Symplectic capacities and their blind spots

joint w/ Yuanpu Liang

arXiv: 2109.01792v1

Symplectic Capacity

$$c : \{ U \subset \mathbb{R}^{2n} \} \to [0, \infty]$$

1) $c(\varphi(U)) = c(U)$ $\quad \varphi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{2n})$

2) $U \subset V \Rightarrow c(U) \leq c(V)$

3) $c(\lambda U) = \lambda^2 c(U)$

4) $0 < c(B^{2n}(1))$, $\quad c(B^2(1) \times \mathbb{R}^{2n-2}) < \infty$.

Ex $\quad c_G(U) = \sup \left\{ \pi R^2 \;\middle|\; \begin{array}{l} \exists\, \varphi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{2n}) \\ \text{s.t. } \varphi(B^{2n}(R)) \subset U \end{array} \right\}$

Ex $\quad c_k^{EH}$ (Ekeland - Hofer) $\quad k \in \mathbb{N}$

$\uparrow$

grading by Fadell- Rabinowitz index

Ex $\quad c_k^{ECH} : \{ U \subset \mathbb{R}^4 \} \to [0,\infty]$ $\quad$ Hutchings

$\uparrow$

grading by Conley- Zehnder index

Gutt-Hutchings Capacities

$c_1 \leq c_2 \leq \dots \leq c_k \leq \dots$

- Defined for star-shaped domains $X \subset \mathbb{R}^{2n}$ using S^1-equivariant symplectic homology

$$CH_*(X) = \begin{cases} \mathbb{Q}, & * \in n-1+2\mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$$c_k(X) = \inf \left\{ L \;\middle|\; \begin{array}{l} \text{image } i_L : CH^L(X) \to CH(X) \\ \text{contains } CH_{n-1+2k}(X). \end{array} \right\}$$

Computations

- $c_1(X)$ = min action of closed characteristic on ∂X.
  = min period of Reeb orbit of $(i_{r\partial/\partial r}\omega)|_{\partial X}$ on ∂X.

$a \geq 1$

- $E(1,a) = \left\{ \pi|z_1|^2 + \dfrac{\pi|z_2|^2}{a} \leq 1 \right\} \subset \mathbb{C}^2$

  $c_k(E(1,a)) = (\mathrm{Sort}\{\mathbb{N} \cup a\mathbb{N}\})[k]$

- $P(1,a) = \{ \pi|z_1|^2 \leq 1 , \pi|z_2|^2 \leq a \}$

  $c_k(P(1,a)) = k.$

<u>Q1</u> Are the c_k independent?

Or do they always depend on the periods of finitely many simple closed Reeb orbits on ∂X?

<u>A1</u> Yes. For each $\ell \in \mathbb{N}$ there are X and Y.

(convex with smooth boundary) s.t.

$$c_k(X) = c_k(Y) \quad \forall\ k \neq \ell.$$

$$c_\ell(X) \neq c_\ell(Y)$$

$$(\mathrm{Vol}(X) = \mathrm{Vol}(Y)).$$

$\underline{Q2}$ If ∂X is smooth, do the $c_k(X)$ see $Vol(X)$?

- Without ∂X smooth obviously no. $c_k(P(1,a)) = k$.

Consider

$$E_p(1,a) = \left\{ (\pi|z_1|^2)^p + \left(\frac{\pi|z_2|^2}{a}\right)^p \leq 1 \right\}$$

$$\begin{tikzcd}
& E_p(1,a) \arrow[dl, "p\to 1"'] \arrow[dr, "p\to\infty"] & \\
E(1,a) \arrow[rr, rightsquigarrow, "c_k \text{ forget } a"] & & P(1,a)
\end{tikzcd}$$

Lemma For each k $\exists$ $p(k)$ s.t. $c_k(E_p(1,a)) = k$

for all $p \geq p(k)$.

Lemma For each $p \in (1, \infty)$ $\exists$ $k(p)$ s.t. $c_k(E_p(1,a))$

depends on a for all $k \geq k(p)$

$\{c_k(E_p(1,a))\}_{k \in \mathbb{N}}$ remembers a for all $p < \infty$.

Nevertheless A2 is NO

$\exists$ X, Y w/ smooth (convex) boundary s.t. $c_k(X) = c_k(Y)$ $\forall k$

+ $\mathrm{Vol}(X) \neq \mathrm{Vol}(Y)$.

