Revisiting and extending PN structures

H. Bursztyn (IMPA)

joint with T. Drummond, C. Netto

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Outline:

(0) What's a Poisson-Nijenhuis structure?

(1) Why? some motivation

(2) Revisiting PN structures

(3) Extending PN structures

(4) Application (to integration in Poisson geom)

(0) Poisson-Nijenhuis structures

Poisson structure: $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$

Lie bracket st $\{f,\cdot\} : C^\infty(M) \longrightarrow C^\infty(M)$ is derivation

$$"X_f$$

Examples: symplectic, Poisson-Lie groups/homog. spaces,

various moduli spaces ...

(0) Poisson-Nijenhuis structures

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Lie bracket st $\{f, \cdot\} : C^\infty(M) \to C^\infty(M)$ is derivation

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{X_f}$

$\therefore \quad \{f, g\} = \pi(df, dg)$, $\pi \in \mathfrak{X}^2(M)$ $\rightsquigarrow$ $\pi^\#: T^*M \to TM$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \alpha \longmapsto \pi(\alpha, \cdot)$

$[\pi, \pi] = 0$

(0) Poisson-Nijenhuis structures

Poisson structure: $\{.,.\}: C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$

Lie bracket st $\{f,.\}: C^\infty(M) \longrightarrow C^\infty(M)$ is derivation

$\quad\quad\quad\quad =: X_f$

$$\therefore \quad \{f,g\} = \pi(df, dg), \quad \pi \in \mathfrak{X}^2(M) \rightsquigarrow \begin{array}{l} \pi^\#: T^*M \to TM \\ \alpha \mapsto \pi(\alpha, \cdot) \end{array}$$

Nijenhuis operator: $r: TM \longrightarrow TM \quad (r \in \Omega^1(M, TM))$

with vanishing Nijenhuis torsion $N_r \in \Omega^2(M, TM)$:

$$N_r(X,Y) := [r(X), r(Y)] - r\big([r(X),Y] + [X, r(Y)] - r([X,Y])\big) = 0$$

We say that $\pi \in \mathfrak{X}^2(M)$ and $r \in \Omega^1(M, TM)$ are compatible if

$$
\begin{tikzcd}
T^*M \arrow[r, "\pi^\#"] \arrow[d, "r^*"'] & TM \arrow[d, "r"] \\
T^*M \arrow[r, "\pi^\#"] & TM
\end{tikzcd}
$$

1. $\pi^\# \circ r^* = r \circ \pi^\#$

$\left(\Rightarrow \pi_r \in \mathfrak{X}^2(M), \; \pi_r^\# = r \circ \pi^\# \right)$

We say that $\pi \in \mathfrak{X}^2(M)$ and $r \in \Omega^1(M, TM)$ are

<u>compatible</u> if

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$$\begin{tikzcd} T^*M \arrow[r, "\pi^\#"] \arrow[d, "r^*"'] & TM \arrow[d, "r"] \\ T^*M \arrow[r, "\pi^\#"] & TM \end{tikzcd}$$

2. $R^r_\pi(X, \alpha) := \pi^\#\left(\mathcal{L}_X r^*\alpha - \mathcal{L}_{r(X)}\alpha\right) - \left(\mathcal{L}_{\pi^\#(\alpha)} r\right)(X) = 0$

$X \in \mathfrak{X}(M)$

$\alpha \in \Omega^1(M)$

Magri-Morosi concomitant

We say that $\pi \in \mathfrak{X}^2(M)$ and $r \in \Omega^1(M, TM)$ are <u>compatible</u> if

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$X \in \mathfrak{X}(M)$
$\alpha \in \Omega^1(M)$

Magri-Morosi concomitant

$\langle \beta, R^r_\pi(X,\alpha) \rangle$
$\shortparallel$
$\langle C^r_\pi(\alpha,\beta), X \rangle$

<u>Rmk</u>: 2 $\Leftrightarrow$

$$C^r_\pi(\alpha,\beta) = [\alpha,\beta]_{\pi_r} - \left([r^*\alpha,\beta]_\pi + [\alpha, r^*\beta]_\pi - r^*[\alpha,\beta]_\pi\right) = 0$$

(1) Why PN structures? 3 reasons ...

