Gravity as a Double Copy of Gauge Theory

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Motivation

1. Feynman rules for GR: expand Einstein-Hilbert Lagrangian [DeWitt '66]

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad -P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],
\end{aligned}$$

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^\sigma p^\tau\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\
&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\
&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad -P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\Big].
\end{aligned}$$

$+$ infinite number of higher-point vertices...

2. GR and YM are fundamental theories.

Gravity $\sim$ YM ^{2}

Free fields

- Polarisation states: $\boxed{\varepsilon^{\mu\nu} = \epsilon^{\mu}\,\tilde{\epsilon}^{\nu}}$ (graviton + dilaton + B-field)
- Degrees of freedom match: $(D-2)^2$.

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Scattering amplitudes

- "Factorisation" of ϵ_μ, $\tilde{\epsilon}_\nu$ preserved by interactions!
- **Double copy** $\boxed{\mathcal{A}_{\text{grav}}(\varepsilon_i^{\mu\nu}) \sim (\text{prop})^{-1}\, \mathcal{A}_{\text{YM}}(\epsilon_i^{\mu}) \times \mathcal{A}_{\text{YM}}(\tilde{\epsilon}_i^{\nu})\,\Big|_{\text{colour stripped}}}$
- Well established at tree level.
- Useful but unproven at loop level.

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Classical solutions

- Suggests correspondence between classical theories.
- Map between solutions?

Outline

Double copy for scattering amplitudes

- KLT relations
- Colour-kinematics duality
- Scattering equations

Double copy for classical solutions

- Exact double copy: Kerr-Schild spacetimes
- Perturbative double copy

Double copy for scattering amplitudes

I. KLT relations: string theory origin

[Kawai, Lewellen, Tye '86]

Vertex operators: $V_{\text{closed}}(\varepsilon^{\mu\nu} = \epsilon^{\mu}\tilde{\epsilon}^{\nu}) \sim V_{\text{open}}(\epsilon^{\mu})\,\bar{V}_{\text{open}}(\tilde{\epsilon}^{\nu})$ $\quad s_{ij} = (k_i + k_j)^2$

$$\mathcal{A}_3^{\text{grav}} = A^{\text{YM}}(123)\,\tilde{A}^{\text{YM}}(123) \qquad \mathcal{A}_4^{\text{grav}} = \frac{\sin \pi\alpha' s_{12}}{\pi\alpha'}\,A^{\text{YM}}(1234)\,\tilde{A}^{\text{YM}}(1243)$$

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Field theory limit is $\alpha' \to 0$.

In general (tree level)

[Bern, Dixon, Perelstein, Rozowsky '98]

$$\mathcal{A}_n^{\text{grav}} = \sum_{P_n, P'_n} A^{\text{YM}}(P_n)\;\; S_{\text{KLT}}[P_n, P'_n]\;\; \tilde{A}^{\text{YM}}(P'_n) \qquad\qquad S_{\text{KLT}} \sim s_{ij}^{n-3}$$

Useful at loop level via unitarity cuts.

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Recall YM colour decomposition: colour traces or colour factors.

$$\mathcal{A}_n^{\text{YM}} = \sum_{\text{non cyclic}} A^{\text{YM}}(1,2,\ldots,n)\, \mathrm{tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) = \sum_{\alpha \in \text{cubic}} N_\alpha\, c_\alpha$$

with $c_\alpha = f^{abc} f^{\cdots} \cdots f^{\cdots}$, $f^{abc} = \mathrm{tr}([T^a, T^b] T^c)$,

but Jacobi identities: $c_\alpha \pm c_\beta \pm c_\gamma = 0$

II. BCJ duality and double copy

[Bern, Carrasco, Johansson '08, '10]

Gauge theory

$$\mathcal{A}_{\text{YM}} = \sum_{\alpha\in\text{cubic}} \frac{n_\alpha c_\alpha}{D_\alpha}$$

kinematic numerators: $n_\alpha(k_i, \epsilon_i)$

colour factors: $c_\alpha = f^{abc} f^{\cdots} \cdots f^{\cdots}$

propagators: $D_\alpha = \prod_r K^2_{\alpha,r}$

Colour-kinematics duality

$$\exists n_\alpha : c_\alpha \pm c_\beta \pm c_\gamma = 0 \;\leftrightarrow\; n_\alpha \pm n_\beta \pm n_\gamma = 0$$

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Double copy to gravity

colour-kinematics satisfying n_α

same propagators

states scattered: $\varepsilon_{\mu\nu} = \epsilon_\mu \tilde{\epsilon}_\nu$

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Stringy understanding from monodromy and pure spinor formalism. [Bjerrum-Bohr, Damgaard, Vanhove; Stieberger '09] [Mafra, Schlotterer, Stieberger '11]

Self-dual YM and gravity: kinematic algebra, off-shell BCJ. [RM, O'Connell '11]

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Tree level: duality true, double copy same as KLT relations.