Q3 If ∂X is smooth, do $c_k(X)$ and $\mathrm{Vol}(X)$ determine X up to symplectomorphism

A3 NO, there are X, Y with smooth boundary (convex) such that $c_k(X) = c_k(Y)$ $\forall k \in \mathbb{N}$ $\mathrm{Vol}(X) = \mathrm{Vol}(Y)$ but $X \not\cong Y$.

(They can be distinguished using c_9^{ECH}.)

Q4 Is $c_k(X) = c_k^{EH}(X)$ $\forall$ $k \in \mathbb{N}$?

This is stated as a conjecture by Gutt-Hutchings

Work by Abbondandolo-Kang and Irie suggest true if X is convex.

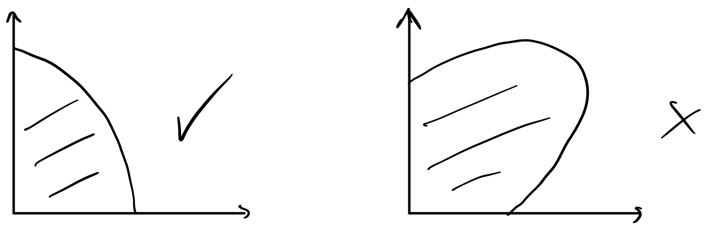
The examples underlying A1, A2, A3 are obtained using a refinement of formulae of Gutt-Hutchings.

Gutt - Hutchings establish a computable formula for $c_k(X)$ when X is a convex/concave toric domain.

$\mu : (z_1, \dots, z_n) \mapsto (\pi |z_1|^2, \dots, \pi |z_n|^2)$, $\Omega \subset (\mathbb{R}_{\geq 0})^n$

- $X_\Omega = \mu^{-1}(\Omega)$ convex if $\hat{\Omega} = \{ x \in \mathbb{R}^n \mid (|x_1|, \dots, |x_n|) \in \Omega \}$ is

i.e.



- X_Ω is concave if the complement of Ω in $\mathbb{R}^n_{\geq 0}$ is convex.

Thm Gutt-Hutchings

If X_Ω is convex then

$$c_k(X_\Omega) = \min \left\{ \|v\|_\Omega \;\middle|\; v \in (\mathbb{Z}_{\geq 0})^n, \; \sum v_j = k \right\}$$

where $\|v\|_\Omega = \max \{ \langle v, w \rangle \mid w \in \Omega \}$

(solve & compare $\binom{k+n-1}{n-1}$ optimization problems.)

If X_Ω is concave then

$$c_k(X_\Omega) = \max \left\{ [v]_\Omega \;\middle|\; v \in \mathbb{N}^n, \sum v_j = k+n-1 \right\}$$

where $[v]_\Omega = \min \{ \langle v, w \rangle \mid w \in \Omega \}$

$\Omega \subset \mathbb{R}^n_{\geq 0}$ is symmetric if $(v_1, \ldots, v_n) \in \Omega \Rightarrow (v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \in \Omega$ for all $\sigma \in S_n$.

<u>Thm</u> (K-Liang)

If Ω is symmetric and X_Ω is convex then

$$c_k(X_\Omega) = \max_{w \in \Omega} \langle V(k,n), w \rangle \quad \text{where}$$

$$V(k,n) = \left(\left\lfloor \tfrac{k}{n} \right\rfloor, \ldots, \left\lfloor \tfrac{k}{n} \right\rfloor, \underbrace{\left\lceil \tfrac{k}{n} \right\rceil, \ldots, \left\lceil \tfrac{k}{n} \right\rceil}_{k \bmod n} \right)$$

If Ω is symmetric and X_Ω concave, then

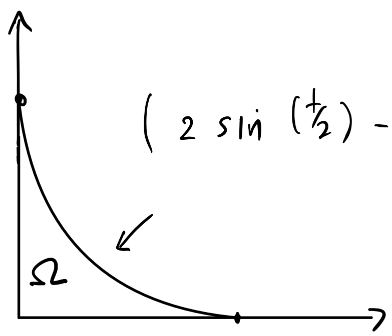
$$c_k(X_\Omega) = \min_{w \in \Omega} \langle \check{V}(k,n), w \rangle$$

$$\check{V}(k,n) = \Big(\underbrace{\left\lceil \frac{k+n-1}{n} \right\rceil, \ldots, \left\lceil \frac{k+n-1}{n} \right\rceil}_{k+n-1 \mod n}, \left\lfloor \frac{k+n-1}{n} \right\rfloor, \ldots, \left\lfloor \frac{k+n-1}{n} \right\rfloor \Big)$$

Example   The Lagrangian Bidisk.

