Homotopy type of equivariant symplectomorphisms of rational ruled surfaces.

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Hirzerbruch surfaces

- We define the symplectic form ω_λ on $S^2 \times S^2$ as $\omega_\lambda := \lambda\sigma \oplus \sigma$ where σ is the standard symplectic form on S^2.Similarly, for $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, we define an analogous form ω_λ which gives area 1 to the fibers and area $\lambda - 1$ to symplectic sections of self-intersection -1.
- Consider the complex submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$ defined by

$$W_m := \{([x_1, x_2], [y_1, y_2, y_3]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \mid {x_1}^m y_2 - x_2^m y_1 = 0\}$$

- Each W_m is equipped with a integrable almost complex structure J_m
- When m is even, we can endow $\mathbb{C}P^1 \times \mathbb{C}P^2$ with the symplectic form $(\lambda - \frac{m}{2})\omega_1 \oplus \omega_2$ and restricting this symplectic form to W_m makes it a symplectic manifold. We can analogously define the form $(\lambda - \frac{m+1}{2})\omega_1 \oplus \omega_2$ when m is odd.
- W_m endowed with these forms are symplectomorphic to $(S^2 \times S^2, \omega_\lambda)$ when m is even or to $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \omega_\lambda)$ when m is odd.

- W_m admits a holomorphic toric action $\mathbb{T}^2_m$ given by

$$(u,v)\cdot([x_1,x_2],[y_1,y_2,y_3]) = ([ux_1,x_2],[u^m y_1,y_2,vy_3])$$

- On $S^2\times S^2$, when $m \leqslant \lceil 2\lambda\rceil - 1$ these T_m actions are hamiltonian and Kähler with respect to the the complex structure J_m and symplectic form ω_λ
- These integers m characterise all $\mathbb{T}^2$ hamiltonian actions on $S^2\times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.
- Every hamiltonian circle action on $S^2\times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ extends to $\mathbb{T}^2$ actions.
- Every triple of numbers $(a,b;m)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_{\geqslant 0}$ corresponds to an S^1 action (on either $S^2\times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$) where (a,b) corresponds to the embedding

$$S^1 \hookrightarrow \mathbb{T}^2_m$$
$$t \mapsto (t^a,t^b)$$

and m corresponds to the torus $\mathbb{T}^2_m$ the S^1 sits inside. We are interested in only effective S^1 actions i.e gcd(a,b)=1.

Topology of symplectomorphism group

Baby example (Smale): The symplectomorphism group of (S^2, ω) is homotopy equivalent to SO(3).

Idea of proof:

$$Stab(j_0) \to \mathrm{Diff}^+ \to \mathcal{J}_\omega$$
$$\mathrm{Symp}(S^2, \omega) \to \mathrm{Diff}^+ \to \Omega^{\mathrm{vol}}_{[\omega]}$$

where $\Omega^{\mathrm{vol}}_{[\omega]}$ is the space of all volume forms in the cohomology class of ω.

- Let $\mathcal{J}_{\omega_\lambda}$ denote the space of ω_λ compatible almost complex structures. Then

Theorem (Abreu-McDuff)

The space $\mathcal{J}_{\omega_\lambda}$ admits a finite decomposition into disjoint Fréchet manifolds of finite codimensions

$$\mathcal{J}_{\omega_\lambda} = U_0 \sqcup U_2 \ldots \sqcup U_{2n}$$

where $2n = \lceil 2\lambda \rceil - 1$ and where

$$U_{2k} := \{J \in \mathcal{J}_{\omega_\lambda} \mid \text{the class } (B - kF) \text{ is represented by a } J\text{-sphere.}\}$$

- The integrable a.c.s J_{2k} on W_{2k} belong to the strata U_{2k}.
- Hence associated to each strata U_{2k} there is a unique Hamiltonian action of a torus $\mathbb{T}^2_{2k}$. The Hamiltonian $\mathbb{T}^2$ actions that $(S^2 \times S^2, \omega_\lambda)$ admits are in one to one correspondance with the strata.

