Commuting Varieties of Reductive Groups

Poincaré Polynomials of Spaces of Commuting Group Elements

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Outline

Commuting varieties (for matrices)

- Fix $r, n \in \mathbb{N}$. Let

$$Comm_n^r := \{(A_1, \cdots, A_r) \in (Mat_{n\times n}\mathbb{C})^r \ : \ A_iA_j = A_jA_i\}$$

Commuting variety of r-tuples of size n matrices.

- Surprisingly many Open Problems:
  - Irreducibility?
  - Dimensions of components?
  - Geometry? Arithmetic?
- Known: (trivial cases: $Comm_1^r = \mathbb{C}^r$; $Comm_n^1 = Mat_{n\times n}\mathbb{C}$)
  - $Comm_n^2$ is irreducible of the expected dimension $n^2 + n$
  - $Comm_n^4$ is reducible, for every $n \geq 4$ (!)
  - $Comm_n^3$ is reducible for $n \geq 32$ (Guralnick, 1992; improved to $n \geq 29$)

The diagonal component

- **Distinguished component** in $Comm_n^r$: those r-tuples which **can be simultaneously diagonalized.**

Let $D_n \subset Mat_{n\times n}\mathbb{C}$ be the vector space of diagonal matrices, and:

$$\phi : GL(n,\mathbb{C})\times D_n^r \to Comm_n^r, \quad (g,B_1,\cdots,B_n) \mapsto (gB_1g^{-1},\cdots,gB_ng^{-1})$$

Then

$$DCom_n^r := \overline{im(\phi)} \qquad \text{is irreducible}$$

Let's call it the "Diagonal Component" in $Comm_n^r$.

Theorem [Motzkin-Taussky, 1955] For $r = 2$

$$Comm_n^2 = DCom_n^2$$

so that $Comm_n^2$ is *irreducible* of dimension $n^2 + n$.

Commuting r-tuples in groups

- Fix $r \in \mathbb{N}$, G a (compact, Lie) group. Let
$$Comm^r(G) = \{(g_1, \cdots, g_r) \in G^r \ : \ g_i g_j = g_j g_i\}$$
Space of (ordered) r-tuples (or r-sequences) of **commuting elements** in G.

V. Kac - A. Smilga, 1999: "The problem of constructing the quantum vacuum states of pure supersymmetric Yang--Mills theories placed on a small 3-dimensional spatial torus T^3 is reduced to a pure mathematical problem of classifying the flat connections on T^3"

For $r \geq 2$, $Comm^r(G)$ is not necessarily connected!!
We have now a **torus component:** Let $T \subset G$ be a **fixed** maximal torus.

$$TCom^r(G) \ := \ \overline{im(\psi)} \subset Comm^r(G)$$
$$\psi : G \times T^r \to Comm^r(G) \qquad (g, t_1, \cdots, t_r) \mapsto (gt_1g^{-1}, \cdots, gt_rg^{-1})$$

Theorem [Kac-Smilga '99] If $r > 2$, and G is simple, then $Comm^r(G)$ is connected (hence equals the torus component) only for $G = SU(n)$ or $G = Sp(n)$.

Commuting varieties (for reductive groups)

- Finally, let G be a reductive (complex algebraic) group

$$Comm^r(G) := \{(g_1, \cdots, g_r) \; : \; g_i g_j = g_j g_i\} \subset G^r$$

and call it Commuting variety of r-tuples in G.

It is an affine algebraic variety, since $G \subset GL(V)$ for some vector space V.
Example: write $GL_n \equiv GL(n, \mathbb{C})$.

$$\begin{aligned} Comm^r(GL_n) &= Comm^r_n \cap (GL_n)^r \\ TCom^r(GL_n) &= DCom^r_n \cap (GL_n)^r. \end{aligned}$$

$Comm^r(G)$ is generally a very singular algebraic variety, again not necessarily irreducible ($TCom^r(G)$ being one component), with intricate topology.
By contrast, the variety of commuting matrices $Comm^r_n$ is contractible!