$$P_L = \{ (z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2) \mid x_1^2 + x_2^2 \le 1,\ y_1^2 + y_2^2 \le 1 \}$$

<u>Thm</u> (Ramos) $\mathring{P}_L$ is symplectomorphic to $\mathring{X}_\Omega$ for



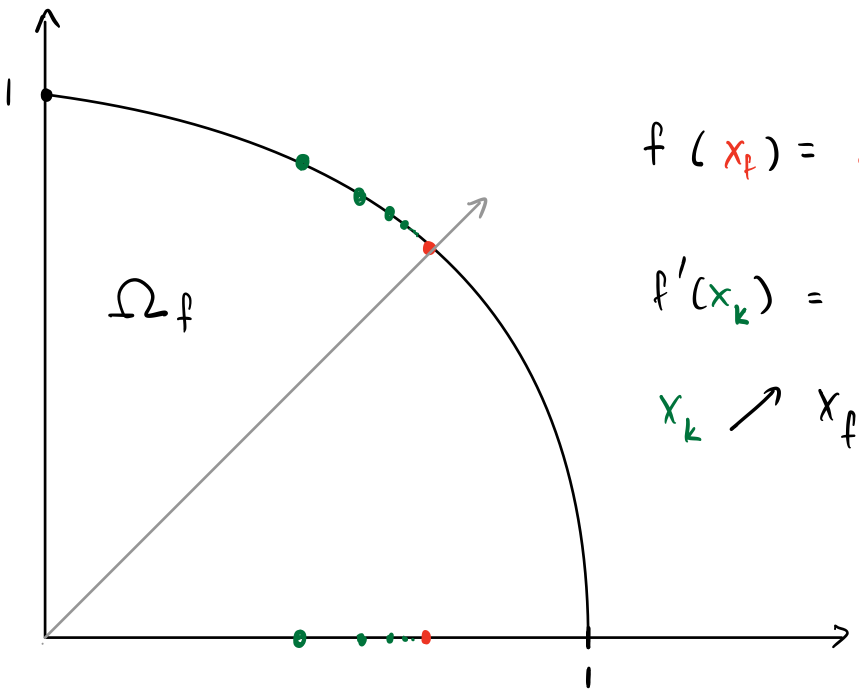
$$\left(2\sin\left(\tfrac{t}{2}\right) - t\cos\left(\tfrac{t}{2}\right),\ 2\sin\left(\tfrac{t}{2}\right) + (2\pi - t)\cos\left(\tfrac{t}{2}\right) \right)$$

$$t \in [0, 2\pi]$$

$$\text{Thm} \Rightarrow c_k(P_L) = \begin{cases} 2k+2, & k \text{ odd} \\ (4k+2)\sin\left(\frac{\pi}{2}\left(\frac{k+2}{k+1}\right)\right), & k \text{ even} \end{cases}$$

Example $f : [0,1] \to [0,1]$, $f(0)=1$, $\underline{f = f^{-1}}$, $f'' < 0$

$$\Omega_f = \{ (x,y) \in \mathbb{R}^2_{\geq 0} \mid y \leq f(x) \}$$



$f(x_f) = x_f$

$f'(x_k) = -\frac{(k-1)}{(k+1)}$ k-odd

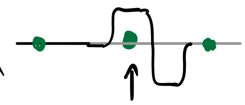
$x_k \nearrow x_f$

Thm $\Rightarrow$ $c_{2n}(X_{\Omega_f}) = 2n\, x_f$

$$c_{2n+1}(X_{\Omega_f}) = n\, x_{2n+1} + (n+1)\, f(x_{2n+1})$$

Blind Spots of the c_k.

- A compact C^2-small perturbation of f away from x_f does not change the c_{2n}'s
- A compact C^2-small perturbation of f away from x_{2k+1} does not change c_{2k+1}.

<u>A1</u>: For $f_\delta = f + \delta \Big($  $\Big)$ + mirror bump

x_{2k+1}

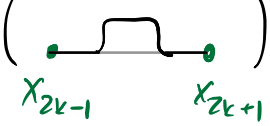
$$\mathrm{Vol}(X_{\Omega_{f_\delta}}) = \mathrm{Vol}(X_{\Omega_f})$$

$$c_{2n}(X_{\Omega_{f_\delta}}) = c_{2n}(X_{\Omega_f})$$

$$c_{2n+1}(X_{\Omega_{f_\delta}}) = c_{2n+1}(X_{\Omega_f}) \qquad n \neq k.$$

$$c_{2k+1}(X_{\Omega_{f_\delta}}) = c_{2k+1}(X_{\Omega_f}) + \delta(k+1)$$

Starting w/ f concave one can also deform any c_{2k}.