Let $\mathcal{J}_{\omega_\lambda}^{S^1}$ be the space of S^1 invariant compatible almost complex structures and $\mathrm{Symp}_h^{S^1}(S^2 \times S^2)$ be the space of equivariant symplectomorphisms that act trivially on homology. We would like to consider the fibration arising from the natural action

$$\begin{aligned} \mathrm{Symp}_h^{S^1}(S^2 \times S^2) &\longrightarrow \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{2k} \\ \psi &\longmapsto \psi^* J_{2k} \end{aligned}$$

- Not a Serre fibration!

Instead we consider

$$\mathrm{Stab}^{S^1}(\overline{D}) \longrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}_{B-kF}^{S^1} \simeq \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_k$$

Theorem 1 (Ch.)

For $(a, b; m) \neq (0, \pm 1; m)$ and for each non-empty stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_k$,

$$\mathrm{Symp}^{S^1}_h(S^2 \times S^2)/\mathbb{T}^2_k \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_k$$

and for $(a, b; m) = (0, \pm 1; m)$,

$$\mathrm{Symp}^{S^1}_h(S^2 \times S^2)/(S^1 \times SO(3)) \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$$

Question

Given a $S^1(a, b, m)$ action on $S^2 \times S^2$, which strata does the space $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersect ?

Suppose that for certain $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects only one strata, say U_m, then

$$\{*\} \simeq \mathcal{J}^{S^1}_{\omega_\lambda} = \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m \simeq \mathrm{Symp}^{S^1}_h(S^2 \times S^2)/\mathit{stabiliser}$$

Equivalent question

Given a $S^1(a,b,m)$ action on $S^2 \times S^2$, which tori T_k does this circle action extend to?

- To answer this question we use Karshon's classification of S^1 actions on 4-manifolds
- An effective Hamiltonian S^1-action on a symplectic 4-manifold is characterized, up to equivariant symplectomorphisms, by a labeled graph which encodes the essential information about the invariant spheres and the cohomology class of the symplectic form.

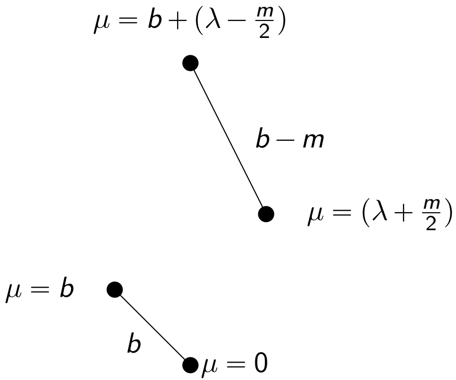
Theorem (Karshon)

The graph determines the Hamiltonian circle action up to equivariant symplectomorphisms.

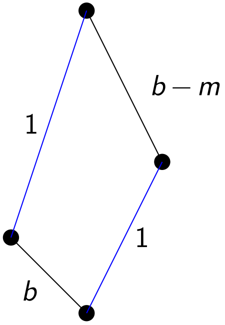
Karshon Graphs

- Each component of the fixed point set corresponds to a unique vertex.
- Each vertex is labeled by the value of the moment map on the corresponding fixed point component. If an extremal vertex corresponds to a symplectic surface S, two additional labels are attached: the genus of that surface, and its symplectic area.
- Two vertices are connected by an edge if and only if the corresponding isolated fixed points are connected by a $\mathbb{Z}_k$-sphere $(k \geqslant 2)$.
- Each edge is labelled by the isotropy weight $k \geqslant 2$ of the corresponding $\mathbb{Z}_k$ sphere.
  In particular, the classification tells that that it is not important to keep track of the spheres with trivial isotropy and hence, these spheres do not appear in the labelled graphs.

