STATISTICAL MECHANICS FOR COMPLEX SYSTEMS

Constantino Tsallis

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Complexity Science Hub Vienna, Austria

Lisboa, Outubro 2021
PILLARS OF CONTEMPORARY THEORETICAL PHYSICS

- Newtonian mechanics

- Maxwell electromagnetism

- Einstein special relativity
  (when the velocities approach that of light)

- Quantum mechanics
  (when masses are very small)

- Boltzmann-Gibbs statistical mechanics (→ Thermodynamics)

Previous four + Theory of probabilities
<table>
<thead>
<tr>
<th>QUANTUM MECHANICS TEXTBOOKS</th>
<th>BOLTZMANN-GIBBS STATISTICAL MECHANICS TEXTBOOKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>square well</td>
<td>YES</td>
</tr>
<tr>
<td>harmonic oscillator</td>
<td>YES</td>
</tr>
<tr>
<td>rigid rotator</td>
<td>YES</td>
</tr>
<tr>
<td>spin 1/2 in magnetic field</td>
<td>YES</td>
</tr>
<tr>
<td>nonionized Hydrogen atom</td>
<td>YES</td>
</tr>
</tbody>
</table>

NO! WHY?
The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.
### Entropic Functionals

<table>
<thead>
<tr>
<th>Entropy $S_q$ ($q$ real)</th>
<th>$p_i = \frac{1}{W}$ (∀$i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BG$ entropy ($q = 1$)</td>
<td>$k \ln W$</td>
</tr>
<tr>
<td></td>
<td>$\forall p_i (0 \leq p_i \leq 1)$</td>
</tr>
<tr>
<td></td>
<td>$-k \sum_{i=1}^{W} p_i \ln p_i$</td>
</tr>
<tr>
<td>$\frac{W^{1-q} - 1}{k(1-q)}$</td>
<td>$\frac{1 - \sum_{i=1}^{W} p_i^q}{kq - 1}$</td>
</tr>
</tbody>
</table>

- Concave
- Extensive
- Lesche-stable
- Finite entropy production per unit time
- Pesin-like identity (with largest entropy production)
- Composable (unique trace form; Enciso-Tempesta)
- Topsoe-factorizable (unique)
- Amari-Ohara-Matsuzoe conformally invariant geometry (unique)
- Biro-Barnafoldi-Van universal thermostat independence (unique)
- Nonadditive (if $q \neq 1$)

Possible generalization of Boltzmann-Gibbs statistical mechanics

**DEFINITIONS**: $q$–logarithm: 
\[ \ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \quad (x > 0; \ \ln_1 x = \ln x) \]

$q$–exponential: 
\[ e_q^x \equiv \left[ 1 + (1-q) x \right]^{\frac{1}{1-q}} \quad (e_1^x = e^x) \]

Hence, the entropies can be rewritten:

<table>
<thead>
<tr>
<th></th>
<th>equal probabilities</th>
<th>generic probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BG entropy</strong></td>
<td>$k \ \ln W$</td>
<td>$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$</td>
</tr>
<tr>
<td>$(q = 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>entropy $S_q$</strong></td>
<td>$k \ \ln_q W$</td>
<td>$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$</td>
</tr>
<tr>
<td>$(q \in \mathbb{R})$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

An entropy is **additive** if, for any two probabilistically independent systems $A$ and $B$,

$$S(A + B) = S(A) + S(B)$$

Therefore, since

$$\frac{S_q(A + B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q) \frac{S_q(A)}{k} \frac{S_q(B)}{k}$$

$S_{BG}$ and $S_q^{\text{Renyi}}$ ($\forall q$) are additive, and $S_q$ ($\forall q \neq 1$) is nonadditive.

---

**EXTENSIVITY:**

Consider a system $\Sigma \equiv A_1 + A_2 + \ldots + A_N$ made of $N$ (not necessarily independent) identical elements or subsystems $A_1$ and $A_2$, ..., $A_N$.

