Minimal surfaces in hyperbolic manifolds

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2. Minimal surface entropy (Calegari-Marques-Neves);
3. Average area ratio (Gromov);
4. Some recent results relating both quantities and a positive answer to a conjecture of Gromov (with Lowe).

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Margulis showed in '69 that the limit below exists (called topological entropy)

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Manning showed in '79 that it is identical to the volume entropy

$$h_{vol}(g) := \lim_{R\to\infty} \frac{\ln \text{vol}_g(\tilde{B}_R(x))}{R}, \quad \text{where } x \in \tilde{M} = \text{universal cover of M}.$$

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A particular case of Besson–Courtois–Gallot '95 implies:

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Conjecture (Agol, Storm, and Thurston)

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Similar characterization for minimal surfaces of M^3?

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- unique in their homotopy class, π_1-injective, or flow trajectories.

So we need to restrict to a suitable subclass of minimal surfaces of (M^3, g)...

$M^3 = \mathbb{H}^3/\Gamma$ closed manifold.

- Π :=homotopy class of π_1-injective surfaces in M
- Given $\Sigma \in \Pi$, Σ lifts to a disc $\Omega \subset \mathbb{H}^3$

- (Bonahon, Thurston) $\Lambda(\Pi) := \partial_\infty \mathbb{H}^3 \cap \bar{\Omega}$ is either a Jordan curve or $\partial_\infty \mathbb{H}^3 \simeq S^2$.

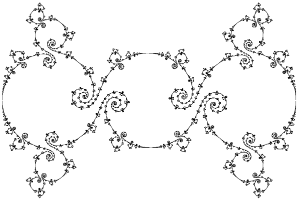

FIGURE 10. A pattern of identifications of a circle, here represented as the equator, whose quotient space is topologically a sphere. This defines, topologically, a sphere-filling curve.

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Why $\mathcal{S}_\varepsilon(M)$?

- Khan-Markovic showed that $\mathcal{S}_\varepsilon(M)$ has roughly h^{2h} elements with genus $\leq h$ if $\varepsilon > 0$. In particular it is non-empty.

- (Uhleneck '83, Seppi '16): If $\varepsilon \ll 1$, minimal surfaces in $\Pi \in S_\varepsilon$ with respect to g_{hyp} are unique in their homotopy class.

Minimal surface entropy.

- $\mathcal{S}_\varepsilon(M) := \{\text{homotopy class } \Pi \text{ where } dim_H(\Lambda(\Pi)) \leq 1 + \varepsilon\}$
- Schoen-Yau shows the existence of $\Sigma_g(\Pi) \in \Pi$ an area-minimizing surface

$$\text{area}_g(\Pi) := \text{area}(\Sigma_g(\Pi)) = \inf\{\text{area}_g(\Sigma)) : \Sigma \in \Pi\}$$

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$$E(g) := \lim_{\varepsilon \to 0} \limsup_{L \to \infty} \frac{\ln \#\{\text{area}_g(\Pi) \leq 4\pi(L-1) : \Pi \in S_\varepsilon(M)\}}{L \ln L}.$$

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Theorem (Calegari–Marques–N.)

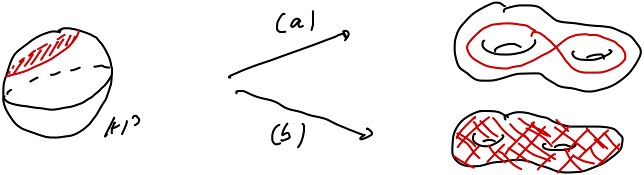
If g has sectional curvature ≤ -1 then

$$E(g) \geq 2 = E(g_{hyp}) \text{ with equality } \iff g \overset{\text{Isometric}}{=} g_{hyp}.$$

- To prove the $\iff$ statement we need to use a classification theorem in homogenous dynamics.

Ratner-Shah classification.

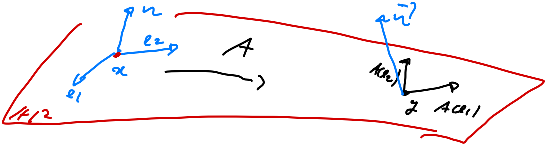
(Ratner, Shah, '91) Totally geodesic discs in $M^3 = \mathbb{H}^3 \setminus \Gamma$ either
(a) cover a closed surface or (b) are dense in the Grassmanian of 2-planes of M.



Set $F(M) =$ frame bundle of M^3. There is an action of $PSL(2, \mathbb{R}) = \text{Isom}^+(\mathbb{H}^2)$ on $F(M)$.