- The only action for which $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects 2 strata are for the circles $(\pm 1, b; m)$ ($b \neq \{\pm m, 0\}$). For all other cases the space $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects only one strata. As $\mathcal{J}^{S^1}_{\omega_\lambda}$ is contractible we can read the homotopy type of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2)$.
- Consider the graph for the circle $(1, b; m)$ and $2\lambda > |2b - m|$ and assume b $>$ m

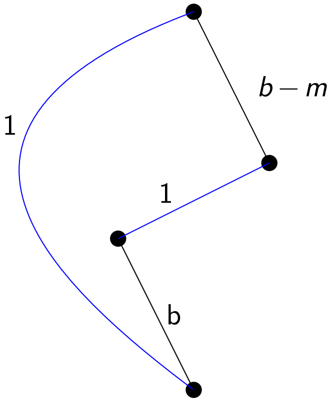


- The above graph corresponds to the following 2 extended graphs.



This corresponds to the intersection with the strata U_m.

- Or it could correspond to the following graph



- This corresponds to the intersection with the strata $U_{|2b-m|}$.

Homotopy type of Symplectomorphism group

$(a, b; m)$	λ	Number of strata $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects	Homotopy type of $Symp^{S^1}_h(S^2 \times S^2)$
$(0, \pm 1; m) \quad m \neq 0$	$\lambda > 1$	1	$S^1 \times SO(3)$
$(0, \pm 1; 0)$ or	$\lambda = 1$	1	$S^1 \times SO(3)$
$(\pm 1, 0; 0)$	$\lambda > 1$	1	$S^1 \times SO(3)$
$(\pm 1, \pm 1, 0)$	$\lambda = 1$	1	$\mathbb{T}^2$
$(\pm 1, 0; m)$ $m \neq 0$	$\lambda > 1$	1	$\mathbb{T}^2$
$(1, \pm m; m)$ $m \neq 0$	$\lambda > 1$	1	$\mathbb{T}^2$
$(1, b; m)$	$\lvert 2b - m \rvert \geqslant 2\lambda \geqslant 1$	1	$\mathbb{T}^2$
$b \neq \{m, 0\}$	$2\lambda > \lvert 2b - m \rvert \geqslant 0$	2	?
$(-1, b; m), b \neq \{-m, 0\}$	$\lvert 2b + m \rvert \geqslant 2\lambda \geqslant 1$	1	$\mathbb{T}^2$
	$2\lambda > \lvert 2b + m \rvert \geqslant 0$	2	?
All other values of $(a, b; m)$	$\forall \lambda$	1	$\mathbb{T}^2$

The case (1,b,m) $b \neq \{\pm m, 0\}$

As mentioned above we are only interested in the case when $2\lambda > |2b - m|$. We would like to first understand the homology of the group $Symp_h^{S^1}(S^2 \times S^2)$ for this specific S^1 action.

- From the above analysis we have the following 2 fibrations
- $\mathbb{T}^2_m \longrightarrow Symp_h^{S^1}(S^2 \times S^2) \longrightarrow Symp_h^{S^1}(S^2 \times S^2)/\mathbb{T}^2_m \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$
- $\mathbb{T}^2_{|2b-m|} \longrightarrow Symp_h^{S^1}(S^2 \times S^2) \longrightarrow Symp_h^{S^1}(S^2 \times S^2)/\mathbb{T}^2_{|2b-m|} \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{|2b-m|}$

Theorem

The inclusion $i : \mathbb{T}^2_m, \mathbb{T}^2_{|2b-m|} \hookrightarrow Symp_h^{S^1}(S^2 \times S^2)$ induces a map which is surjective in cohomology (with coefficients in any field k).

Hence by Leray Hirsch theorem we have

- $H^*(Symp_h^{S^1}(S^2 \times S^2)) \cong H^*(V_m) \otimes H^*(\mathbb{T}^2)$
- $H^*(Symp_h^{S^1}(S^2 \times S^2)) \cong H^*(V_{|2b-m|}) \otimes H^*(\mathbb{T}^2)$

where $V_m := \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_m$ and $V_{|2b-m|} := \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{|2b-m|}$.