Moduli space of Commuting matrices/sequences in G

We wish to consider the space of r-sequences of commuting linear operators on a vector space $V \cong \mathbb{C}^n$ without a preferred basis.

- Fix $r, n \in \mathbb{N}$. $G = GL_n$ acts on $Comm_n^r$ by conjugation:
$$MC_n^r = Comm_n^r / G.$$
- Motivation: Hilbert scheme of points on $\mathbb{C}^r$.
- Open problems: Irreducibility? Dimension of components? Geometry?
- Remarks: In the complex reductive case (but not in the compact case), we need the (affine) **geometric invariant theory** (GIT) quotient: $MC_n^r := Comm_n^r /\!\!/ G$, and similarly for the moduli space of r-sequences in G:
$$MC^r(G) := Comm^r(G) /\!\!/ G.$$
- Again, there is a diagonal component for matrices: $DCom_n^r /\!\!/ G$, and a torus component in the group case $TCom^r(G) /\!\!/ G$.

The symmetric group and invariant polynomials

Symmetric group S_n = group of **permutations** of the n element set $\{1, 2, \cdots, n\}$.

- S_n acts on a Cartesian product X^n by permuting the variables:

  $\mathrm{Sym}^n X := X^n / S_n \quad = \quad \{\text{unordered } n\text{-tuples of elements of } X\}$

Example: What is $\mathrm{Sym}^n \mathbb{C}$?
Let V be a $\mathbb{C}$-vector space of dimension n, then $\mathrm{Sym}^n \mathbb{C} = V/S_n$.
Consider the action of S_n on polynomials in V.
If G acts on X, then G acts on $\mathcal{F}(X, Y) = \{\text{maps } X \to Y\}$ by:
$(g \cdot f)(x) := f(g^{-1} \cdot x) \in Y$.

Algebra: (1st) Fundamental Theorem of Invariant Theory

$$\mathbb{C}[x_1, \cdots, x_n]^{S_n} \cong \mathbb{C}[e_1, \cdots, e_n]$$

where $e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$. Therefore $\mathrm{Sym}^n \mathbb{C} \cong \mathbb{C}^n$!

More generally, symmetric products of Riemann surfaces (smooth algebraic curves) are smooth. Example: $\mathrm{Sym}^n(\mathbb{C}P^1) \cong \mathbb{C}P^n$.

The case GL_n and Symmetric Products

- In the case $G = GL_n \equiv GL(n,\mathbb{C}) \subset Mat_{n\times n}\mathbb{C}$, commuting varieties have an interesting relation with symmetric products.

The **maximal torus** of GL_n is $T \cong (\mathbb{C}^*)^n$ and the Weyl group is $W := NT/T \cong S_n$. We have:

$$\begin{aligned} MC_n^r := Comm_n^r /\!\!/ GL_n &= (\mathbb{C}^n)^r/W = (\mathbb{C}^r)^n/S_n = \mathrm{Sym}^n(\mathbb{C}^r) \\ MC^r(GL_n)/\!\!/ GL_n &= T^r/W = ((\mathbb{C}^*)^n)^r/S_n = \mathrm{Sym}^n((\mathbb{C}^*)^r). \end{aligned}$$

Molien's formula for the Hilbert-Poincaré series (generating function for the dimensions of the graded components of $\mathbb{C}[x_1,\cdots,x_n]^{S_n} = \mathbb{C}[\mathrm{Sym}^n(\mathbb{C})]$)

$$\frac{1}{n!}\sum_{\sigma\in S_n}\frac{1}{\det\left(I - q\,A_\sigma\right)} = \prod_{k=1}^{n}\frac{1}{1-q^k},$$

where A_σ is the action of $\sigma \in S_n$ on $V^* \cong \mathbb{C}^n$. Example: $n = 2$ we have the series: $1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + \cdots$ $(\{e_1\},\ \{e_1^2, e_2\},\ \{e_1^3, e_1 e_2\},\ \{e_1^4, e_1^2 e_2, e_2^2\},\ \ldots)$