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I) Completely integrable systems (Magri - Morosi)

(M^{2n}, ω) $\quad f_1, \dots, f_n$ indep., $\{f_i, f_j\} = 0$

$f_1 = H$

Arnold - Liouville:

(action-angle variables)

quasi-periodic motion ...

(1) Why PN structures ? 3 reasons ...

I) Completely integrable systems (Magri - Morosi)

(M, π, r) P.N. : • $\pi_r^{\#} = r \circ \pi^{\#}$ is Poisson, $[\pi, \pi_r] = 0$ "BIHAMILTONIAN"

• $I_k := \frac{1}{k} tr(r^k)$, $k \geq 0$, in pairwise INVOLUTION.

(1) Why PN structures? 3 reasons ...

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P.N.

• $I_k := \frac{1}{k} tr(r^k)$, $k \geq 0$, in pairwise INVOLUTION.

Various examples: Kepler, Toda, Calogero-Moser, ∞-dim (KdV, ...)

Gelfand-Tsetlin ...

II) Poisson-Lie theory and quantization

G compact s.s. , $\mathcal{O}$ coadj orbit $\left\langle \begin{array}{l} \omega_{KKS} \\ \pi_{Bruhat} \end{array} \right.$

$$\Rightarrow \quad \left(\omega_{KKS}^{-1},\ r = \pi_{Bruhat}^{\#} \circ \omega_{KKS}^{\flat} \right) \text{ is PN}$$

($\mathcal{O}$ hermit. symmetric sp)

Khoroshkin - Radul - Rubtsov

Foth

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$\Rightarrow \quad \left(\omega_{KKS}^{-1},\ N = \pi^{\#}_{Bruhat} \circ \omega^{b}_{KKS}\right)$ is PN ($\mathcal{O}$ hermit. symmetric sp)

$\leftrightarrow$ quantization via symplectic groupoids (Weinstein, Hawkings, Bonechi et al)

(quantum homog. spaces as groupoid C^*-algebras)$_{\text{(Shen)}}$

II) Poisson-Lie theory and quantization

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$\hookrightarrow$ quantization via symplectic groupoids (Weinstein, Hawkings, Bonechi et al)

(quantum homog. spaces as groupoid C^*-algebras) (Sheu)

III) Holomorphic Poisson geometry (Laurent-Stienon-Xu)

(M, J) : $\quad \Pi = \pi + i\,\pi' \quad \begin{cases} \pi' = \pi_J \quad (\pi_J^{\#} = J \circ \pi^{\#}) \\ (\pi, J) \text{ is P.N. !} \end{cases}$

complex mfd $\qquad$ Holomorphic Poisson

Beyond P.N.?

Poisson structures $\longleftrightarrow$ Dirac structures

(symplectic structures $\longleftrightarrow$ closed 2-forms)

(= presymplectic)

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(symplectic structures $\longleftrightarrow$ closed 2-forms
(= presymplectic))

Q: What's Dirac-Nijenhuis structure?

Clemente-Gallardo / Nunes da Costa

Cariñena / Grabowski / Marmo

He - Liu

(Motivated e.g. by holomorphic Dirac structures...)

(2) Revisiting P.N. structures

① $\pi^\# \circ r^* = r \circ \pi^\#$

? $\longrightarrow$ ② $R^r_\pi(X,\alpha) := \pi^\#\left(\mathcal{L}_X r^*\alpha - \mathcal{L}_{r(X)}\alpha\right) - \left(\mathcal{L}_{\pi^\#(\alpha)} r\right)(X) = 0$

(Kosmann - Schwarzbach)