Assume $b > m \implies 2b - m > m$, then by "Alexander-Eells duality" and the long exact sequence of pairs we have the following result

Theorem(Eells)

$H^p(V_{|2b-m|}) \cong H^{p+c-1}(V_m)$ where c is the real codimension of $V_{|2b-m|}$ in $\mathcal{J}_{\omega_\lambda}^{S^1}$.

Calculating the codimension

- As $b > m$, we would like to calculate the codimension of the space $V_{|2b-m|} = \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{|2b-m|}$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$.

Question

Why is $V_{|2b-m|}$ a "submanifold" ?

Main issue:

$$\begin{array}{ccc} & & \mathcal{M}(B - \frac{|2b-m|}{2}F, \mathcal{J}_{\omega_\lambda}) \\ & & \downarrow \pi \\ \mathcal{J}^{S^1}_{\omega_\lambda} & \xrightarrow{\quad i \quad} & \mathcal{J}_{\omega_\lambda} \end{array}$$

The maps i and π are not transverse.

Solution:

Redo construction of universal moduli space under presence of group action

$$\mathcal{M}^{S^1}(B-kF,\mathcal{J}^{S^1}_{\omega_\lambda,l}) := \{(u,J) \mid \text{u is equivariant, somewhere injective, J- holomorphic and represents the class } B-kF\}$$

where $u : S^2 \to S^2 \times S^2$ is an equivariant map for the fixed action $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ is the space of equivariant C^l almost complex structures.

We have the following projection

$$\begin{array}{c} \mathcal{M}^{S^1}(B - \frac{|2b-m|}{2}F, \mathcal{J}^{S^1}_{\omega_\lambda, l}) \\ \downarrow \pi \\ \mathcal{J}^{S^1}_{\omega_\lambda, l} \end{array}$$

whose image is $V_{|2b-m|}$. We can then argue that the image of π is a smooth banach submanifold

Let $I^{S^1}_{\omega, l}$ denote the space of S^1 invariant integrable ω_λ compatible complex structures on $S^2 \times S^2$. i.e

$$\begin{array}{ccc} & & \mathcal{M}^{S^1}(B - \frac{|2b-m|}{2}F, \mathcal{J}^{S^1}_{\omega_\lambda, l}) \\ & & \downarrow \pi \\ I^{S^1}_{\omega, l} & \xrightarrow{i} & \mathcal{J}^{S^1}_{\omega_\lambda, l} \end{array}$$

Then we have the following theorem

Theorem (Ch.)

For any of the Hirzerbruch surfaces, and any hamiltonian S^1 action

- $i \pitchfork \pi$
- the infinitesimal complement (i.e the fibre of the normal bundle) of $U_{|2b-m|,l} \cap \mathcal{J}_{\omega_\lambda}^{S^1}$ at $J_m \in I_{\omega,l}^{S^1}$ can be identified with $H^{0,1}(S^2 \times S^2, T^{1.0}_{J_m}(S^2 \times S^2))^{S^1}$.

where $H^{0,1}(S^2 \times S^2, T^{1.0}_{J_m}(S^2 \times S^2))^{S^1}$ is the space of S^1 invariant $(0,1)$ forms with values in the holomorphic tangent bundle of $S^2 \times S^2$ with respect to the complex structure J_m.

- Hence to calculate the codimension we only need to calculate the dimension of $H^{0,1}(S^2 \times S^2, T^{1.0}_{J_m}(S^2 \times S^2))^{S^1}$.

- By a result of Kodaira we have, $H^{0,1}(S^2 \times S^2, T^{1,0}_{J_m}(S^2 \times S^2)) \cong \mathbb{C}^{m-1}$.
- $S^1(a,b;m) \subset \mathbb{T}^2_m = S^1 \times S^1 \subset S^1 \times SO(3)$. Let $J_m \in U_m$ then,

Theorem (Abreu-Granja-Kitchloo)

$S^1 \times SO(3)$ acts on $H^{0,1}(S^2 \times S^2, T^{1,0}_{J_m}(S^2 \times S^2))$ via the representation $\text{Det} \otimes \text{Sym}^{m-2}(\mathbb{C}^2)$.

where Det is the standard action of $S^1 = U(1)$ on $\mathbb{C}$, and where $\text{Sym}^{m-2}(\mathbb{C}^2)$ is the representation of SO(3) induced by the action of $SU(2)$ on the space of all homogeneous polynomial of degree $(m-2)$ on $\mathbb{C}^2$.