Loop level (integrand): duality conjectural, double copy gets unitarity cuts.

Gravity $\sim$ YM $\times$ $\widetilde{YM}$: multiplication table

$$(\mathcal{N}=4\ \text{SYM})\ \times\ (\mathcal{N}=4\ \text{SYM})\ \sim\ (\mathcal{N}=8\ \text{SUGRA})$$

$$(\mathcal{N}=4\ \text{SYM})\ \times\ (\mathcal{N}=2\ \text{SYM})\ \sim\ (\mathcal{N}=6\ \text{SUGRA})$$

$$(\mathcal{N}=4\ \text{SYM})\ \times\ (\mathcal{N}=1\ \text{SYM})\ \sim\ (\mathcal{N}=5\ \text{SUGRA})$$

$$(\mathcal{N}=4\ \text{SYM})\ \times\ (\mathcal{N}=0\ \text{YM})\ \sim\ (\mathcal{N}=4\ \text{SUGRA})$$

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etc.

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$$S_{\mathcal{N}=0\text{ SUGRA}} = \int d^Dx\sqrt{-g}\left[\frac{2}{\kappa^2}R - \frac{1}{2(D-2)}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{6}e^{-\frac{2\kappa\varphi}{D-2}}H^{\lambda\mu\nu}H_{\lambda\mu\nu}\right], \quad H = dB.$$

Pure Einstein gravity?

- tree level: yes if external particles are all gravitons
- loop level: need to project out dilaton φ and B-field in loops

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UV surprises in 4D SUGRA: finite for $\mathcal{N} > 4$? [Bern, Carrasco, Dixon, Johansson, Roiban '08...]

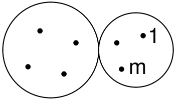
$\mathcal{N}=8$ SUGRA as same D_{crit} as $\mathcal{N}=4$ SYM up to 4 loops.

III. Scattering equations and CHY formulas [Cachazo, He, Yuan '13]

Consider n massless particles, $k_i^2 = 0$, $i = 1, \ldots, n$

$$E_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0, \quad \forall i$$

- kinematic invariants $s_{ij} = 2\, k_i \cdot k_j \quad \longrightarrow \quad$ points $\sigma_i \in \mathbb{CP}^1$
- $\sum_i k_i = 0$: $SL(2, \mathbb{C})$ invariance, $\sigma \to \dfrac{A\,\sigma + B}{C\,\sigma + D}$
- $(n-3)!$ solutions $\sigma_i^{(A)}$
- factorisation: $(k_1 + \ldots + k_m)^2 \to 0$ gives $\sigma_1, \ldots, \sigma_m \to \sigma_\star$



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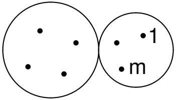

Also apear in high-energy fixed-angle string scattering. [Gross, Mende '88]

III. Scattering equations and CHY formulas [Cachazo, He, Yuan '13]

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- measure is universal

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- $\mathcal{I}$ specifies the theory

$$\mathcal{I}_{\text{YM}} = \text{Pf}'M(\epsilon) \times \left(\frac{\text{tr}(T^{a_1} T^{a_2} \dots T^{a_n})}{\sigma_{12}\sigma_{23} \cdots \sigma_{n1}} + \text{non-cyclic perm} \right) \qquad \sigma_{rs} = \sigma_r - \sigma_s$$

$$\mathcal{I}_{\text{grav}} = \text{Pf}'M(\epsilon) \times \text{Pf}'M(\tilde{\epsilon}) \quad \Rightarrow \quad \text{Gravity} \sim \text{YM}^2$$

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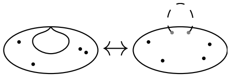
Predecessor: 4D formula from twistor-string theory. [Witten 03; Roiban, Spradlin, Volovich 04]

Worldsheet model for CHY: ambitwistor-string theory. [Mason, Skinner 13]

Loop-level formulas!

