu\nu\rangle$.

Operator Schmidt decomposition of a unitary U of size d^2

$U = d^2 \sum_{i=1}^{d^2} \sqrt{\lambda_i} A_i \otimes B_i$, where $\mathrm{Tr}\left\{A_i^\dagger A_j\right\} = \mathrm{Tr}\left\{B_i^\dagger B_j\right\} = \delta_{ij}$.
Then the Schmidt vector λ normalized as $\sum_{i=1}^{d^2} \lambda_i = 1$ is given by the spectrum of a positive matrix $\sigma = \frac{1}{d^2} U^R (U^R)^\dagger$

a) local $U_{\rm loc} = V_A \otimes V_B$, $\lambda = (1, 0, \ldots, 0)$; entropy $E(U) = E(\lambda) = 0$
b) U^R is unitary, $\lambda = (1, 1, \ldots, 1)/d^2$; entropy $E(U) = E(\lambda) = 1 - 1/d^2$.

Two-qubit quantum gates: unitary $U \in U(4)$

Reshuffled matrix: blocks converted into vectors - color entries exchanged:

$$X^R_{kj} := \left[\begin{array}{cc|cc} \mathbf{X_{11}} & \mathbf{X_{12}} & X_{21} & X_{22} \\ X_{13} & X_{14} & \mathbf{X_{23}} & \mathbf{X_{24}} \\ \hline \mathbf{X_{31}} & \mathbf{X_{32}} & X_{41} & X_{42} \\ X_{33} & X_{34} & \mathbf{X_{43}} & \mathbf{X_{44}} \end{array}\right].$$

Consider a unitary matrix invariant with respect to reshuffling:

$$U_\theta := \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin\theta & \cos\theta \end{array}\right] = U^R_\theta,$$

so that the Schmidt vector is uniform, $\lambda = (1,1,1,1)/4$, and the entropy $E(\lambda)$ is maximal. For any phase θ these gates are **maximally nonlocal** (**Musz, Kuś, K.Ż.** 2013). For $\theta = 0$ we arrive at the SWAP gate S. The gates U_θ belong to the class of *dual unitary* gates, (**Bertini, Kos, Prosen** 2019), such that U and U^R are unitary.

Reshuffling of a matrix $U_D \in U(9)$

$$U = \left(\begin{array}{ccc|ccc|ccc} \bullet & \bullet & \bullet & x & x & x & y & y & y \\ x & x & x & \bullet & \bullet & \bullet & z & z & z \\ y & y & y & z & z & z & \bullet & \bullet & \bullet \\ - & - & - & - & - & - & - & - & - \\ \bullet & \bullet & \bullet & x & x & x & y & y & y \\ x & x & x & \bullet & \bullet & \bullet & z & z & z \\ y & y & y & z & z & z & \bullet & \bullet & \bullet \\ - & - & - & - & - & - & - & - & - \\ \bullet & \bullet & \bullet & x & x & x & y & y & y \\ x & x & x & \bullet & \bullet & \bullet & z & z & z \\ y & y & y & z & z & z & \bullet & \bullet & \bullet \end{array}\right) \in U(9)$$

x, y, z denote entries exchanged by reshuffling,

so to arrive at $U = U^R$ they can be replaced by 0,

while $\bullet$ do not change,

so they can be filled in with numbers satisfying unitarity conditions,

Two-qutrit Dual unitary gate $U_D \in U(9)$

$$D_D = \left(\begin{array}{ccc|ccc|ccc}
{\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} & x & x & x & y & y & y \\
x & x & x & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & z & z & z \\
y & y & y & z & z & z & {\color{green}\bullet} & {\color{green}\bullet} & {\color{green}\bullet} \\
- & - & - & - & - & - & - & - & - \\
{\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} & x & x & x & y & y & y \\
x & x & x & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & z & z & z \\
y & y & y & z & z & z & {\color{green}\bullet} & {\color{green}\bullet} & {\color{green}\bullet} \\
- & - & - & - & - & - & - & - & - \\
{\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} & x & x & x & y & y & y \\
x & x & x & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & z & z & z \\
y & y & y & z & z & z & {\color{green}\bullet} & {\color{green}\bullet} & {\color{green}\bullet}
\end{array}\right) = D_D^R$$