An entropy is **extensive** if

$$0 < \lim_{N \to \infty} \frac{S(N)}{N} < \infty \quad \text{i.e.,} \quad S(N) \propto N \quad (N \to \infty)$$
EXTENSIVITY OF THE ENTROPY \((N \to \infty)\)

\(W \equiv \) total number of possibilities with nonzero probability, assumed to be equally probable

If \(W(N) \sim \mu^N \) \((\mu > 1)\)

\[\Rightarrow S_{BG}(N) = k_B \ln W(N) \propto N \quad \text{OK!}\]

If \(W(N) \sim N^\rho \) \((\rho > 0)\)

\[\Rightarrow S_q(N) = k_B \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)} \]

\[\Rightarrow S_{q=1-1/\rho}(N) \propto N \quad \text{OK!}\]

If \(W(N) \sim \nu^{N^\gamma} \) \((\nu > 1; 0 < \gamma < 1)\)

\[\Rightarrow S_\delta(N) = k_B \left[ \ln W(N) \right]^\delta \propto N^{\gamma \delta} \]

\[\Rightarrow S_{\delta=1/\gamma}(N) \propto N \quad \text{OK!}\]

**IMPORTANT:** \[\mu^N >> \nu^{N^\gamma} >> N^\rho \quad \text{if} \ N >> 1\]

*All happy families are alike; each unhappy family is unhappy in its own way.*

Leo Tolstoy (Anna Karenina, 1875-1877)
<table>
<thead>
<tr>
<th>SYSTEMS ( W(N) )</th>
<th>ENTROPY ( S_{BG} ) (ADDITIVE)</th>
<th>ENTROPY ( S_q ) ((q \neq 1)) (NONADDITIVE)</th>
<th>ENTROPY ( S_\delta ) ((\delta \neq 1)) (NONADDITIVE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{e.g., } \mu^N ) ((\mu &gt; 1))</td>
<td>EXTENSIVE</td>
<td>NONEXTENSIVE</td>
<td>NONEXTENSIVE</td>
</tr>
<tr>
<td>( \text{e.g., } N^\rho ) ((\rho &gt; 0))</td>
<td>NONEXTENSIVE</td>
<td>EXTENSIVE ((q = 1 - 1/\rho))</td>
<td>NONEXTENSIVE</td>
</tr>
<tr>
<td>( \text{e.g., } \nu^N \gamma ) ((\nu &gt; 1; 0 &lt; \gamma &lt; 1))</td>
<td>NONEXTENSIVE</td>
<td>NONEXTENSIVE</td>
<td>EXTENSIVE ((\delta = 1/\gamma))</td>
</tr>
</tbody>
</table>
A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability. Therefore the deep impression that classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of applicability of its basic concepts, it will never be overthrown.

Albert Einstein (1949)
**COMPOSITION OF VELOCITIES OF INERTIAL SYSTEMS (d=1)**

\[ v_{13} = v_{12} + v_{23} \]  
(Galileo)

\[ v_{13} = \frac{v_{12} + v_{23}}{1 + \frac{v_{12}v_{13}}{c^2}} \]  
(Einstein)

**Newton mechanics:**
It satisfies Galilean additivity but violates Lorentz invariance (hence mechanics can not be unified with Maxwell electromagnetism)

**Einstein mechanics (Special relativity):**
It satisfies Lorentz invariance (hence mechanics is unified with Maxwell electromagnetism) but violates Galilean additivity

**Question:** which is physically more fundamental, the additive composition of velocities or the unification of mechanics and electromagnetism?
On a $q$-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanley Steinberg

See also:

H. J. Hilhorst, JSTAT P10023 (2010)


M. Jauregui, C. T. and E. M. F. Curado, JSTAT P10016 (2011)

A. Plastino and M. C. Rocca, Physica A 392, 3952 (2013)

A. Plastino and M. C. Rocca, Physica A 392, 3952 (2013)

S. Umarov and C. T., J Phys A 49, 415204 (2016)

Generalization of symmetric $\alpha$-stable Lévy distributions for $q > 1$

Sabir Umarov, Constantino Tsallis, Murray Gell-Mann, and Stanley Steinberg

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2Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
3Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA
4Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, USA

(Received 10 November 2009; accepted 4 January 2010; published online 3 March 2010)
### CENTRAL LIMIT THEOREM

$N^{1/\alpha(2-q)}$-scaled attractor $\mathbb{F}(x)$ when summing $N \to \infty$ $q$-independent identical random variables with symmetric distribution $f(x)$ with $\sigma_Q = \int dx \ x^2 [f(x)]^Q / \int dx \ [f(x)]^Q$ \(Q = 2q - 1, q_1 = \frac{1+q}{3-q}\)