[Geyer, Mason, RM, Tourkine 15, 16]

[Adamo, Casali, Skinner 13]



Summary for amplitudes

$$\mathcal{A}_{\text{grav}}(\varepsilon_i^{\mu\nu}) \sim (\text{prop})^{-1}\, \mathcal{A}_{\text{YM}}(\epsilon_i^{\mu}) \times \mathcal{A}_{\text{YM}}(\tilde{\epsilon}_i^{\nu}) \,\Big|_{\text{colour stripped}}$$

KLT relations

$$\mathcal{A}_{\text{grav}} = \sum_{P_n, P_n'} A_{\text{YM}}(\epsilon, P_n)\;\; S_{\text{KLT}}[P_n, P_n']\;\; A_{\text{YM}}(\tilde{\epsilon}, P_n')$$

BCJ double copy

$$\mathcal{A}_{\text{YM}} = \sum_{\alpha\in\text{cubic}} \frac{n_\alpha(\epsilon)\, c_\alpha}{D_\alpha} \qquad\qquad \mathcal{A}_{\text{grav}} = \sum_{\alpha\in\text{cubic}} \frac{n_\alpha(\epsilon)\, n_\alpha(\tilde{\epsilon})}{D_\alpha}$$

CHY formulas

$$\mathcal{A} = \int d\mu\, \mathcal{I} \qquad \mathcal{I}_{\text{YM}} = \text{Pf}\,'M(\epsilon) \times \mathcal{C} \qquad \mathcal{I}_{\text{grav}} = \text{Pf}\,'M(\epsilon) \times \text{Pf}\,'M(\tilde{\epsilon})$$

Double copy for classical solutions

Double copy for black holes?

[RM, O'Connell, White '14]

Question: is there a double copy for classical solutions?

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Challenges:

- What is "graviton" in exact solution?
- Non-perturbative double copy?
- Relation of diffeo. choice in gravity to gauge choice in YM?

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Still...

- Should work in perturbation theory.

  Some solutions can be constructed perturbatively.

  Examples: Schwarzchild [Duff 73; Neill, Rothstein 13], shockwave [Saotome, Akhoury '12].

- Direct map of exact solutions? Need miracle!

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Stationary Kerr-Schild spacetimes

"Exact perturbation" $$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu$$

where k_μ is null and geodesic wrt $\eta_{\mu\nu}$ and $g_{\mu\nu}$. $\quad (k^\mu = g^{\mu\nu} k_\nu = \eta^{\mu\nu} k_\nu)$

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Einstein equations linearise:

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Stationary vacuum case (take $k_0 = 1$): $R^0{}_0 = \frac{1}{2}\nabla^2 \phi = 0$

$$R^i{}_0 = \frac{1}{2}\,\partial_\ell \left[\partial^i \left(\phi k^\ell\right) - \partial^\ell \left(\phi k^i\right)\right] = 0$$

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$$0 = D_\mu F^{a\mu\nu} = c^a \begin{cases} -\nabla^2\phi & \nu = 0 \\ -\partial_\ell \left[\partial^i \left(\phi k^\ell\right) - \partial^\ell \left(\phi k^i\right)\right] & \nu = i \end{cases} \qquad \checkmark$$

Simplest example: point charge

Check spherically symmetric solutions sourced by point charge.

Einstein theory: Schwarzschild solution

- $$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu, \qquad \phi(r) = \frac{2M}{r}, \qquad k = dt + dr$$

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YM theory: Coulomb solution

- $$A^a_\mu = \phi\, k_\mu\, c^a, \quad \phi(r) = \frac{q}{r}$$

$$A^a \;\rightarrow\; A^a + d(-q\, c^a \log r) = \frac{q}{r}\, c^a\, dt$$

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Schwarzschild $\sim$ (Coulomb)^2

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$$\text{Schwarzschild} \sim (\text{Coulomb})^2$$

Makes sense!

However, expect $(\text{YM})^2 \;\sim\;$ Einstein $g_{\mu\nu}$ + dilaton φ + B-field $B_{\mu\nu}$

Why vacuum? How to add dilaton? See later.

Stationary Kerr-Schild: more examples

Kerr solution: (M, a), $a = J/M$. Schwarzschild is $a = 0$.

- $$\phi(r,z) = \frac{2\,M\,r^3}{r^4 + a^2 z^2}, \qquad \frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$$
- $$k = dt + \frac{rx+ay}{r^2+a^2}\,dx + \frac{ry-ax}{r^2+a^2}\,dy + \frac{z}{r}\,dz$$

Single copy: Maxwell field generated by certain rotating charged disk.