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For $r \in \Omega^1(M, TM)$, consider

$$D^r : \mathfrak{X}(M) \longrightarrow \Omega^1(M, TM), \qquad D^r_X(Y) = (\mathcal{L}_Y r)(X) \overset{= i_X(D^r(Y))}{}$$

$$= [Y, r(X)] - r([X,Y])$$

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$$D^{r,*} : \Omega^1(M) \longrightarrow \Omega^1(M, T^*M),$$

$$D^{r,*}_X(\alpha) = \mathcal{L}_X r^*(\alpha) - \mathcal{L}_{r(X)}\alpha = i_X d(r^*\alpha) - i_{r(X)} d\alpha$$

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$$\therefore \quad R^{r}_{\pi}(X,\alpha) = \pi^{\#}\left(D^{r,*}_X(\alpha)\right) - D^{r}_X\left(\pi^{\#}(\alpha)\right)$$

Compatibility (π, r) :

- $\pi^{\#} \circ r^{*} = r \circ \pi^{\#}$
- $\pi^{\#} \circ D_X^{r,*} = D_X^{r} \circ \pi^{\#}$

$$\begin{tikzcd}
\Omega^1(M) \arrow[r, "\pi^{\#}"] \arrow[d, "D_X^{r^*}"', "r^*"] & \mathfrak{X}(M) \arrow[d, "r"', "D_X^{r}"] \\
\Omega^1(M) \arrow[r, "\pi^{\#}"] & \mathfrak{X}(M)
\end{tikzcd}
\qquad \forall X \in \mathfrak{X}(M)$$

Compatibility (π, r) :

- $\pi^{\#} \circ r^{*} = r \circ \pi^{\#}$
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$$\begin{tikzcd} \Omega^1(M) \arrow[r, "\pi^{\#}"] \arrow[d, "r^*", "D_X^{r*}"'] & \mathfrak{X}(M) \arrow[d, "D_X^{r}", "r"'] \\ \Omega^1(M) \arrow[r, "\pi^{\#}"] & \mathfrak{X}(M) \end{tikzcd} \qquad \forall X \in \mathfrak{X}(M)$$

Where do D^{r} and $D^{r,*}$ come from ?

↳ "1-derivations" on vector bundles ...

Interlude: "Derivations" on vector bundles

$E \xrightarrow{q} M$ vector bundle

Interlude: "Derivations" on vector bundles

$E \xrightarrow{q} M$ vector bundle

$$\begin{tikzcd} TE \arrow[r] \arrow[d, "Tq"'] & E \arrow[d, "q"] \\ TM \arrow[r] & M \end{tikzcd}$$

Interlude: "Derivations" on vector bundles

$E \xrightarrow{q} M$ vector bundle

Derivations

(D, X), $D : \Gamma(E) \to \Gamma(E)$

$\mathbb{R}$-linear

$X \in \mathfrak{X}(M)$

such that

$$D(fu) = \underbrace{f}_{C^\infty(M)} D(u) + (\mathcal{L}_X f)u$$

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Linear vector fields

$$Z \in \mathfrak{X}_{Lin}(E)$$

$\Leftarrow$ flow is by VB automorphisms

$$\Updownarrow$$

$$\begin{tikzcd} E \arrow[r, "Z"] \arrow[d] & TE \arrow[d, "Tq"] \\ M \arrow[r, "X"] & TM \end{tikzcd}$$

VB-map

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VB-map

Prop:

$$\boxed{\mathrm{Der}(E) \overset{1-1}{\rightleftarrows} \mathfrak{X}_{Lin}(E)}$$

This can be generalized to higher degrees:

Thm: $\mathrm{Der}_K(E) \overset{1-1}{\rightleftarrows} \Omega^K_{\mathrm{lin}}(E, TE)$

$$\Omega^0_{\mathrm{lin}}(E,TE) = \Gamma_{\mathrm{lin}}(TE) = \mathfrak{X}_{\mathrm{lin}}(E)$$

This can be generalized to higher degrees:

$$\mathrm{Der}_k(E) \overset{1-1}{\rightleftharpoons} \Omega^k_{Lin}(E, TE)$$

$\Omega^0_{Lin}(E,TE)$
$= \Gamma_{Lin}(TE)$
$= \mathfrak{X}_{Lin}(E)$

$\underline{k=1}$:

1-Derivation (D, ℓ, r):

- $D : \Gamma(E) \longrightarrow \Omega^1(M, E)$ $\mathbb{R}$-Linear
- $\ell : E \longrightarrow E$ $(\ell \in \mathrm{End}(E))$
- $r : TM \longrightarrow TM$ $(r \in \Omega^1(M, TM))$

such that, $\forall X \in \mathfrak{X}(M)$

$D_X : \Gamma(E) \to \Gamma(E)$,

$$D_X(fu) = f D_X(u) + (\mathcal{L}_X f)\, \ell(u) - (\mathcal{L}_{r(X)} f)\, u$$

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$$\mathrm{Der}_k(E) \overset{1-1}{\rightleftharpoons} \Omega^k_{Lin}(E, TE)$$

$\Omega^0_{Lin}(E, TE)$

$= \Gamma_{Lin}(TE)$

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$K \in \Omega^1_{Lin}(E, TE)$:

$$\begin{tikzcd} TE \arrow[r, "K"] \arrow[d] & TE \arrow[d] \\ TM \arrow[r, "r"] & TM \end{tikzcd}$$

VB-map

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$$\Omega^0_{Lin}(E,TE) = \Gamma_{Lin}(TE) = \mathfrak{X}_{Lin}(E)$$

<u>$k=1$</u> :

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$$D_X(fu) = f D_X(u) + (\mathcal{L}_X f)\,\ell(u) - (\mathcal{L}_{r(X)} f)\, u$$

$$\xrightarrow{1-1} \qquad \longleftarrow$$

$K \in \Omega^1_{Lin}(E, TE)$:

$$\begin{tikzcd} TE \arrow[r, "K"] \arrow[d] & TE \arrow[d] \\ TM \arrow[r, "r"] & TM \end{tikzcd}$$

VB-map

Examples of "$\mathrm{Der}_1(E) \rightleftharpoons \Omega^1_{\mathrm{lin}}(E, TE)$"

1) **Connections on $E \to M$**

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$D : \Gamma(E) \to \Omega^1(M, E)$ connection

$$\downarrow\uparrow$$

$(D, \mathrm{id}_E, 0)$ 1-derivation

Examples of "$\mathrm{Der}_1(E) \rightleftharpoons \Omega^1_{\mathrm{Lin}}(E, TE)$"

1) Connections on $E \to M$

$D : \Gamma(E) \to \Omega^1(M, E)$ connection $\rightleftharpoons$ $\underbrace{H}_{=\ker\Theta} \subseteq TE$, $TE = \ker Tq \oplus H$

$$\downarrow\uparrow \qquad\qquad\qquad\qquad \downarrow\uparrow$$

$(D, \mathrm{id}_E, 0)$ 1-derivation

$\Theta : TE \to TE$ linear

$\Theta^2 = \Theta$, $\operatorname{Im}\Theta = \ker Tq$

Examples of "$\mathrm{Der}_1(E) \rightleftharpoons \Omega^1_{\mathrm{lin}}(E, TE)$"

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$$\downarrow\uparrow \qquad\qquad\qquad\qquad \downarrow\uparrow$$

$$(D, \mathrm{id}_E, 0) \text{ 1-derivation} \quad \rightleftharpoons \quad \Theta : TE \to TE \text{ linear}$$

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Examples of "$\mathrm{Der}_1(E) \rightleftharpoons \Omega^1_{lin}(E, TE)$"

1) Connections on $E \to M$

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$\downarrow\uparrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow\uparrow$

$(D, \mathrm{id}_E, 0)$ 1-derivation $\rightleftharpoons$ $\Theta : TE \to TE$ linear