· $A \in PSL(2,\mathbb{R})$

· $p = (x, \{e_1, e_2, n\}) \in F(M)$



A closed surface in M induces a probability measure on $F(M)$.

(Ratner, '91) Let μ be an ergodic $PSL(2, \mathbb{R})$-invariant measure on $F(M)$. Then either μ is the homogeneous measure on $F(M)$ or is supported on some closed totally geodesic surface.

Minimal surface entropy.

The general principle started by Calegari–Marques–Neves is to use Ratner's classification theorem to understand minimal surfaces in negatively curved manifolds.

Theorem (Lowe–N.)

Assume $R(g) \geq -6$. *Then* $E(g) \leq 2 = E(g_{hyp})$ *and equality holds if and only if* g *is isometric to* g_{hyp}.

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The general principle started by Calegari–Marques–Neves is to use Ratner's classification theorem to understand minimal surfaces in negatively curved manifolds.

Theorem (Lowe–N.)

g_0

Assume $R(g) \geq -6$. Then $E(g) \leq 2 = E(g_{hyp})$ and equality holds if and only if g is isometric to g_{hyp}.

1: Run (unnormalized) Ricci flow $(g_t)_{t\geq 0}$ where $g_t \to g_0$ and $R(g_t) \geq -6$ all $t \geq 0$

2: If $E(g) > E(g_0) \Rightarrow \exists \delta > 0$, $\Pi_m \in S_{\epsilon}(M)$ so that
$\text{area}_g(\Pi_m) \leq (1-\delta)\, \text{area}_{g_0}(\Pi_m)$ all $m \in \mathbb{N}$.

3: the measures $\mu_{m,t} := \frac{1}{|\text{area}_{g_0}(\Pi_m)|} \int_{\Sigma_{g_t}(\Pi_m)} dA_{g_t} \xrightarrow[m\to+\infty]{} \mu_t$ measure on $F(M)$

4: $\mu_t(1) = \lim_{m\to+\infty} \text{area}_{g_t}(\Pi_m) / \text{area}_{g_0}(\Pi_m) \Rightarrow \mu_0(1) \leq 1-\delta$

ODE comparison $(+)$ $R(g_t) \geq -6 \Rightarrow \boxed{\mu_t(1) \leq 1 - \delta e^{-t}}$

5: $g_t = g_0 + h e^{-t} + o(e^{-t})$ and $\mu_t(1) = 1 + \mu_{-\infty}(h) e^{-t} + o(e^{-t})$

6: $\mu_{-\infty}$ is $PSL(2,\mathbb{R})$-invariant $\Rightarrow$ Ratner's classification shows
$\mu_{-\infty}(h) = 0 \Rightarrow \mu_t(1) = 1 + o(e^{-t})$. CONTRADICTION!

Average area ratio

$M^3 = \mathbb{H}^3/\Gamma$ with metric g. Gromov defined in '91 the average area ratio as

$$A(g/g_{hyp}) = \frac{1}{\mathrm{vol}_{hyp}(Gr_2(M))} \int_{Gr_2(M)} \frac{|\tau|_g}{|\tau|_{hyp}} dV_{hyp}(\tau).$$

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Motivation: In the same way that the unit tangent bundle of M has a foliation where each leaf is a geodesic, $Gr_2(M)$ admits a foliation $\mathcal{L}$ where each leaf corresponds to a totally geodesic plane of M.

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- He conjectured that if $R(g) \geq -6$ then $A(g/g_{hyp}) \geq 1$.

Average area ratio

[illegible]: $I(g/g_{hyp}) = \frac{1}{|T^1M|_{hyp}} \int_{T^1M} \sqrt{g_x(v,v)}\, dVol(x,v)$

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- Schoen conjectured that if $R(g) \geq -6$ then $\mathrm{vol}_g(M)/\mathrm{vol}_{hyp}(M) \geq 1$ and this was proven by Perelman using Ricci flow.

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Theorem (Lowe–N.)

For every metric g we have $E(g)A(g/g_{hyp}) \geq 2$ and equality if and only if $g = g_{hyp}$.

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For instance, if $g \geq g_{hyp}$ *and* $R(g) \geq -6$ *then* $g = g_{hyp}$

Proof: If $R(g) \geq -6$ then $E(g) \leq 2$ and so $A(g/g_{hyp}) \geq 2/E(g) \geq 1$.

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Proof of Theorem:

1:

2:

3:

4:

Thank you!