A2 For $f_\delta = f + \delta\Big($ 

$\Big)$ + mirror bump

$$c_n\left(X_{\Omega_{f_\delta}}\right) = c_n\left(X_{\Omega_f}\right) \quad \forall\ k \in \mathbb{N}$$

$$\mathrm{Vol}\left(X_{\Omega_{f_\delta}}\right) \neq \mathrm{Vol}\left(X_{\Omega_f}\right).$$

To establish A3 we need the ECH capacities of Hutchings in dim 4.

Choose $f = f^{-1}$, $f'' > 0$. $\Rightarrow X_{\Omega_f}$ concave.

One can compute $c_k^{ECH}(X_{\Omega_f})$ algorithmically using the ordered weight expansions of Ω_f:

Sort $\{ \tau_0, \tau_1, \tau_2, \tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \dots \}$

Hutchings, Choi + Cristofaro-Gardiner + Frenkel + Hutchings + Ramos

ex

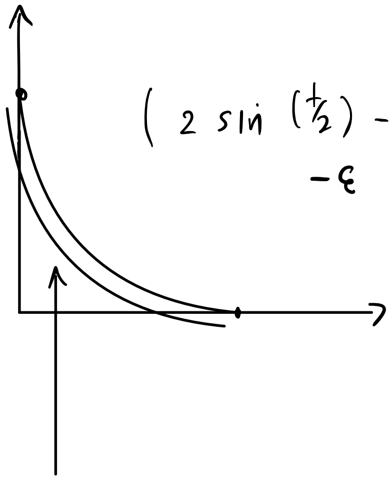
$$\tau_{22} = 3x_{22} + f(x_{22}) - \tau - \tau_2 \quad , \quad f'(x_{22}) = -3$$

Idea

- The points x_* used to define the T_*'s of X_{Ω_f} are dense.

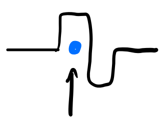
- Deform f near an x_* which lies away from x_f and the x_{2n}

- Changes $c_\ell^{ECH}(X_{\Omega_{f_\delta}})$ but not c_k or Vol.



$$\left(2\sin\left(\tfrac{t}{2}\right) - t\cos\left(\tfrac{t}{2}\right) - \varepsilon,\ 2\sin\left(\tfrac{t}{2}\right) + (2\pi - t)\cos\left(\tfrac{t}{2}\right) - \varepsilon \right)$$

$f_\varepsilon(x)$ $\rightsquigarrow$ $X_{\Omega_{f_\varepsilon}}$ has smooth boundary. for $\varepsilon > 0$

- x_{22} lies away from x_{f_ε} and x_{2n} determining the c_n
- $f_{\varepsilon,\delta} = f_\varepsilon(x) + \delta($  $)$ + mirror bump.

x_{22}

A3

$$c_k(X_{\Omega_{f_{\varepsilon,\delta}}}) = c_k(X_{\Omega_{f_\varepsilon}}) \quad \forall\, k \in \mathbb{N}$$

$$\mathrm{Vol}(X_{\Omega_{f_{\varepsilon,\delta}}}) = \mathrm{Vol}(X_{\Omega_{f_\varepsilon}})$$

$$c_q^{ECH}(X_{\Omega_{f_{\varepsilon,\delta}}}) = c_q^{ECH}(X_{\Omega_{f_\varepsilon}}) + \delta$$

This is a general phenomenon in dim 4.

Yuanpu's Proof of the simplified formula(s)

Given Ω symmetric s.t. X_Ω is convex. Need

$$c_k(X_\Omega) \overset{G\text{-}H}{=} \min \left\{ \|v\|_\Omega \;\middle|\; v \in (\mathbb{Z}_{\geq 0})^n,\ \sum v_j = k \right\}$$

$$\overset{L\text{-}K}{=} \|V(k,n)\|_\Omega$$

for $V(k,n) = \left(\lfloor \frac{k}{n} \rfloor, \dots, \lfloor \frac{k}{n} \rfloor, \underbrace{\lceil \frac{k}{n} \rceil, \dots, \lceil \frac{k}{n} \rceil}_{k \bmod n} \right)$

and $\|v\|_\Omega = \max\limits_{w \in \Omega} \langle v, w \rangle$

$v \in \mathbb{R}^n$ is ordered if $v_1 \leq v_2 \leq \dots \leq v_n$

<u>Ex</u> $V(k,n)$ is ordered.