Character varieties

- Γ – a finitely presented group:

$$\Gamma = \langle \gamma_1, \cdots, \gamma_n \,|\, r_1, \cdots, r_m \rangle$$

  Ex: fundamental group $\pi_1(M)$ of a manifold/variety M.
- G – a Lie group.
  Typically, G is a real or complex reductive group
  Ex: $G = SL_n\mathbb{C}$, $GL_n\mathbb{C}$, $SL_n\mathbb{R}$, $U(n)$, Sp_n, ...
- $\mathrm{R}_\Gamma G := \hom(\Gamma, G)$ – the **G-representation variety of Γ** (affine algebraic variety, given $G \subset GL_n\mathbb{C}$)
- $\mathrm{X}_\Gamma G := \hom(\Gamma, G)/\!\!/G$ – the **G-character variety of Γ.**

It is the GIT quotient, under *conjugation*: $g \in G$, $\rho \in \hom(\Gamma, G)$:

$$(g \cdot \rho)(\gamma) := g\,\rho(\gamma)\,g^{-1}, \qquad \gamma \in \Gamma.$$

Example: with $\Gamma = \mathbb{Z}^r$ (free abelian group) we have $\mathrm{R}_{\mathbb{Z}^r} G = Comm^r(G)$ and $\mathrm{X}_{\mathbb{Z}^r} G = MC^r(G)$ we let $\mathrm{R}^0_{\mathbb{Z}^r} G$ denote the torus component $TComm^r(G)$.

Motivation

- (Topology/Diff. Geometry) Space of Flat G-connections on a manifold M with $\pi_1(M) = \Gamma$.
- (Algebra) Matrix invariants under **simultaneous conjugation**.
- (Knot theory) The ***A-polynomial*** is defined by the image of a morphism between character varieties: $\mathsf{X}_\Gamma SL_2\mathbb{C} \to \mathsf{X}_{\mathbb{Z}^2} SL_2\mathbb{C}$.
- Non-abelian Hodge correspondence:

Theorem ([Hitchin, Donaldson, Corlette, Simpson 1986-90])

*Let M be a Riemann surface and G be real/complex reductive Lie group. Then $\mathsf{X}_\Gamma G = \hom(\Gamma, G)/\!\!/G$ is **homeomorphic** to $\mathcal{H}_M G$, a moduli space of G-**Higgs bundles** over M.*

The case of Surface groups

Let Γ =(central extension of) $\pi_1(\Sigma_g)$, the fund. group of a genus g compact orientable (Riemann) surface.

- N. Hitchin ('87): Poincaré polynomials for $G = SL_2\mathbb{C}$
- P. Gothen ('94): Poincaré polynomials for $G = SL_3\mathbb{C}$
- T. Hausel - F. Rodriguez-Villegas ('08): Hodge-Deligne polynomials for $SL_2\mathbb{C}$, conjectures for higher n.
- M. Logares, V. Muñoz, P. Newstead ('13): E-polynomials for $SL_2\mathbb{C}$ and low g.
- O. Schiffman ('16), A. Mellit ('17): Poincaré polynomials for all $G = SL_n\mathbb{C}$.

Actually, most of these results are for smooth (twisted) character varieties.

Very little is known for $\hom(\pi_1\Sigma_g, G)/\!/G$ even for SL_2 or GL_2 (LMN, Baraglia-Hekmati '17: E-polynomials) as these are very singular spaces.

Free groups & hyperbolic surface groups

Let $\Gamma = F_r$ the free group of rank r, and note $\mathfrak{X}_{F_r}G \cong G^r /\!\!/ G$
Recall: $Y \subset X$ is a **strong deformation retract** (of X) if there is a homotopy $H : [0,1] \times X \to X$ with $H_1 = id_X$, $H_0(X) = Y$ and $H_t|_Y = id_Y$.