$x, y, z = 0$ denote entries exchanged by reshuffling which are set to zero

Two-qutrit Dual unitary gate $U_D \in U(9)$

determined by three unitary matrices $V, W, Y \in U(3)$

$$D_D = \left(\begin{array}{ccc|ccc|ccc}
V_{11} & V_{12} & V_{13} & x & x & x & y & y & y \\
x & x & x & W_{11} & W_{12} & W_{13} & z & z & z \\
y & y & y & z & z & z & Y_{11} & Y_{12} & Y_{13} \\
- & - & - & - & - & - & - & - & - \\
V_{21} & V_{22} & V_{23} & x & x & x & y & y & y \\
x & x & x & W_{21} & W_{22} & W_{23} & z & z & z \\
y & y & y & z & z & z & Y_{21} & Y_{22} & Y_{23} \\
- & - & - & - & - & - & - & - & - \\
V_{31} & V_{32} & V_{33} & x & x & x & y & y & y \\
x & x & x & W_{31} & W_{32} & W_{33} & z & z & z \\
y & y & y & z & z & z & Y_{31} & Y_{32} & W_{33}
\end{array}\right) = D_D^R$$

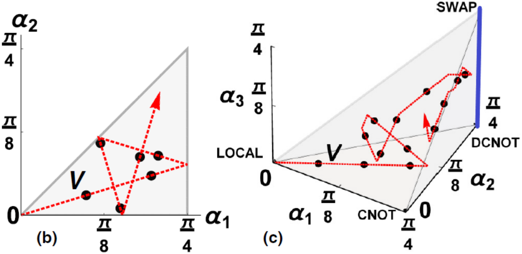
$x, y, z = 0$ denote entries exchanged by reshuffling which are set to zero

Canonical form of Two-qubit gates

Any $U \in U(4)$ can be written in the **Cartan form**,

$$U = (V_1 \otimes V_2) \exp\Big(i \sum_{j=1}^{3} \alpha_j \sigma_j \otimes \sigma_j\Big)(V_3 \otimes V_3),$$

where V_i represent single-qubit gates and σ_j stand for 3 Pauli matrices. The vector *information content* α can be chosen from a Weyl chamber, $\pi/4 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0$ (**Kraus, Cirac** 2001)



- maximaly nonlocal gates with $\alpha = (\pi/4, \pi/4, \alpha_3)$ interpolate between SWAP and DCNOT gates – blue line
- time evolution for $\alpha(V^t)$ leads to a billiard like trajectory in the chamber, ergodic for a generic initial point, (**Mandarino, Linowski, K.Ż.** 2018)

Entangling power of a bi-partite gate

Any local gate, $U_{\mathrm{loc}} = V_A \otimes V_B$, cannot produce entanglement.

Does it mean that any **strongly non-local** gate always produces entanglement?

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Does it mean that any **strongly non-local** gate always produces entanglement?

No! SWAP gate, $S|x,y\rangle = |y,x\rangle$ is maximally non-local, $E(S) = E_{\rm max} = 1 - 1/d^2$ and it cannot change entanglement....

Another useful measure: **entangling power** $e_p(U) = \langle E_L(U|x,y\rangle)\rangle_{x,y}$ where the averaging is done over random product states $|x,y\rangle$.

Zanardi showed (2001) that

$$e_p(U) := [E(U) + E(US) - E(S)]/(d^2-1)^2.$$

With *gate typicality*,

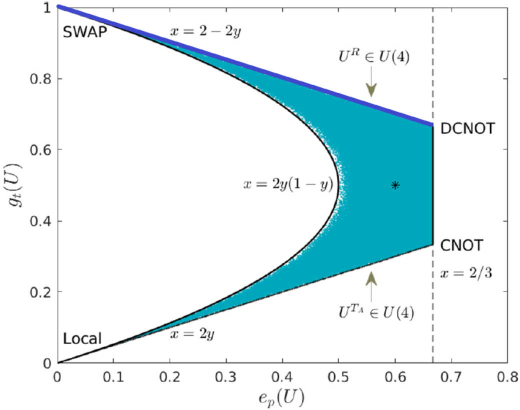
$$g_t(U) := d^2[E(U) - E(US) + E(S)]/(d^2-1)^2$$

they span a plane, (e_p, g_t), useful to study the set of all the gates,

Jonnadula, Mandayam, K.Ż., Lakshminarayan (2017)