Extends to $D > 4$: Myers-Perry black holes (M, a_i) are also Kerr-Schild.

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Cosmological constant $\leftrightarrow$ constant charge density. [Luna, RM, O'Connell, White '15]

NUT charge $\leftrightarrow$ magnetic monopole: multi-Kerr-Schild

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi\,k_\mu k_\nu + \psi\,\ell_\mu \ell_\nu\,, \quad \phi \propto M\,, \quad \psi \propto N \quad \Rightarrow \quad A_\mu^{\text{dyon}} = \phi\,k_\mu + \psi\,\ell_\mu$$

Radiation from an accelerated particle: correct Bremsstrahlung.
[Luna, RM, Nicholson, O'Connell, White '16]

Alternative formulation

[in preparation, Luna et al]

Try double copy of curvatures:

$$A_\mu = \epsilon_\mu \, e^{ik\cdot x}, \quad F_{\mu\nu} = i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu) \, e^{ik\cdot x}$$

$$h_{\mu\nu} = \epsilon_\mu \epsilon_\nu \, e^{ik\cdot x}, \quad R_{\mu\nu\rho\lambda} = \frac{1}{2}(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu)(k_\rho \epsilon_\lambda - k_\lambda \epsilon_\rho) \, e^{ik\cdot x}$$

Obvious relation: $e^{ik\cdot x} R_{\mu\nu\rho\lambda} \sim F_{\mu\nu} F_{\rho\lambda}$

More general? Not so simple: symmetries of $R_{\mu\nu\rho\lambda}$, non-linear gauge, ...

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Spinorial approach to GR ($D = 4$)

[Penrose '60]

Basic object is $\sigma^\mu_{A\dot{A}}$ such that $$\left(\sigma^\mu_{A\dot{A}} \sigma^\nu_{B\dot{B}} + \sigma^\nu_{A\dot{A}} \sigma^\mu_{B\dot{B}}\right) \varepsilon^{\dot{A}\dot{B}} = g^{\mu\nu} \varepsilon_{AB}$$

Translation spacetime indices $\leftrightarrow$ spinor indices: $V_\mu \to V_{A\dot{A}} = \sigma^\mu_{A\dot{A}} V_\mu$.

Like spinor-helicity, but curved.

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Like spinor-helicity, but curved.

Want formula: curvature $R \sim \dfrac{1}{\text{scalar}}$ (curvature F)^2

Weyl spinor and algebraic classification

Weyl curvature $W_{\mu\nu\rho\lambda}$:

$$W_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} + \text{terms}(R_{\mu\nu}, g_{\mu\nu}) = R_{\mu\nu\rho\lambda} \text{ in vacuum as } R_{\mu\nu} = 0$$

Weyl spinor C_{ABCD}:

$$W_{\mu\nu\rho\lambda} \rightarrow W_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = C_{ABCD}\,\varepsilon_{\dot{A}\dot{B}}\,\varepsilon_{\dot{C}\dot{D}} + \bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\,\varepsilon_{AB}\,\varepsilon_{CD}$$

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Can decompose into four rank 1 spinors: $C_{ABCD} = \mathbf{a}_{(A}\mathbf{b}_B\mathbf{c}_C\mathbf{d}_{D)}$

$\rightarrow$ Four *principal null directions*: $a_{A\dot{A}} = \mathbf{a}_A\,\bar{\mathbf{a}}_{\dot{A}}$ and same for $b_{A\dot{A}}, c_{A\dot{A}}, d_{A\dot{A}}$.

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Algebraic classification of spacetimes [Petrov '54]

How many principal null directions are aligned? Types I, II, D, III, N, O.

Type D: $\mathbf{a}_A \propto \mathbf{c}_A$, $\mathbf{b}_A \propto \mathbf{d}_A$, then $C_{ABCD} \propto y_{AB}\,y_{CD}$, where $y_{AB} = \mathbf{a}_{(A}\mathbf{b}_{B)}$.

Spinorial double copy

Take Minkowski space: $\sigma^a = \frac{1}{\sqrt{2}}(\mathbb{1}, \sigma^i).$

Maxwell spinor f_{AB}: $F_{\mu\nu} \;\rightarrow\; F_{A\dot{A}B\dot{B}} \;=\; f_{AB}\,\varepsilon_{\dot{A}\dot{B}} \;+\; \bar{f}_{\dot{A}\dot{B}}\,\varepsilon_{CD}$

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- Applies to all previous Kerr-Schild examples (type D).
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- More examples?
  C-metric $\leftrightarrow$ Lienard-Weichert potential for uniform acceleration
- Extension to $D > 4$?