Theorem (F.-Lawton-Casimiro-Oliveira '09-'15)

Let G a real/complex reductive group, with maximal compact subgroup K. Then, $\mathfrak{X}_{F_r}K$ is a strong deformation retract of $\mathfrak{X}_{F_r}G$ (hence, Betti numbers agree $b_k(\mathfrak{X}_{F_r}K) = b_k(\mathfrak{X}_{F_r}G)$, for all k, r).

A concrete formula: Tom Baird, 2007, computed $P_t(\mathfrak{X}_{F_r}SU(2))$:

$$P_t(\mathfrak{X}_{F_r}SU(2)) = 1 + t - \frac{t(1+t^3)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^r}{1-t^2} - \frac{(1-t)^r}{1+t^3}\right)$$

As far as I know, in this very singular case, no computation was done for SL_n, $n > 2$. **Note:** For F_r the representation variety $\mathfrak{R}_{F_r}G$ is trivial !!

The case Γ abelian/nilpotent

From now on, Γ is **nilpotent**, that is, with $\Gamma_{k+1} = [\Gamma_k, \Gamma_k]$:

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_n = \{e\}.$$

Examples:

- $\Gamma = \mathbb{Z}^r$, free abelian; more generally, any abelian group.
- $\Gamma = H(\mathbb{Z})$, the (discrete) Heisenberg group; more generally, unipotent upper triangular matrices with $\mathbb{Z}$ entries.

Theorem (F.-Lawton '09-'14)

Let $\Gamma = \mathbb{Z}^r$ and G a complex reductive group, with maximal compact K. Then, $\mathsf{X}_\Gamma K$ is a strong deformation retract of $\mathsf{X}_\Gamma G$.

Theorem (Bergeron '15)

$\mathsf{X}_\Gamma K$ is a strong deformation retract of $\mathsf{X}_\Gamma G$, for any finitely generated nilpotent Γ.

Reduction from nilpotent to abelian

The abelianization of Γ is $\Gamma_{Ab} := \Gamma/[\Gamma,\Gamma]$, and we say that the *abelian rank of Γ is $r \in \mathbb{N}_0$* when

$$\Gamma_{Ab} \cong \mathbb{Z}^r \oplus F, \qquad F \text{ finite abelian group.}$$

For K compact, let $\mathsf{R}^0_\Gamma K$ be the *identity component* of $\mathsf{R}_\Gamma K = \hom(\Gamma, K)$.

Theorem (Bergeron-Silberman '16)

For Γ nilpotent of abelian rank r, and K compact Lie group:

$$\mathsf{R}^0_\Gamma K \cong \mathsf{R}^0_{\mathbb{Z}^r} K, \qquad \textit{and} \qquad \mathsf{X}^0_\Gamma K \cong \mathsf{X}^0_{\mathbb{Z}^r} K.$$

This implies that $\mathsf{R}^0_\Gamma K$ and $\mathsf{X}^0_\Gamma K$ actually equal the torus component (Baird '09).

Theorem (F-Lawton-Silva '21)

For $\mathbb{Z}^r$, and G complex reductive, the torus component coincide with the identity component.

Summary of Topological and algebraic invariants

Let X be a space with finite cohomology (eg, a compact manifold, a finite CW complex, an algebraic variety, etc).

Let $b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$. The Poincaré polynomial of X is:

$$P_X(t) := \sum_{k \geq 0} b_k(X)\ t^k$$

Euler characteristic: $\chi(X) := P_X(-1) = \sum_{k\geq 0}(-1)^k\ b_k(X)$

Example: If G is a connected Lie group of positive dimension, then $\chi(G) = 0$.

Now, let X be a quasi-projective algebraic variety X. Its cohomology decomposes into "Hodge pieces" of dimensions $h^{k,p,q}(X)$, $k, p, q \in \{0, \cdots, 2d\}$. Mixed Hodge polynomial:

$$\mu_X(t, u, v) := \textstyle\sum_{k,p,q} h^{k,p,q}(X)\, t^k u^p v^q.$$

Then: $P_X(t) = \mu_X(t, 1, 1)$ and the Serre (E-) polynomial is $E_X(u, v) := \mu_X(-1, u, v)$.