Two-qubit gates:, $d^2 = 4$ no absolute maximum of e_p

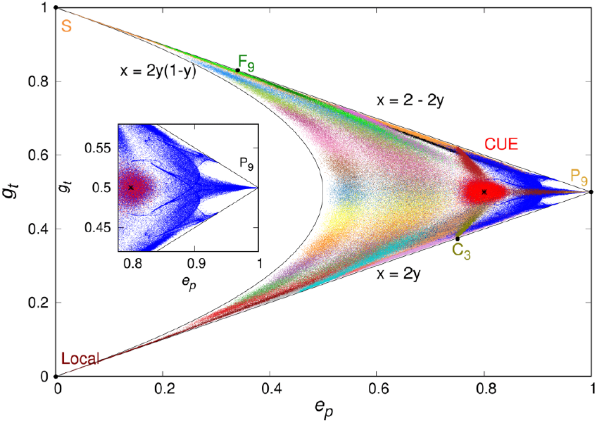
projection of the set $U(4)$ of two-qubit unitary gates onto the plane (e_p, g_t):



Upper blue line represents the maximally nonlocal gates (**dual unitary**). Maximal entangling power e_p is attained for gates interpolating between CNOT and DCNOT, but the absolute maximum, $e_p = 1$ is not achieved.

Two-qutrit gates:, $d^2 = 9$, absolute maximum $e_p = 1$

projection of the set $U(9)$ of two-qutrit unitary gates onto the plane (e_p, g_t):



Maximal entangling power $e_p = 1$ is achieved for a particular permutation matrix P_9 such that reshuffled matrix P_9^R and partial transpose, P_9^Γ are unitary. **Jonnadula, Mandayam, K.Ż., Lakshminarayan** (2020)

$U(9)$ gate maximizing the entangling power

permutation matrix of size $9 = 3^2$

$$P_9 = U_{\substack{ij\\ml}} = \left(\begin{array}{ccc|ccc|ccc} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{array}\right) \in U(9)$$

Furthermore, also two reordered matrices
(by partial transposition, P_9^Γ and by reshuffling, P_9^R) remain **unitary**:

$$U^{\Gamma} = U_{\substack{il\\mj}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in U(9)$$

$$U^{R} = U_{\substack{im\\jl}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \in U(9)$$

Wawel Castle, **Cracow**, Poland

Multipartite pure quantum states

are determined by a **tensor**:

e.g. $|\Psi_{ABCD}\rangle = \sum_{i,j,k,l} T_{i,j,k,l} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \otimes |l\rangle_D$.

Mathematical problem: in general for a **tensor** there is no (unique) **Singular Value Decomposition** and it is not simple to find the **tensor rank** or **tensor norms** (nuclear, spectral)

see **Bruzda, Friedland, K.Ż. arXiv:1912.06854**

Open question: Which state of N subsystems with d–levels each is the **most entangled** ?

Absolutely maximally entangled (AME) states

Definition. State $|\psi\rangle \in \mathcal{H}_d^{\otimes M}$ with $M = 2k$ is called **AME state** if it is maximally entangled for all possible symmetric splittings of the system into two parts of k subsystems each so the reduced states are maximally mixed (**Scott 2001**), **Facchi et al.** (2008,2010), **Arnaud & Cerf** (2012)

Applications: quantum error correction codes, teleportation, etc...

Some examples of AME states:

simplest case, $d = 2$:

There exist no AME states for 4 qubits

Higuchi & Sudbery (2000) - **frustration** like in spin systems – **Facchi, Florio, Marzolino, Parisi, Pascazio** (2010) – to many constraints to be simmultaneously satisfied
$\Rightarrow$ no 2-qubit unitary $U \in U(4)$ achives the absolute bound $e_p = 1$

higher dimension, $d = 3$,

There exists an AME state of 4 qutrits

state AME(4,3) $|\Psi_3^4\rangle \in \mathcal{H}_3^{\otimes 4}$ by **Popescu**:

$$
\begin{aligned}
|\Psi_3^4\rangle = \; & |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

and corresponds to the optimal permutation matrix P_9

AME(4,3) state of four qutrits, $N = 4$ and $d = 3$

$$
\begin{aligned}
|\Psi_3^4\rangle \;=\; & |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