Perturbative double copy [Luna, RM, Nicholson, Ochirov, O'Connell, Westerberger, White '16]

In general, no exact solutions, no hope for exact double copy.

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- First map linearised solutions. *talk by* **Silvia Nagy** [Anastasiou et al, Cardoso et al]
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Recent work: radiation from point charges [Goldberger, Ridgway '16],
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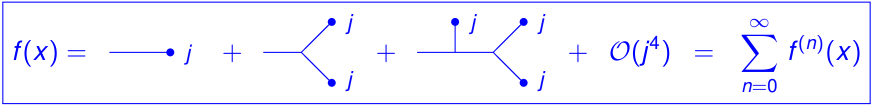
Goals

- Rewrite gravitational perturbation theory in BCJ-ish way. [Bern, Grant '99] [Cheung, Remmen '16]
- Clarify splitting into graviton, dilaton, B-field.

Perturbative double copy: setup

Construct solutions perturbatively:

- start with linearised solution j, $\Box\, j = 0$.
- proceed order by order: $f^{(0)} = j, \quad \Box\, f^{(1)} \sim g\, j^2, \quad \Box\, f^{(2)} \sim g^2\, j^3, \quad \ldots$



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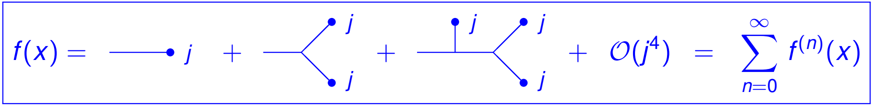


Gauge theory field A_μ ,

admits BCJ Lagrangian enforcing colour-kinematics duality.

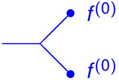
Gravity want field $H_{\mu\nu} \sim$ graviton + dilaton + B-field , **"fat graviton"**

whose vertices are double copy of YM with BCJ Lagrangian.

Perturbative double copy: setup

Example: 3-point vertex

$$f^{(1)}(x) \; =$$



Gauge theory

$$A^{(1)a\mu}(-p_1) = \frac{i}{2p_1^2}\, f^{abc} \int \bar{d}^D p_2 \bar{d}^D p_3 \bar{\delta}^D(p_1 + p_2 + p_3)$$
$$\times \left[(p_1 - p_2)^\gamma \eta^{\mu\beta} + (p_2 - p_3)^\mu \eta^{\beta\gamma} + (p_3 - p_1)^\beta \eta^{\gamma\mu}\right] A^{(0)b}_\beta(p_2) A^{(0)c}_\gamma(p_3)$$

Gravity

$$H^{(1)\mu\mu'}(-p_1) = \frac{1}{4p_1^2} \int \bar{d}^D p_2 \bar{d}^D p_3 \bar{\delta}^D(p_1 + p_2 + p_3)$$
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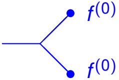

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BCJ Lagrangian for YM has infinite sequence of non-local ($\Box^{-1}$) vertices.

[Bern, Dennen, Huang, Kiermaier '10]

Fat graviton: linearised theory

Definition $$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^{q}_{\mu\nu}(\varphi - \mathfrak{h})$$

- φ is the dilaton. $B_{\mu\nu}$ is the B-field. $\partial^\mu B_{\mu\nu} = 0.$
- $\kappa\,\mathfrak{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}\,g^{\mu\nu} = \kappa\,(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) + \mathcal{O}(h^2).$ $\partial^\mu \mathfrak{h}_{\mu\nu} = 0.$
- Projector: $P^{q}_{\mu\nu} = \frac{1}{D-2}\left(\eta_{\mu\nu} - \frac{q_\mu\partial_\nu + q_\nu\partial_\mu}{q\cdot\partial}\right)$, where $q^\mu = \text{const.}$

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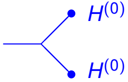
Properties

- $H^{\mu\nu}$ has $(D-2)^2$ propagating dof's. $\partial^\mu H_{\mu\nu} = 0.$
- EOMs: $\partial^2 H_{\mu\nu} = 0 \quad \Leftrightarrow \quad \partial^2 \mathfrak{h}_{\mu\nu} = 0\,,\ \partial^2\varphi = 0\,,\ \partial^2 B_{\mu\nu} = 0$
- Projector origin: $\sum_{i=1}^{D-2} \epsilon^i_\mu \epsilon^i_\nu = \eta_{\mu\nu} - \frac{q_\mu k_\nu + q_\nu k_\mu}{q\cdot k}$, where $\epsilon^i \cdot q = 0.$
- Invertible: can get "skinny fields" $\mathfrak{h}_{\mu\nu}\,, \varphi\,, B_{\mu\nu}$ from $H_{\mu\nu}$.