Theorem

For the circle action $S^1(1, b, m)$ $b > m$ and $2\lambda > |2b - m|$, the codimension of the strata $U_{|2b-m|}$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ is 2.

Hence we can now calculate the rational homology of the centraliser.

Theorem (Ch.)

For $S^1(\pm 1, b) \subset \mathbb{T}^2_m$ when ($b \neq 0$ or $\pm m$), and $2\lambda > |2b - m|$ then we have

$$H^p(\mathrm{Symp}^{S^1}_h(S^2 \times S^2), \mathbb{R}) = \begin{cases} \mathbb{R}^4 & p \geqslant 2 \\ \mathbb{R}^3 & p = 1 \\ \mathbb{R} & p = 0 \end{cases}$$

$$\begin{array}{ccc} S^1 & \xrightarrow{(1,b')} & T^2_{|m-2b|} \\ {\scriptstyle (1,b)}\downarrow & & \downarrow \\ T^2_m & \longrightarrow & P \end{array}$$

$$\begin{array}{cccc} H_*(S^1;k) & \xrightarrow{(1,b')} & H_*(T^2_{|m-2b|};k) & \\ {\scriptstyle (1,b)}\downarrow & & \downarrow & \searrow{\scriptstyle i_{|m-2b|}} \\ H_*(T^2_m;k) & \longrightarrow & H_*(P;k) & \\ & \searrow{\scriptstyle i_m} & & \overset{\Upsilon}{\dashrightarrow} \ H_*(\mathrm{Symp}^{S^1}_h(S^2\times S^2,\omega_\lambda);k) \end{array}$$

We can use the computation of the ranks of homology of $\mathrm{Symp}^{S^1}_h(S^2\times S^2)$ for any field k, to conclude that the map

$$\Upsilon : H_*(P;k) \to H_*(\mathrm{Symp}^{S^1}_h(S^2\times S^2,\omega_\lambda);k)$$

is an isomorphism. Hence $\mathrm{Symp}^{S^1}_h(S^2\times S^2) \simeq \Omega S^3 \times S^1 \times S^1 \times S^1$

Homotopy type of Symplectomorphism group

$(a, b; m)$	λ	Number of strata $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects	Homotopy type of $Symp^{S^1}(S^2 \times S^2)$
$(0, \pm 1; m) \quad m \neq 0$	$\lambda > 1$	1	$S^1 \times SO(3)$
$(0, \pm 1; 0)$ or $(\pm 1, 0; 0)$	$\lambda = 1$ $\lambda > 1$	1 1	$S^1 \times SO(3)$ $S^1 \times SO(3)$
$(\pm 1, \pm 1, 0)$	$\lambda = 1$	1	$\mathbb{T}^2 \times \mathbb{Z}_2$
$(\pm 1, 0; m)$ $m \neq 0$	$\lambda > 1$	1	$\mathbb{T}^2$
$(1, \pm m; m)$ $m \neq 0$	$\lambda > 1$	1	$\mathbb{T}^2$
$(1, b; m)$ $b \neq \{m, 0\}$	$\lvert 2b - m \rvert \geqslant 2\lambda \geqslant 1$ $2\lambda > \lvert 2b - m \rvert \geqslant 0$	1 2	$\mathbb{T}^2$ $\Omega S^3 \times S^1 \times S^1 \times S^1$
$(-1, b; m), b \neq \{-m, 0\}$	$\lvert 2b + m \rvert \geqslant 2\lambda \geqslant 1$	1	$\mathbb{T}^2$
	$2\lambda > \lvert 2b + m \rvert \geqslant 0$	2	$\Omega S^3 \times S^1 \times S^1 \times S^1$
All other values of $(a, b; m)$	$\forall \lambda$	1	$\mathbb{T}^2$