This state is also encoded in a pair of orthogonal Latin squares of size 3,

$$
\begin{array}{|c|c|c|}
\hline
0\alpha & 1\beta & 2\gamma \\ \hline
1\gamma & 2\alpha & 0\beta \\ \hline
2\beta & 0\gamma & 1\alpha \\ \hline
\end{array}
=
\begin{array}{|c|c|c|}
\hline
A\spadesuit & K\clubsuit & Q\diamondsuit \\ \hline
K\diamondsuit & Q\spadesuit & A\clubsuit \\ \hline
Q\clubsuit & A\diamondsuit & K\spadesuit \\ \hline
\end{array} .
$$

Corresponding **Quantum Code**:

$$
\begin{aligned}
|0\rangle &\to |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\
|1\rangle &\to |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\
|2\rangle &\to |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle
\end{aligned}
$$

AME(4,3) state of four qutrits, $N = 4$ and $d = 3$

AME state $|\Psi_3^4\rangle = \sum_{i,j=0}^{2} |i\rangle \otimes |j\rangle \otimes |i + j_{\text{mod }3}\rangle \otimes |i + 2j_{\text{mod }3}\rangle$

$$= \sum_{i,j=0}^{2} |i,j\rangle \otimes |\phi_{ij}\rangle = \sum_{i,j=0}^{2} |i,j\rangle \otimes U|i,j\rangle$$

$U = P_9 \in U(9)$, where U acts as an isometry between the basis (i,j) and $(l,k) = (i+j, i+2j)$ denoting: **rows, columns, suits, honors**

i	j	$i+j$	$i+2j$
0	0	0	0
0	1	1	2
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	2
2	0	2	2
2	1	0	1
2	2	1	0

Mutually orthogonal Latin Squares (MOLS)

♣) $d = 2$. There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)
♡) $d = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(d-1)$ MOLS.
♠) $d = 6$. Only a **single** Latin Square exists (No OLS!).

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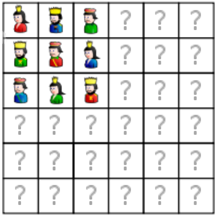
Euler's problem: **36** officers of six different ranks from six different units come for a **military parade**. Arrange them in a square such that in each row / each column all uniforms are different.



No solution exists ! (**1799** conjecture by **Euler**), proof (121 years later) **Gaston Tarry** "Le Probléme de 36 Officiers". *Compte Rendu* **(1900)**.

Mutually orthogonal Latin Squares (MOLS)

An apparent solution of the $d = 6$ **Euler's** problem of **36 officers**
36cuBe by **D. C. Niederman**, (2008):
the World's Most Challenging Puzzle

Why do we care about AME states?

Since they can be used for various purposes
(e.g. **Quantum codes,** **teleportation,**...)

Resources needed for **quantum teleportation**:

a) **2-qubit** **Bell state** allows one to teleport **1 qubit** from A to B

b) **2-qudit generalized** **Bell state** allows one to teleport **1 qudit**

c) **3-qubit** **GHZ state** allows one to teleport **1 qubit** between any users

d) **4-qutrit** **GHZ state** allows one to teleport **1 qutrit**
between any two out of four users

f) **4-qutrit** **state AME(4,3)** allows one to teleport **2 qutrits** between
any pair chosen from four users to the other pair!
- say from the pair (A & C) to (B & D)

relations between **AME states** and **multiunitary matrices,**
perfect tensors **and** **holographic codes**

4–party AME state and two–unitary matrices

Consider an **AME state** of four parties A, B, C, D with d levels each,

$$|\psi\rangle = \sum_{i,j,l,m=1}^{d} T_{ijlm}|i,j,l,m\rangle$$

It is **maximally entangled** with respect to all **three** partitions:

$$AB|CD \text{ and } AC|BD \text{ and } AD|BC.$$

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions are **maximally mixed**,
$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD} = \rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD} = \rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD} = \mathbb{1}_{d^2}/d^2$
Thus matrices $U_{\mu,\nu}$ of order d^2 obtained by reshaping the tensor T_{ijkl} are **unitary** for three reorderings:

a) $\mu,\nu = ij, lm,$ b) $\mu,\nu = im, jl,$ c) $\mu,\nu = il, jm.$

Such a tensor T is called **perfect**,

Pastawski, Yoshida, Harlow, Preskill (2015)