<table>
<thead>
<tr>
<th>$q = 1$ [independent]</th>
<th>$q \neq 1$ (i.e., $Q = 2q - 1 \neq 1$) [globally correlated]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_Q &lt; \infty$ ((\alpha = 2))</td>
<td>$\mathbb{F}(x) = G_q(x) = G_{(3q_1-1)/(1+q_1)}(x)$, with same $\sigma_Q$ of $f(x)$</td>
</tr>
<tr>
<td>$\alpha = 2$</td>
<td>$G_q(x) \sim \begin{cases} G(x) &amp; \text{if }</td>
</tr>
<tr>
<td>Classic CLT</td>
<td>(\lim_{q \to 1} x_c(q, 2) = \infty)</td>
</tr>
</tbody>
</table>

| $\sigma_Q \to \infty$ \((0 < \alpha < 2)\) | $\mathbb{F}(x) = L_{q,\alpha}$, with same $|x| \to \infty$ asymptotic behavior |
|-----------------------------------------------|---------------------------------------------------------------------|
| $\alpha > 2$ | $G_{2(1-q)-\alpha(1+q)}^{\alpha(1+3-q)}(x) \sim C_{q,\alpha}^* / |x|^{2(1-q)-\alpha(3-q)}$ |
| Levy-Gnedenko CLT | (intermediate regime) |
| \(\lim_{\alpha \to 2} x_c(1, \alpha) = \infty\) | $L_{q,\alpha} \sim \begin{cases} G_{2\alpha q-\alpha+3} / |x|^{1+\alpha} & \text{if } |x| >> x_c(1, \alpha) \end{cases}$ |
| S. Umarov, C. T., M. Gell-Mann and S. Steinberg |

### Levy-Gnedenko CLT

- $\mathbb{F}(x) = L_{q,\alpha}$, with same $|x| \to \infty$ asymptotic behavior
- $G_{2(1-q)-\alpha(1+q)}^{\alpha(1+3-q)}(x) \sim C_{q,\alpha}^* / |x|^{2(1-q)-\alpha(3-q)}$
  - (intermediate regime)
- $L_{q,\alpha} \sim \begin{cases} G_{2\alpha q-\alpha+3} / |x|^{1+\alpha} & \text{if } |x| >> x_c(1, \alpha) \end{cases}$
  - (distant regime)
The standard map: From Boltzmann-Gibbs statistics to Tsallis statistics

Ugur Tirnakli1,* & Ernesto P. Borges2,3,*

As well known, Boltzmann-Gibbs statistics is the correct way of thermostatically approaching ergodic systems. On the other hand, nontrivial ergodicity breakdown and strong correlations typically drag the system into out-of-equilibrium states where Boltzmann-Gibbs statistics fails. For a wide class of such systems, it has been shown in recent years that the correct approach is to use Tsallis statistics instead. Here we show how the dynamics of the paradigmatic conservative (area-preserving) standard map exhibits, in an exceptionally clear manner, the crossing from one statistics to the other. Our results unambiguously illustrate the domains of validity of both Boltzmann-Gibbs and Tsallis statistical distributions. Since various important physical systems from particle confinement in magnetic traps to autoionization of molecular Rydberg states, through particle dynamics in accelerators and comet dynamics, can be reduced to the standard map, our results are expected to enlighten and enable an improved interpretation of diverse experimental and observational results.
STANDARD MAP (Chirikov 1969)

\[ p_{i+1} = p_i - K \sin x_i \pmod{2\pi} \]
\[ x_{i+1} = x_i + p_{i+1} \pmod{2\pi} \]
\[ i = 0, 1, 2, \ldots \]

(area-preserving)

Particle confinement in magnetic traps, particle dynamics in accelerators, comet dynamics, ionization of Rydberg atoms, electron magneto-transport
In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).
CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

\[ V(r) \sim -\frac{A}{r^\alpha} \quad (r \to \infty) \quad (A > 0, \ \alpha \geq 0) \]

integrable if \( \alpha / d > 1 \) (short-ranged)

non-integrable if \( 0 \leq \alpha / d \leq 1 \) (long-ranged)
Validity and failure of the Boltzmann weight

L. J. L. Cirto\(^1\), A. Rodríguez\(^2\), F. D. Nobre\(^{1,3}\) and C. Tsallis\(^{1,3,4,5}\)

\(^1\) Centro Brasileiro de Pesquisas Físicas - Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
\(^2\) Departamento de Matemática Aplicada a la Ingeniería Aeroespacial, Universidad Politécnica de Madrid, Plaza Cardenal Cisneros s/n, 28040 Madrid, Spain
\(^3\) National Institute of Science and Technology for Complex Systems - Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
\(^4\) Santa Fe Institute - 1399 Hyde Park Road, Santa Fe, 87501 NM, United States
\(^5\) Complexity Science Hub Vienna - Josefstädter Strasse 39, 1080 Vienna, Austria

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published online 24 August 2018
$d$-DIMENSIONAL XY MODEL

\[ P(\bar{p}_i)/P_0 = e^{-\beta_p (\bar{p}_i P_0)^2} \]

\[ \alpha/d = 0.90 \]
\[ q_p = 1.59 \]
\[ \beta_p = 5.6 \]