Finite cyclic group actions

- Ideally we would like to have a uniform approach to understanding the homotopy type of equivariant symplectomorphisms that works for all compact abelian group actions. This is the primary reason we use holomorphic curve techniques rather than moment map techniques in the S^1 case.
- Due to work of R.Chiang and L.Kessler we know that every hamiltonian $\mathbb{Z}_n$ action on $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ extends to hamiltonian S^1 actions. Hence as before we now have a triple of numbers $(a, b; r)$ arising from each hamiltonian $\mathbb{Z}_n$ action on these manifolds.
- Most of the techniques we use in the S^1 case go through mutatis mutandis in the $\mathbb{Z}_n$ case as well and we again have when $\lambda > 1$ and $r \neq 0$,

$$\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2)/\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_k) \simeq \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k$$

(Whenever $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k$ is non-empty).

As before we would like to now answer the question

Question

Fix a hamiltonian $\mathbb{Z}_n$ action. Which strata does $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}$ intersect?

In the S^1 case, we used momentum map techniques to establish an equivalent formulation of the above question in terms of extensions of group actions. Hence, these proofs do not generalise to the $\mathbb{Z}_n$ case. We the classification of complex structures to resolve this issue.

Theorem

Fix a $\mathbb{Z}_n(a, b, k)$ action on $(S^2 \times S^2, \omega_\lambda)$, then
$\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)/\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_k) \simeq I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k \simeq \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k$.

Theorem

For any $\mathbb{Z}_n(a, b; r)$ action, the space $I_{\omega_\lambda}^{\mathbb{Z}_n}$ is contractible.

Question

Fix a hamiltonian $\mathbb{Z}_n$ action. Which strata does $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersect?

Theorem(Ch.)

Let $\mathbb{Z}_n(a, b, r)$ be a symplectic action on $(S^2 \times S^2, \omega_\lambda)$. Then the space of $\mathbb{Z}_n$-equivariant complex structures $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects the strata $U_{r'}$ iff $\mathbb{Z}_n$ extends to torus action $\mathbb{T}^2_{r'}$.

- In the S^1 case, we used Karshon graphs to determine when a given $S^1(a, b; r)$ action is symplectomorphic a different $S^1(a', b', r')$ action. Unfortunately, we do not have such a classification for $\mathbb{Z}_n$ actions on $S^2 \times S^2$ (and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$) up to equivariant symplectomorphisms.
- However due to the work of Weimin Chen and D.Wilczynski, there is a classification of $\mathbb{Z}_n$ action up to equivariant diffeomorphisms.

There are 6 standard diffeomorphisms $c_1, \cdots, c_6$ which give an equivariant diffeomorphism between the Hirzerbruch surfaces W_r and $W_{r'}$, with $\mathbb{Z}_n$ actions $(a, b; r)$ and $(a', b'; r')$ respectively provided the triples satisfy the following conditions:

Type c_1: When $a' = -a, b' = -b$ and $r' = r$

Type c_2: When $a' = -a, b' = b + ra$ and $r' = r$

Type c_3: When $a' = a, b' = -b$ and $r' \equiv -r$ (mod 2n)

Type c_4: When $r' = r = 0$, $a' = b$, and $b' = a$

Type c_5: When $a' = a$, $b' = b$, and $r' \equiv r$ (mod 2n).

Type c_6: When $a' = a$, $b' = b$, and $r'a' \equiv 2b - ra$ (mod 2n)

Theorem (W.Chen)

Two $\mathbb{Z}_n$-Hirzebruch surfaces are orientation-preserving equivariantly diffeomorphic iff there is a composition of standard equivariant diffeomorphism between them.

We use the above result to derive a partial result about $\text{Symp}^{\mathbb{Z}_n}(S^2 \times S^2)$ for specific $\mathbb{Z}_n$ actions.