$\Theta^2 = \Theta$, $\mathrm{Im}\,\Theta = \ker Tq$

2) Holomorphic structures on $E \to M$

- complex st $r : TM \to TM$
- $\ell : E \to E$, $\ell^2 = -\mathrm{Id}_E$
- ∇ flat $T^{1,0}$-connection on (E, ℓ)

Examples of "$\mathrm{Der}_1(E) \rightleftharpoons \Omega^1_{lin}(E, TE)$"

1) Connections on $E \to M$

$$\underset{}{D : \Gamma(E) \to \Omega^1(M,E)} \text{ connection} \quad \rightleftharpoons \quad H \overset{=\ker\Theta}{\subseteq} TE, \; TE = \ker Tq \oplus H$$

$$\downarrow\uparrow \qquad\qquad\qquad\qquad \downarrow\uparrow$$

$$(D, \mathrm{id}_E, 0) \text{ 1-derivation} \quad \rightleftharpoons \quad \Theta : TE \to TE \text{ linear}, \; \Theta^2 = \Theta, \; \mathrm{Im}\,\Theta = \ker Tq$$

2) Holomorphic structures on $E \to M$

- complex st $r : TM \to TM$
- $\ell : E \to E$, $\ell^2 = -\mathrm{Id}_E$
- ∇ flat T^{10}-connection on (E, ℓ)

$\rightleftharpoons$ (D, ℓ, r) 1-derivation

$$D_X(u) := \ell\left(\nabla_{X + i r(X)} u\right)$$

Examples of "$\mathrm{Der}_1(E) \rightleftharpoons \Omega^1_{\mathrm{lin}}(E, TE)$"

1) Connections on $E \to M$

$D : \Gamma(E) \to \Omega^1(M,E)$ connection $\rightleftharpoons$ $H \subseteq TE$, $TE = \ker Tq \oplus H$ ($= \ker\Theta$)

$\Updownarrow$ $\Updownarrow$

$(D, \mathrm{id}_E, 0)$ 1-derivation $\rightleftharpoons$ $\Theta : TE \to TE$ linear, $\Theta^2 = \Theta$, $\mathrm{Im}\,\Theta = \ker Tq$

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- complex st $r : TM \to TM$
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$\rightleftharpoons$ (D, ℓ, r) 1-derivation $\rightleftharpoons$ $J : TE \to TE$ linear complex struct.

$$D_X(u) := \ell\left(\nabla_{X + i r(X)} u\right)$$

Back to D^r and $D^{r,*}$:

$$
\begin{tikzcd}
 & r^{tg} \in \Omega^1(TM, T(TM)) \\
r \in \Omega^1(M, TM) \arrow[ur, "\text{Tangent lift}"] \arrow[dr, "\text{Cotangent lift}"'] & \\
 & r^{cotg} \in \Omega^1(T^*M, T(T^*M))
\end{tikzcd}
$$

(Yano - Ishihara)

Back to D^r and $D^{r,*}$:

$$r \in \Omega^1(M, TM) \xrightarrow{\text{Tangent lift}} r^{tg} \in \Omega^1(TM, T(TM))$$

$$r \in \Omega^1(M, TM) \xrightarrow{\text{Cotangent lift}} r^{cotg} \in \Omega^1(T^*M, T(T^*M))$$

(Yano - Ishihara)

<u>Prop</u> (Drummond '20)

- $r^{tg} \in \Omega^1_{Lin}(TM, T(TM)) \rightleftharpoons (D^r, r, r)$ 1-derivation
- $r^{cotg} \in \Omega^1_{Lin}(T^*M, T(T^*M)) \rightleftharpoons (D^{r,*}, r^t, r)$ 1-derivation

<u>D^r and $D^{r,*}$ are dual</u> : $\langle D^{r,*}_X(\alpha), Y \rangle = \mathcal{L}_X \langle \alpha, r(Y) \rangle - \mathcal{L}_{r(X)} \langle \alpha, Y \rangle - \langle \alpha, D^r_X(Y) \rangle$