Corresponding **unitary matrix** U of order d^2 is called **two–unitary**
if reordered matrices U^R and U^Γ remain **unitary.**

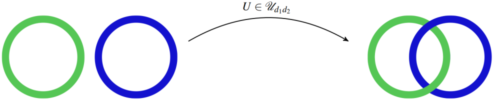
Goyeneche, Alsina, Latorre, Riera, K. Ż. (2015)

2-unitarity (U^R and U^Γ unitary) is stronger than *dual unitary* (U^R unitary)

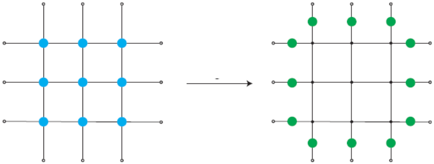
In hunt for an $|AME(4,6)\rangle$ state of 4 quhex, $d = 6$

To find the state
$|AME(4,6)\rangle = (U_{AB} \otimes \mathbb{I}_{CD})|\Psi^+_{AC|BD}\rangle = \sum_{i,j,k,\ell=1}^{6} t_{ijk\ell}|i,j,k,\ell\rangle$

we look for a 2–**unitary** matrix $U_{AB} \in U(36)$, which remains unitary after reorderings, maximizes the **entangling power** $e_p(U)$



(average entanglement of $U_{AB}|\psi_A\rangle \otimes |\psi_B\rangle$)
and leads to a perfect tensor $t_{ijk\ell}$ used for models of bulk/boundary duality



Optimization over the space $U(36)$ of dimension $36^2 - 1 = 1295$
is not easy...

Numerical Search

$$U_0 \mapsto U_0^{\mathrm{R}} \mapsto \left(U_0^{\mathrm{R}}\right)^{\Gamma} := U_0^{\Gamma\mathrm{R}} \mapsto U_1$$

$\tilde{P} =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
54	63	26	15	42	31
65	56	12	21	33	44

=

A♠	K♣	Q♦	J♥	10✿	9✱
K♦	A♥	J✿	Q✱	9♠	10♣
Q♣	J♠	9♥	10♦	A✱	K✿
J✱	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✱	A✿	J♣	Q♠
9✿	10✱	A♣	K♠	Q♦	J♥

$$e_p(\tilde{P}) = \frac{314}{315} \simeq .9968 \qquad e_p(\tilde{P}e^{iH\varepsilon}) \to .9991$$

$$e_p(\tilde{P}) \to \frac{419}{420} \simeq .9976$$

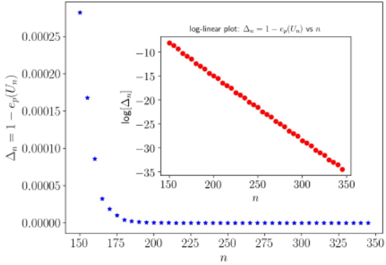
$$e_p(\tilde{P}_s) = \frac{104}{105} \simeq .9905 \qquad e_p(\tilde{P}_s e^{iH\varepsilon}) \to 1$$

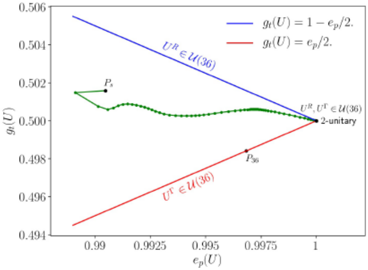
$\tilde{P}_s =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
64	56	26	15	43	31
55	63	12	21	42	34