Fat graviton: first correction

Example

Simple starting point: $H^{(0)}_{\mu\nu} = \frac{\kappa}{2}\,\frac{M}{4\pi r}\,u_\mu u_\nu$, with $u_\mu = (1,0,0,0)$.

Compute using double copy of YM 3-point vertex: $H^{(1)}(x) =$ 

Result is: $H^{(1)}_{\mu\nu} = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2}\,k_\mu k_\nu$, with $k_\mu = (0, x^i/r)$.

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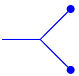
Compute using double copy of YM 3-point vertex: $H^{(1)}(x) =$ 

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Exact spherically symmetric, static solution of Einstein-dilaton gravity known!

Translate the result for comparison:

$$H^{(1)}_{\mu\nu} = \mathfrak{h}^{(1)}_{\mu\nu} + B^{(1)}_{\mu\nu} + P^q_{\mu\nu}(\phi^{(1)} - \mathfrak{h}^{(1)}) + \mathcal{T}^{(1)}_{\mu\nu}$$

where $\mathcal{T}^{(1)}_{\mu\nu}(\mathfrak{h}^{(0)}, \phi^{(0)}, B^{(0)})$ encodes gauge transf / field redefinition.

To be avoided!

Dilatonic point mass: JNW solution

Start with linearised solution: graviton $\sim M$, dilaton $\sim Y$.

Construct order-by-order in perturbation theory.

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Exact solution (M, Y) was found by Janis, Newman, Winicour '68:

$$ds^2 = -\left(1 - \frac{\rho_0}{\rho}\right)^{\gamma} dt^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{-\gamma} d\rho^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{1-\gamma} \rho^2 d\Omega^2$$

$$\varphi = \frac{Y}{\rho_0} \log\left(1 - \frac{\rho_0}{\rho}\right) \qquad \rho_0 = 2\sqrt{M^2 + Y^2} \qquad \gamma = \frac{M}{\sqrt{M^2 + Y^2}}$$

- $Y = 0$: pure Einstein gravity $\to$ Schwarzschild black hole (Kerr-Schild)
- $Y \neq 0$: naked singularity at origin $\rho = \rho_0$, cf. no-hair theorems

  $Y = M$: matches previous example $H^{(0)}_{\mu\nu} \sim M/r$.

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General JNW not vacuum and not Kerr-Schild.

Exact double copy?

Beyond vacuum solutions

Back to question: $(\text{YM})^2 \sim$ Einstein $g_{\mu\nu}$ + dilaton φ + B-field

Spherically symetric, static solution sourced by point charge:

- Yang-Mills: Coulomb solution.
- Pure Einstein gravity: Schwarschild solution.
- Einstein + dilaton: JNW solution. More general double copy of Coulomb.

$$A_\mu^{(0)\,a} = \frac{g\,c^a}{4\pi r}\,u_\mu \quad \leftrightarrow \quad H_{\mu\nu}^{(0)} = \frac{\kappa}{2}\frac{1}{4\pi r}\left(M\,u_\mu u_\nu + (M-Y)\,\frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right),\; q\cdot l = 1$$

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Analogue for scattering amplitudes:

- not all gravity states are of the form $\varepsilon^{\mu\nu} = \epsilon^\mu\,\tilde{\epsilon}^\nu$,
- but double copy gives full gravity theory, $\varepsilon^{\mu\nu}$ is linear comb. of above.
- Eg. dilaton case $\varepsilon_\varphi^{\mu\nu} \propto \sum_{i=1}^{D-2} \epsilon_\mu^i \epsilon_\nu^i = \eta_{\mu\nu} - \frac{q_\mu k_\nu + q_\nu k_\mu}{q\cdot k}$

Of course, pure Einstein gravity is the most interesting sector.

Conclusion

- Gravity is double copy from gauge theory, at least perturbatively.
- Major tool in scattering amplitudes.
- Double copy of classical solutions is possible.
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Many open questions!

- Amplitudes: loop-level structure?
- Exact solutions: Non Kerr-Schild? Non vacuum? Spinorial for $D > 4$?
- New GR solutions from gauge theory?
- Perturbative story: phenomenological applications?