Back to compatibility: (π, r)

$$\pi^{\#} : T^*M \longrightarrow TM$$

$r^{\cot g}$ $\qquad\qquad$ $r^{\ast g}$

Back to compatibility: (π, r)

$$\pi^{\#} : T^*M \longrightarrow TM$$

$$r^{\cot g} \qquad\qquad r^{*g}$$

$$\begin{tikzcd}
T(T^*M) \arrow[r, "T\pi^{\#}"] \arrow[d, "r^{\cot g}"'] & T(TM) \arrow[d, "r^{tg}"] \\
T(T^*M) \arrow[r, "T\pi^{\#}"] & T(TM)
\end{tikzcd}$$

Back to compatibility: (π, r)

$$\pi^{\#} : T^*M \longrightarrow TM$$

$$r^{cotg} \qquad\qquad r^{tg}$$

$$T\pi^{\#} \circ r^{cotg} = r^{tg} \circ T\pi^{\#}$$

$$\Updownarrow$$

$$\pi^{\#} \circ r^{*} = r \circ \pi^{\#}$$

$$\pi^{\#} \circ D_X^{r,*} = D_X^{r} \circ \pi^{\#} \qquad \left(\Leftrightarrow R_{\pi}^{r} = 0 \right)$$

(3) Extending PN structures

- symplectic str $\longleftrightarrow$ closed 2-forms (= presymplectic)
- Poisson str $\longleftrightarrow$ "Dirac structures"

$\leftrightarrow$ submfds / pull backs
quotients / reduction ...

(3) Extending PN structures

- symplectic str $\longleftrightarrow$ closed 2-forms (= presymplectic)
- Poisson str $\longleftrightarrow$ "Dirac structures"

Consider $\mathbb{T}M = TM \oplus T^*M$ equipped with

$$\langle (X,\alpha),(Y,\beta)\rangle = \beta(X) + \alpha(Y)$$

Courant-Dorfman $\rightsquigarrow$ $[\![(X,\alpha), (Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha)$

(3) Extending PN structures

- symplectic str $\hookrightarrow$ closed 2-forms (= presymplectic)
- Poisson str $\hookrightarrow$ "Dirac structures"

Consider $\mathbb{T}M = TM \oplus T^*M$ equipped with

$$\langle (X,\alpha),(Y,\beta)\rangle = \beta(X) + \alpha(Y)$$

Courant-Dorfman $\rightsquigarrow$ $[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha)$

Dirac structure : subbundle $L \subseteq \mathbb{T}M$ such that

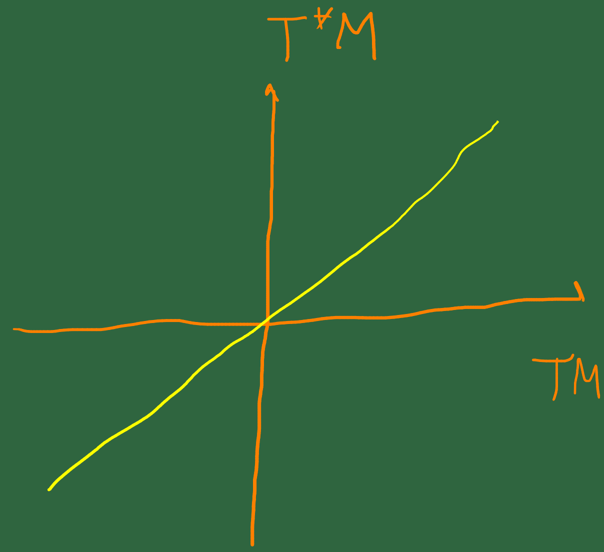
- $L = L^\perp$
- $[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L)$