Examples and properties

Polynomial invariants associated to a **topological space**:

Space M	Poincaré polynomial $P_M(t)$	Euler char. $\chi(M)$
$\mathbb{R}^n$	1	1
Σ_g	$1+2gt+t^2$	$2-2g$
S^n	$1+t^n$	$1+(-1)^n$
$\mathbb{CP}^n$	$1+t^2+\cdots+t^{2n}$	$n+1$
$X\times Y$	$P_X(t)\,P_Y(t)$	$\chi(X)\chi(Y)$
$X\sqcup Y$	?	$\chi(X)+\chi(Y)$

Polynomial invariants associated to a **quasi-projective variety**:

Space M	Mixed Hodge $\mu_X(t,u,v)$	Serre (E-) polynomial
Σ_g	$1+gt(u+v)+t^2uv$	$1-gu-gv+uv$
$\mathbb{CP}^n$	$1+t^2uv+\cdots+t^{2n}u^nv^n$	$1+uv+\cdots+(uv)^n$
toric	$\sum_{j=0}^d a_j(t^2uv-1)^j$	$\sum_{j=0}^d a_j(x-1)^j$
$GL_n\mathbb{C}$	$\prod_{j=1}^n(1+t^{2j-1}u^jv^j)$	$\prod_{j=1}^n(1-x^j)$
$X\times Y$	$\mu_X(t)\,\mu_Y(t)$	$\chi(X)\chi(Y)$
$X\sqcup Y$	?	$\chi(X)+\chi(Y)$

Char. var. of free Abelian groups - Irreducible components

Theorem (Sikora, F.-Lawton '15)

For $G = GL_n\mathbb{C}$, $SL_n\mathbb{C}$ and $Sp_n\mathbb{C}$,

$$\mathsf{X}_{\mathbb{Z}^r}G \cong T_G^r / W_G.$$

G is complex simple, and $\mathsf{X}_{\mathbb{Z}^r}G$ irreducible $\Rightarrow G = SL_n\mathbb{C}$ or $Sp_n\mathbb{C}$.

Corollary (F.-Lawton-Silva '21)

For $G = GL_n$, $G = SL_n\mathbb{C}$ or $G = Sp_n\mathbb{C}$ we have $\mathsf{X}^0_{\mathbb{Z}^r}G = \mathsf{X}_{\mathbb{Z}^r}G$ (= the identity component).

Open problems for general Γ, G: Irreducibility, Singularities, Topology, etc...

Char. var. of free Abelian groups - invariants

Theorem (F-Silva, '18; F-Lawton-Silva '21)

Let G be the complexification of a compact Lie group. Let X^0 be the irreducible component of the identity in $\mathsf{X}_{\mathbb{Z}^r}G$. Then:

$$\mu_{\mathsf{X}^0}(t,u,v) = \frac{1}{|W|}\sum_{\sigma\in W}\det(I + tuvM_\sigma)^r.$$

We recover:

Theorem (Stafa, '17)

Let $\mathsf{X}^0_{\mathbb{Z}^r}K$ be the connected component of the identity in $\mathsf{X}_{\mathbb{Z}^r}K = \mathrm{hom}(\mathbb{Z}^r, K)/K$ ("real" character variety). Then:

$$P_{\mathsf{X}^0_{\mathbb{Z}^r}K}(t) = \frac{1}{|W|}\sum_{\sigma\in W}\det(I + tM_\sigma)^r.$$

$GL_n\mathbb{C}$ character varieties and symmetric products

Now take $G = GL_n\mathbb{C}$, the group of invertible matrices. It has $T_G = (\mathbb{C}^*)^n$ as **maximal torus**, and S_n as **Weyl group**:

$$\mathsf{X}_{\mathbb{Z}^r} GL_n\mathbb{C} = MC^r(G) = T_G^r/W = ((\mathbb{C}^*)^n)^r/S_n = \mathrm{Sym}^n((\mathbb{C}^*)^r)$$