Symmetry of Ω $\Rightarrow$ $\|(v_1, \dots, v_n)\|_\Omega = \|(v_{\sigma(1)}, \dots, v_{\sigma(n)})\|_\Omega$

$\forall \ \sigma \in S_n$

$$\vec{S}(k,n) = \{ v \in (\mathbb{Z}_{\geq 0})^n \mid \Sigma v_j = k, \ v \text{ ordered} \}$$

$$c_k(X_\Omega) \overset{G\text{-}H}{=} \min \{ \|v\|_\Omega \mid v \in \vec{S}(k,n) \} \quad \checkmark.$$

Consider the map $D : \underline{\vec{S}(k,n)} \to \vec{S}(k,n)$

$$V = (\underbrace{V_1, V_1, \dots, V_1}_{t}, \dots, \underbrace{V_n, \dots, V_n}_{T})$$

$$\downarrow$$

$$\begin{cases} (\underbrace{V_1, \dots, V_1}_{t-1}, V_1+1, \dots, V_n-1, \underbrace{V_n \cdots V_n}_{T-1}) & \text{if } V_n > V_1+1 \\ V & \text{otherwise} \end{cases}$$

- $\mathrm{Fix}(D) = \{V(k,n)\}$ and $D^j(V) = V(k,n)$ for $j >> 1$

Prop $\quad \| D(v) \|_\Omega \leq \| v \|_\Omega$

G-H w/ symmetry

$$C_k(X_\Omega) = \min \left\{ \|v\|_\Omega \;\middle|\; v \in \vec{S}(k,n) \right\}$$

$$\Rightarrow C_k(X_\Omega) = \| \hat{v} \|_\Omega .$$

$$\Rightarrow C_k(X_\Omega) = \| D^j(\hat{v}) \|_\Omega \qquad \forall j \in \mathbb{N}$$

$$\Rightarrow C_k(X_\Omega) = \| V(k,n) \|_\Omega \qquad \checkmark$$

<u>lemma</u> $\| v \|_\Omega = \langle v, w \rangle$ for some w ordered

<u>Pf</u> Assume $\| v \|_\Omega = \langle v, w \rangle$ and $w_j > w_{j+1}$

Set $\tilde{w} = (w_1, \dots, w_{j+1}, w_j, \dots, w_n)$

$$\langle v, \tilde{w} \rangle - \langle v, w \rangle = v_j w_{j+1} + v_{j+1} w_j - v_j w_j - v_{j+1} w_{j+1}$$

$$= \underbrace{(v_j - v_{j+1})}_{-\text{ve}} \underbrace{(w_{j+1} - w_j)}_{-\text{ve}}$$

$$\geq 0$$

Since $\langle v, w \rangle = \max_{w \in \Omega} \langle v, w \rangle$ we have $\langle v, \tilde{w} \rangle = \langle v, w \rangle$

Proof of Prop: $\| D(v) \|_\Omega \leq \| v \|_\Omega$

lemma $\Rightarrow$ $\| D(v) \|_\Omega = \langle D(v), w \rangle$ for w ordered

$$\| v \|_\Omega - \| D(v) \|_\Omega \geq \langle v, w \rangle - \langle D(v), w \rangle$$

$$= \left(v_1 w_t + v_n w_{n-\tau} \right) - (v_1 + 1) w_t - (v_n - 1) w_{n-T}$$

$$= w_{n-T} - w_t$$

≥ 0 since w is ordered.

Next Questions

How independent are the c_k?

1) Does there exist a starshaped $X \subset \mathbb{R}^{2n}$ such that no $c_k(X)$ is an integer multiple of any other?

2) Does there exist a star shaped $X \subset \mathbb{R}^{2n}$ s.t for each $l \in \mathbb{N}$ $\exists$ $Y_l \subset \mathbb{R}^{2n}$ with

$$c_k(X) = c_k(Y_l) \quad \forall\ k \neq l$$

$$c_l(X) \neq c_l(Y_l)$$

How independant are the c_k from Vol ?

$$IVR(X) = \sup \frac{vol(Y)}{vol(Z)}$$

where $c_k(X) = c_k(Y) = c_k(Z) \quad \forall\, k \in \mathbb{N}$.

- In dim 2 $IVR(X) = 1$
- $IVR(P(1,a)) = \infty$

3) Is $IVR(E(1,a)) = 1$?

4) If ∂X is smooth is $IVR(X) < C_n$?

$c_4 \geq 6 - 2\sqrt{6} \approx 1.10102$.