Examples

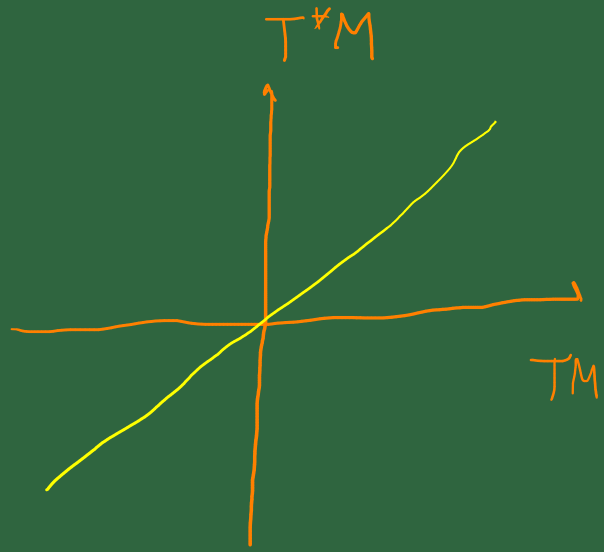
- $\pi \in \mathfrak{X}^2(M)$, $\quad L = \mathrm{graph}(\pi)$

$(T^*M \longrightarrow TM)$ $\quad \pi$ Poisson $\iff$ L Dirac

$(L \cap TM = \{0\})$



Examples



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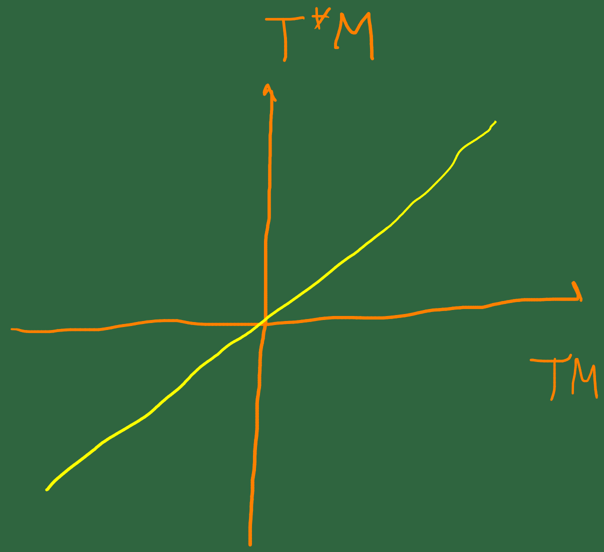

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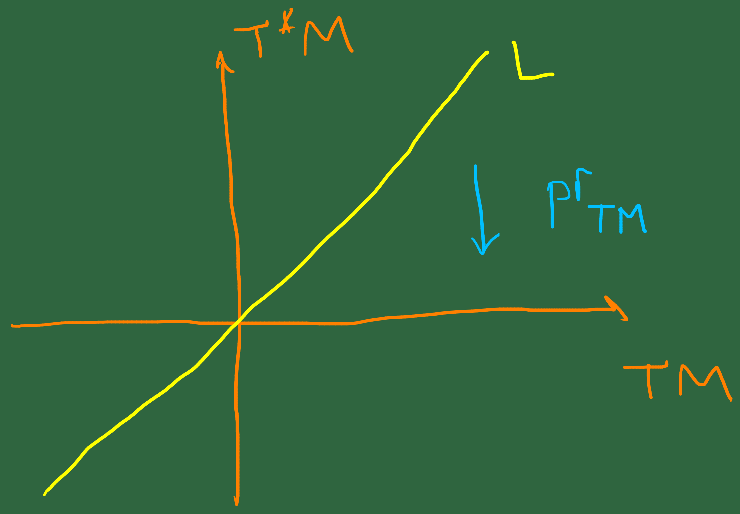
Other examples:

foliations $(L = F \oplus F^\circ)$, submanifolds of Poisson submfds,

"affine" Dirac on Poisson Lie groups $(\mathfrak{l} \subseteq \mathfrak{d}$ Lagrangian$)$, Cartan-Dirac (q-Hamiltonian)

Feature: $pr_{TM}(L) \subseteq TM$ defines leaves $\vartheta \hookrightarrow M$ with

$\omega_\vartheta \in \Omega^2(\vartheta)$, $d\omega_\vartheta = 0$.