Theorem (Ch.)

Consider a $\mathbb{Z}_n(a,b;r)$ action on $S^2 \times S^2$ which satisfies the following numberical conditions $n > 2\lambda > r > 1$, $\gcd(a,b) = 1$ and $r \neq 0$. Then we have the following cases:

1. If $(a,b) \in \{(\pm 1, 0), (\pm 1, \pm r)\}$ then $\text{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq \mathbb{T}^2$.
2. if $(a,b) = (0, \pm 1)$ then, $\text{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq S^1 \times SO(3)$.
3. Further if
   - Either $a = 1$, $b \neq \{0, r\}$ and $2\lambda > |r - 2b| > 1$ or
   - $a = -1$, $b \neq \{0, -r\}$ and $2\lambda > |r + 2b| > 1$

   then :

   In the first case $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects exactly 2 strata U_r and $U_{|r-2b|}$ and in the second it intersects the 2 strata U_r and $U_{|r+2b|}$.

Codimension calculation

WLOG if we assume that $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r$ is the strata of positive codimension in $I_{\omega_\lambda}^{\mathbb{Z}_n}$. Then the complex codimension of $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r$ in $I_{\omega_\lambda}^{\mathbb{Z}_n}$ is given by the number of $k \in \{1, \cdots, r-1\}$ such that $k \equiv b \pmod{n}$.

We can also work out the cohomology $H^*(\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda), \mathbb{R})$ in case 2 of the above theorem.

Theorem (Ch.)

Consider the following $\mathbb{Z}_n$ actions on $S^2 \times S^2$.

- (i) $a = 1$, $b \neq \{0, r\}$, $n > 2\lambda > 1$ and $2\lambda > |2b - r| > 1$; or
- (ii) $a = -1$, $b \neq \{0, -r\}$, $n > 2\lambda > 1$ and $2\lambda > |2b + r| > 1$.

Then, $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects 2 strata. The codimension of the positive codimension strata in $I_{\omega_\lambda}^{\mathbb{Z}_n}$ is 2 and hence,

$$H^p(\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda), \mathbb{R}) = \begin{cases} \mathbb{R}^4 & p \geqslant 2 \\ \mathbb{R}^3 & p = 1 \\ \mathbb{R} & p = 0 \end{cases}$$

As before we can then show that
$\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq \Omega S^3 \times S^1 \times S^1 \times S^1$.

Also, using separate techniques one can also obtain the following result.

Theorem

Under the numerical condition $\gcd(a, n) \neq 1$, the Hamiltonian finite cyclic group $\mathbb{Z}_n(a, b; r) \subset S^1(a, b; r)$ action can only extend to circles $S^1(a', b', r')$ with $r' = r$.

and as a consequence we have,

Theorem(Ch.)

For Hamiltonian finite cyclic groups $\mathbb{Z}_n(a, b; r) \subset S^1(a, b; r)$ actions such that $\gcd(a, n) \neq 1$, $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq \mathbb{T}^2_r$

Future directions

- Given two $\mathbb{Z}_n$ actions $(a, b; r)$ and (a', b', r') on $S^2 \times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ when are they equivariantly symplectomorphic to one another? A classification in this flavour would help us understand the homotopy type of $\mathbb{Z}_n$ equivariant symplectomorphisms when $gcd(a, n) = 1$.
- Proof for the S^1 case using only "soft" techniques. This would give us the homotopy type of S^1 equivariant symplectomorphism for all 4-manifolds with hamiltonian S^1 actions.
- Understand the homotopy type of space of equivariant embeddings. In particular given a compact group G acting symplectically on $(S^2 \times S^2, \omega_\lambda)$ (and analogously for $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$). Let S be a connected component of the fixed point set. Then is the space of equivariant symplectic embeddings $i : B^4(r) \hookrightarrow S^2 \times S^2$ such that $i(0) \in S$ connected?
- What happens for non-abelian groups?