Similarly, for $K = U(n)$ we have $T_K = (S^1)^n$ and $W = S_n$ as well:

$$\mathsf{X}_{\mathbb{Z}^r} U(n) = MC^r(K) = T_K^r/W = ((S^1)^n)^r/S_n = \mathrm{Sym}^n((S^1)^r)$$

Example

If $r = 2$ we have (over Σ_1 an elliptic curve):

$$\mathsf{X}_{\mathbb{Z}^2} U(n) \cong \text{moduli space of rank } n \text{ vector bundles on } \Sigma_1$$

$$\mathsf{X}_{\mathbb{Z}^2} GL_n \cong \text{moduli space of rank } n \text{ Higgs bundles on } \Sigma_1$$

Hence, these are, respectively, nth symmetric products of a real torus $(S^1)^2$, resp. of $(\mathbb{C}^*)^2$.

The space of commuting r-tuples in K

Let K be a compact Lie group, and recall the space of commuting r-sequences: $\mathrm{R}_{\mathbb{Z}^r}K \equiv \hom(\mathbb{Z}^r, K)$ (not necessarily connected). By Kac-Smilga '99, $\mathrm{R}_{\mathbb{Z}^r}K$ is connected for $K = U(n)$ (disconnected in general for $K = SO(n)$ and other examples).

Theorem (Baird, '09)

If $K = SU(n)$, then:

$$P_{\mathrm{R}_{\mathbb{Z}^r}K}(t) = \begin{cases} \frac{1}{2}\left((1+t^2)(1+t)^r + (1-t^2)(1-t)^r\right), & n=2 \\ \frac{1}{6}(1+2t^2+2t^4+t^6)(1+t)^{2r} + \frac{1}{2}(1-t^6)(1-t^2)^r \\ \quad +\frac{1}{3}(1-t^2-t^4+t^6)(1-t+t^2)^r, & n=3 \end{cases}$$

Theorem: Ramras-Stafa formula ('17):

$$P_{\mathrm{R}^0_\Gamma K}(t) = \frac{1}{|W|}\prod_{i=1}^{m}(1-t^{2d_i})\sum_{g\in S_n}\frac{\det(I + t\,A_g)^r}{\det(I - t^2\,A_g)}.$$

The commuting variety of r-tuples in G

Now, denote by $\mathsf{R}^0_\Gamma G$ the identity component of $\mathsf{R}_\Gamma G$ for any **nilpotent group** Γ of abelian rank r. We generalize Ramras-Stafa formula as follows (abbreviate $x = uv$).

Theorem (F.-Lawton, Silva '21)

For G reductive, all $r \geq 1$ we have:

$$\mu_{\mathsf{R}^0_\Gamma G}(t, x) = \frac{1}{|W|} \prod_{i=1}^{m} (1 - t^{2d_i} x^{d_i}) \sum_{g \in S_n} \frac{\det (I + tx\, A_g)^r}{\det (I - t^2 x\, A_g)}.$$

Additionally, the factor $\frac{1}{|W|} \sum_{g \in S_n} \ldots$ is the mixed Hodge series for the G-equivariant cohomology:

$$H^*_G(\mathsf{R}^0_\Gamma G) \cong [H^*(T) \otimes H^*(BT)]^W .$$

Moreover, in the GL_n case the generating function $\sum_{n \geq 0} P_{\mathsf{R}^0_n}(t)\, y^n$ is a plethystic exponential, so we obtain a recursion: $P_{\mathsf{R}^0_n}(t)$ from $P_{\mathsf{R}^0_m}(t)$, $m \leq n$.