=

A♠	K♣	Q♦	J♥	10✿	9✱
K♦	A♥	J✿	Q✱	9♠	10♣
Q♣	J♠	9♥	10♦	A✱	K✿
J✱	Q✿	10♠	9♣	K♥	A♦
9♥	10✱	K✱	A✿	J♦	Q♠
10✿	9♦	A♣	K♠	J♣	Q♥

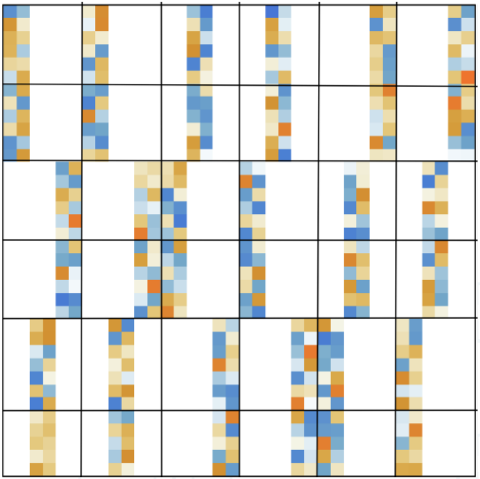




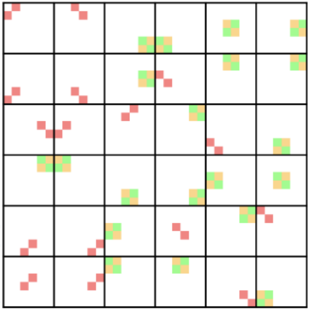
Numerical Cleaning

$$\left(U_6^{(1)} \otimes U_6^{(2)}\right) U_{36} \left(U_6^{(4)} \otimes U_6^{(3)}\right)$$

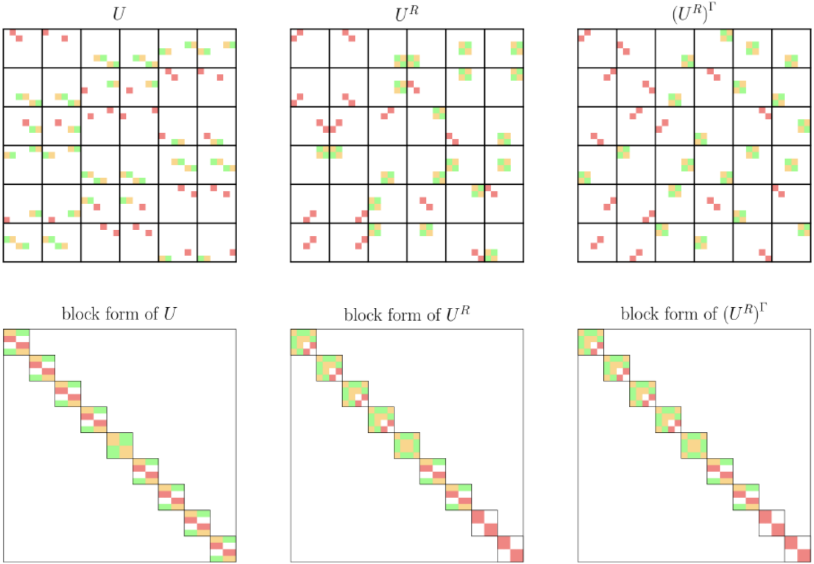
$$\left(\cancel{U_6} \otimes U_2^{\otimes 3}\right) U_{36} \left(U_2^{\otimes 3} \otimes U_2^{\otimes 3}\right)$$



0.75	0	0	0	0	0	0.25	0
0	0.25	0	0	0	0	0	0.25
0	0	0.25	0	0.25	0	0	0
0	0	0	0.75	0	0.25	0	0
0	0	0.25	0	0.25	0	0	0
0	0	0	0.25	0	0.75	0	0
0.25	0	0	0	0	0	0.75	0
0	0.25	0	0	0	0	0	0.25



SOLUTION FOUND



Rather, Burchardt, Bruzda, Rajchel, Lakshminarayan, K.Ż.
preprint arXiv:2102.07787

Solution Found

(1,1)	(2,2)	(1,2)	(2,1)	
$a\,\omega^{10}$	a	$b\,\omega^{15}$	$b\,\omega^{5}$	(6,3)
		c	c	(1,1)
c	c			(5,6)
$b\,\omega^{10}$	b	$a\,\omega^{5}$	$a\,\omega^{15}$	(4,2)

(1,3)	(2,4)	(1,4)	(2,3)	
$a\,\omega^{2}$	$a\,\omega^{14}$	$b\,\omega$	$b\,\omega^{5}$	(2,5)
		$c\,\omega^{5}$	$c\,\omega^{19}$	(3,3)
$c\,\omega^{17}$	$c\,\omega^{19}$			(1,2)
$b\,\omega^{14}$	$b\,\omega^{6}$	$a\,\omega^{3}$	$a\,\omega^{7}$	(6,4)