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Dirac mfd $\rightleftharpoons$ presymplectic foliation

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Holomorphic version: M complex mfd

$$L \subseteq \mathbb{T}^{1,0}M = T^{1,0}M \oplus (T^{1,0}M)^*$$

Lagrangian + Courant-involutive

Compatibility

$L \subseteq \mathbb{T}M$ Dirac , $r \in \Omega^1(M, TM)$

$$\left\{ \begin{array}{l} r : TM \longrightarrow TM \\ r^* : T^*M \longrightarrow T^*M \end{array} \right.$$

$\mathbb{D}^r : \Gamma(\mathbb{T}M) \longrightarrow \Omega^1(M, \mathbb{T}M)$, $\mathbb{D}^r = (D^r, D^{r,*})$

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L and r are compatible if : ① $(r, r^*)(L) \subseteq L$

② $\mathbb{D}^r_X(\Gamma(L)) \subseteq \Gamma(L)$ $\forall X \in \mathfrak{X}(M)$

(Dirac-Nijenhuis if $N_r = 0$)

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Note: $L = \operatorname{graph}(\pi)$:

(1) $\iff \pi^\sharp \circ r^* = r \circ \pi^\sharp$

(2) $\iff \pi^\sharp \circ D^{r,*}_X = D^r_X \circ \pi \iff R^r_\pi \equiv 0$

= PN conditions

Moreover:

- $L = \mathrm{graph}(\omega)$:

$\omega \in \Omega^2_{cl}(M)$

① $\iff$ $\omega^\flat \circ r = r^* \circ \omega^\flat$ $\quad (\Rightarrow \omega_r \in \Omega^2(M))$

② $\iff$ $d(\omega_r) = 0$

(ω, r) is Magri-Morosi's "ΩN" structures

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- Holomorphic Dirac : (M, r) complex mfd

$$\Phi : \mathbb{T}M \longrightarrow \mathbb{T}^{1,0}M \ , \quad \Phi(X, \alpha) = \left(\tfrac{1}{2}(X - i r(X)), \alpha - i r^* \alpha\right)$$

hol. str : $(\mathbb{D}^r, (r, r^*), r)$

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$\mathbb{T}M \supseteq L \xleftrightarrow{1-1} \mathcal{L} = \Phi(L) \subseteq \mathbb{T}^{1,0}M$

Prop : (L, r) Dirac-Nij $\iff$ $\mathcal{L}$ holomorphic Dirac

Dirac-Nijenhuis have natural features:

- leaves are presympl.-Nijenhuis
- quotients are Poisson-Nijenhuis
- Hierarchies of Dirac-Nijenhuis:

$$L_{(n,0)} := (r^n, \mathrm{id}_{T^*})(L) \qquad \left(\pi_n^\# = r^n \circ \pi^\#\right) \quad \text{"commute"}$$

$$L_{(0,n)} := (\mathrm{id}_T, (r^*)^n)(L) \qquad \left(\omega_n^\flat = \omega^\flat \circ r^n\right)$$

(4) Applications (to integration in PG)

Poisson mfds $\rightleftharpoons$ symplectic groupoids

(M, π) $\qquad$ $(\mathcal{G} \rightrightarrows M, \omega)$

$d\omega = 0$

$\omega : T\mathcal{G} \xrightarrow{\sim} T^*\mathcal{G}$

(Weinstein, Karasev, Mackenzie-Xu, Cattaneo-Felder, Crainic-Fernandes ...)

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compatible (1,1) tensor

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$\Rightarrow$ Holomorphic Dirac mfds $\rightleftharpoons$ Holomorphic presymplectic groupoids

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Applications to construct symplectic groupoids of Poisson-homogeneous spaces (in holomorphic category):

$$\begin{array}{c} G \\ \downarrow \\ G/H \end{array}$$

via "affine" Dirac struct.

(B.-Iglesias-Lu)

Final comments:

- Dirac-Nijenhuis on general Courant algebroids
- role on integrable systems? ∞-dim? (Dorfman)

⋮

Thank you!

Obrigado!