Corollary: The Euler characteristic of R^0 is zero for all $r, n > 0$

A "trivial" example

Let $G = GL_n$, $r = 1$, and $\Gamma = \mathbb{Z}$. Then

$$\mathrm{R}_\Gamma GL_n = \hom(\mathbb{Z}, GL_n) = GL_n$$

and the exponents are $d_i = i$ for $i = 1, 2, \cdots, n$. With $x = uv$:

$$\begin{aligned}
\mu_{GL_n}(t,x) &= \frac{1}{n!}\prod_{i=1}^{n}(1-t^{2i}x^i)\sum_{g\in S_n}\frac{\det(I+tx\,A_g)}{\det(I-t^2x\,A_g)} \\
&= \prod_{i=1}^{n}(1+t^{2i-1}x^i),
\end{aligned}$$

a well known result, that follows from a **generalization of the Molien formula** for the Hilbert-Poincaré series of the graded ring of invariant polynomials in n variables $\mathbb{C}[x_1, \cdots, x_n]^{S_n}$:

$$\frac{1}{n!}\sum_{\sigma\in S_n}\frac{\det(I-z\sigma)}{\det(I-q\,\sigma)} = \prod_{k=1}^{n}\frac{1-zq^{k-1}}{1-q^k}.$$

A non-trivial (new) example

Let $G = Sp_n = Sp(n, \mathbb{C})$, $\dim_{\mathbb{C}} G = 10$, and exponents are $\{2, 4\}$.
Let $r = 2$, and $\Gamma = \mathbb{Z}^2$.

$$\mathsf{R}_\Gamma^0 G = Comm^2(Sp_n).$$

Mixed Hodge polynomial, with $x = uv$:

$$\mu_{\mathsf{R}_\Gamma^0 G}(t, x) = 1 + t^2x^2 + t^4x^4 + 2(t^3 + t^7)(x + t^2x^3) + 2t^6x^2 + 3t^{10}x^2.$$

Poincaré polynomial ($P(1) = 16 = (2^2)^2$):

$$P_{\mathsf{R}_\Gamma^0 G}(t) = 1 + t^2 + t^4 + 2(t^3 + t^5 + t^6 + t^7 + t^9) + 3t^{10}.$$

Serre (E-)polynomial ($\chi = 0$)

$$E_{\mathsf{R}_\Gamma^0 G}(t) = 1 - 4x + 6x^2 - 4x^3 + x^4 = (1 - x)^4.$$

The diagram in the proof

G reductive with maximal compact K, maximal torus T, and Weyl group W. $T_K = T \cap K$.

$$\begin{array}{ccccc}
(G/T) \times_W T^r & \xrightarrow{\varphi_G} & \hom^0(\Gamma_{Ab}, G) & \longrightarrow & \hom^0(\Gamma, G) \equiv \mathrm{R}^0_\Gamma G \\
\uparrow & & \uparrow FL & & \uparrow PS \\
(K/T_K) \times_W T_K^r & \xrightarrow{\varphi_K} & \hom^0(\Gamma_{Ab}, K) & \xrightarrow{\cong BS} & \hom^0(\Gamma, K) \equiv \mathrm{R}^0_\Gamma K.
\end{array}$$

The recursion formula for $GL_n\mathbb{C}$

For $G = GL_n$, both $\mathsf{R}^0_\Gamma G$ and $\mathsf{X}^0_\Gamma G$ are related to symmetric products: all computations are based on **Macdonald's theorem.**

Theorem (Macdonald '1962)

Let $P_X(t) = b_0 + b_1 t + b_2 t^2 + \cdots$. *Then,* $P_{Sym^n X}(t)$ *is the coefficient of* y^n *in the rational function:*

$$\frac{(1+ty)^{b_1}(1+t^3y)^{b_3}\cdots}{(1-y)^{b_0}(1-t^2y)^{b_2}\cdots}$$

Corollary (? F '21)

$$P_{Sym^n X}(-t) = \frac{1}{n}\sum_{k=1}^{n} P_X(-t^k)\, P_{Sym^{n-k} X}(-t)$$

Proof: these infinite products are *plethystic exponentials*, which are related to Polya's famous cycle index of S_n, which satisfy a recurrence relation of the above form.

Some references (please ask!)

Thank you!
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