(1,5)	(2,6)	(1,6)	(2,5)	
$a\,\omega$	$a\,\omega^{19}$	$b\,\omega^{14}$	$b\,\omega^{16}$	(4,1)
$a\,\omega$	$a\,\omega^{3}$	$b\,\omega^{10}$	$b\,\omega^{4}$	(3,4)
$b\,\omega^{4}$	$b\,\omega^{18}$	$a\,\omega^{3}$	$a\,\omega^{9}$	(2,6)
$b\,\omega^{2}$	$b\,\omega^{8}$	$a\,\omega^{5}$	$a\,\omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)	
$a\,\omega^{4}$	$a\,\omega^{10}$	$b\,\omega^{17}$	$b\,\omega^{7}$	(4,5)
		$c\,\omega^{2}$	$c\,\omega^{2}$	(3,2)
$c\,\omega^{10}$	$c\,\omega^{6}$			(2,4)
$b\,\omega^{7}$	$b\,\omega^{13}$	$a\,\omega^{10}$	a	(5,3)

(3,3)	(4,4)	(3,4)	(4,3)	
a	a	$b\,\omega^{15}$	$b\,\omega^{15}$	(4,6)
		c	$c\,\omega^{10}$	(6,1)
c	$c\,\omega^{10}$			(5,4)
b	b	$a\,\omega^{5}$	$a\,\omega^{5}$	(1,5)

(3,5)	(4,6)	(3,6)	(4,5)	
$a\,\omega^{2}$	a	$b\,\omega^{19}$	$b\,\omega^{13}$	(2,3)
		$c\,\omega^{16}$	c	(6,2)
$c\,\omega^{8}$	$c\,\omega^{16}$			(3,1)
$b\,\omega^{14}$	$b\,\omega^{12}$	$a\,\omega$	$a\,\omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)	
$a\,\omega^{3}$	$a\,\omega^{7}$	b	b	(1,4)
		c	$c\,\omega^{10}$	(2,1)
$c\,\omega^{13}$	$c\,\omega^{7}$			(3,5)
$b\,\omega^{9}$	$b\,\omega^{13}$	$a\omega^{16}$	$a\omega^{16}$	(6,6)

(5,3)	(6,4)	(5,4)	(6,3)	
$a\,\omega^{12}$	$a\,\omega^{14}$	$b\,\omega^{15}$	$b\,\omega$	(3,6)
		$c\,\omega^{14}$	$c\,\omega^{10}$	(5,1)
$c\,\omega^{7}$	$c\,\omega^{19}$			(2,2)
$b\,\omega^{14}$	$b\,\omega^{16}$	$a\omega^{7}$	$a\omega^{13}$	(4,3)

(5,5)	(6,6)	(5,6)	(6,5)	
$a\,\omega^{18}$	$a\,\omega^{18}$	$b\,\omega^{3}$	$b\,\omega^{3}$	(1,3)
		c	$c\,\omega^{10}$	(5,2)
$c\,\omega$	$c\,\omega^{11}$			(6,5)
$b\,\omega^{10}$	$b\,\omega^{10}$	$a\omega^{5}$	$a\omega^{5}$	(4,4)

AME(4,6) state

$$\frac{1}{6}\sum_{i,j,k,\ell=1}^{d} t_{i,j,k,\ell}\,|i\rangle\,|j\rangle\,|k\rangle\,|\ell\rangle$$

$$a = \frac{1}{\sqrt{2}(\omega+\overline{\omega})} = \frac{1}{\sqrt{5+\sqrt{5}}}$$

$$b = \frac{1}{\sqrt{2}(\omega^3+\overline{\omega}^3)} = \sqrt{\frac{5+\sqrt{5}}{20}}$$

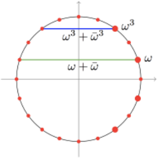
$$c = \frac{1}{\sqrt{2}}$$

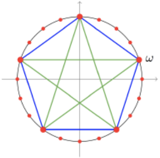
Pythagoras theorem

$$a^2 + b^2 = c^2 = \frac{1}{2}$$

Golden ratio

$$b/c = \varphi = = \frac{1+\sqrt{5}}{2}$$





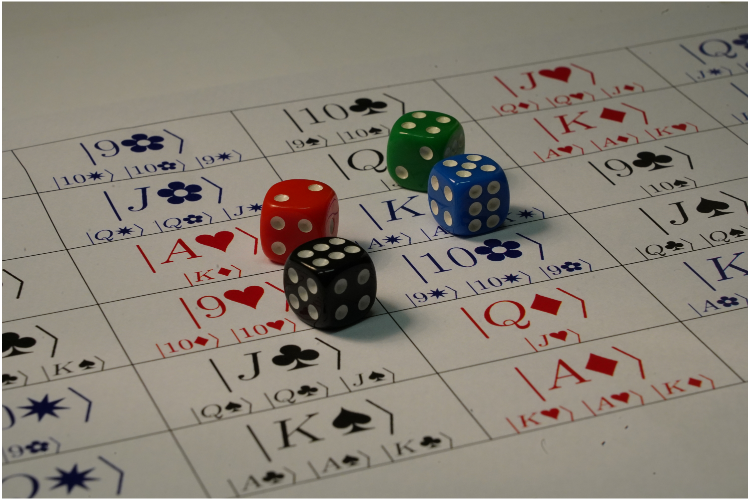
Quantum Officers of Euler

|A♣⟩ |K♠⟩	|A♦⟩ |K♥⟩	|10✿⟩ |10✱⟩ |9✿⟩ |9✱⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩	|Q✿⟩ |Q✱⟩ |J✿⟩ |J✱⟩
|10♣⟩ |9♠⟩	|10♦⟩ |9♥⟩	|Q✿⟩ |Q✱⟩ |J✿⟩ |J✱⟩	|Q♠⟩ |J♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩
|Q✿⟩ |J✱⟩	|Q♣⟩ |J♠⟩	|A♥⟩ |K♦⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|10♠⟩ |9♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩
|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩	|10✿⟩ |10✱⟩ |9✿⟩ |9✱⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩
|10♥⟩ |9♦⟩	|10✱⟩ |9✿⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |J♥⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|A♠⟩ |K♣⟩
|Q♥⟩ |J♦⟩	|Q✱⟩ |J✿⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|10✿⟩ |9✱⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩

$A/K \to A$
$D/J \to B$
$10/9 \to C$

♠/♣ $\to \alpha$
♦/♥ $\to \beta$
✿/✱ $\to \gamma$

$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$	$B\beta$	$B\gamma$
$C\alpha$	$C\beta$	$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$
$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$	$C\alpha$	$C\beta$
$A\gamma$	$A\alpha$	$C\beta$	$C\gamma$	$B\alpha$	$B\beta$
$C\beta$	$C\gamma$	$B\alpha$	$B\beta$	$A\gamma$	$A\alpha$
$B\beta$	$B\gamma$	$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$



Four dice in the golden $|AME(4,6)\rangle$ state corresponding to *36 entangled officers* of **Euler**. Any pair of dice is unbiased, although their outcome determines the state of the other two.

multi–unitary matrices and AME states

Consider an **AME state** of $2k$ parties with d levels each.

It is **maximally entangled** with respect to all possible symmetric partitions, so all its k-party reductions are **maximally mixed**.

Unitary matrix U of order d^k with the property that it remains unitary for any choice of k indices out of $2k$ is called ***k*–unitary**

Example: 3–unitary matrix of order $2^3 = 8$ remains unitary for any of $\binom{6}{3} = 20$ possible reorderings,

$$O_8 = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

Such matrices optimize 3-party entangling power, **Linowski, Rajchel, K.Ż.**

Concluding Remarks

Strongly entangled extremal multipartie quantum states can be useful for quantum error correction codes, multiuser quantum communication and other protocols.

Theorem. Absolutely maximally entangled states $|AME(4,6)\rangle$ of 4 subsystems with 6 levels each **do** exist !

This implies existence of

1. solution of the quantum analogue of the 36 officers problem of **Euler**,
2. optimal bi-partite unitary gate U_{36} with maximal **entangling power**
3. **perfect tensor** $t_{ijk\ell}$ with 4 indices, each running from 1 to 6, to be applied for tensor networks and bulk/boundary correspondence,
4. nonadditive **quantum error correction code** $((3,6,2))_6$ which allows one to encode a single quhex in three quhexes (it does not belong to the class of stabilizer codes).

$\Longrightarrow$ such *extremal* quantum states & the corresponding **multi-unitary matrices** can be useful...