$d$-DIMENSIONAL XY MODEL

Connecting complex networks to nonadditive entropies

R. M. de Oliveira¹, Samuráí Brito², L. R. da Silva¹,³ & Constantino Tsallis³,⁴,⁵,⁶

11, 1130 (2021)
RANDOM GEOMETRY $\leftrightarrow q$ -THERMOSTATISTICS

All links are equally weighted:

$$P(w) = \delta(w - 1) \quad \rightarrow \quad \text{degree distribution } P(k)$$

S. Thurner and C. T., EPL 72, 197 (2005)
S. Brito, L.R. da Silva and C. T., Scientific Reports 6, 27992 (2016)

All links are randomly weighted:

$$P(w) = \frac{\eta}{w_0 \Gamma \left( \frac{1}{\eta} \right)} e^{-\left( \frac{w}{w_0} \right)^\eta} \quad (w_0 > 0; \eta > 0)$$

geographic localization of sites:

\[ p(r) \propto \frac{1}{r^{d+\alpha_G}} \quad (\alpha_G > 0, \ r \geq 1) \]

'energy' \( \rightarrow \) \( \varepsilon_i \equiv \sum_{j=1}^{k_i} \frac{w_{ij}}{2} \quad (w_{ij} \geq 0) \)

preferential attachment:

\[ \Pi_{ij} \propto \frac{\varepsilon_i}{d_{ij}^{\alpha_A}} \quad (\alpha_A \geq 0) \]
\[ \varepsilon_i = \sum_j w_{ij} / 2 \]
\[ p_q(\varepsilon) = \frac{e_q^{\beta_q \varepsilon}}{Z_q} \]

where  \[ e_q^z \equiv [1 + (1 - q)z] \frac{1}{1-q} \quad (e_1^z = e^z) \]

and  \[ \ln_q z \equiv \frac{z^{1-q} - 1}{1-q} \quad (\ln_1 z = \ln z) \]
Limoges - France
Strain-profile determination in ion-implanted single crystals using generalized simulated annealing

Alexandre Boulle\textsuperscript{a*} and Aurélien Debelle\textsuperscript{b}

\textsuperscript{a}Science des Procédés Céramiques et de Traitements de Surface (SPCTS), CNRS UMR 6638, Centre Européen de la Céramique, 12 rue Atlantis, 87068 Limoges, France, and \textsuperscript{b}Centre de Spectrométrie Nucléaire et de Spectrométrie de Masse (CSNSM, UMR 8609), CNRS - IN2P3 – Université Paris-Sud 11, Bâtiment 108, 91405 Orsay Cedex, France. Correspondence e-mail: alexandre.boulle@unilim.fr

A novel least-squares fitting procedure is presented that allows the retrieval of strain profiles in ion-implanted single crystals using high-resolution X-ray diffraction. The model is based on the dynamical theory of diffraction, including a B-spline-based description of the lattice strain. The fitting procedure relies on the generalized simulated annealing algorithm which, contrarily to most common least-squares fitting-based methods, allows the global minimum of the error function (the difference between the experimental and the calculated curves) to be found extremely quickly. It is shown that convergence can be achieved in a few hundred Monte Carlo steps, \textit{i.e.} a few seconds. The method is model-independent and allows determination of the strain profile even without any ‘guess’ regarding its shape. This procedure is applied to the determination of strain profiles in Cs-implanted yttria-stabilized zirconia (YSZ). The strain and damage profiles of YSZ single crystals implanted at different ion fluences are analyzed and discussed.
Acoustic emissions in compression of building materials: $q$-statistics enables the anticipation of the breakdown point

Annalisa Greco\textsuperscript{1}, Constantino Tsallis\textsuperscript{2,3,4}, Andrea Rapisarda\textsuperscript{4,5,6}, Alessandro Pluchino\textsuperscript{5,6,a}, Gabriele Fichera\textsuperscript{1}, and Loredana Contrafatto\textsuperscript{1}
\[
\frac{P(t)}{P(0)} = e^{-\beta_q t}
\]
A novel automatic microcalcification detection technique using Tsallis entropy & a type II fuzzy index

Mohanalin*, Beenamol, Prem Kumar Kalra, Nirmal Kumar

Department of Electrical Engineering, IIT Kanpur, UP-208016, India

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Keywords:
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Type II fuzzy index
Shannon entropy
Mammograms
Microcalcification

ABSTRACT

This article investigates a novel automatic microcalcification detection method using a type II fuzzy index. The thresholding is performed using the Tsallis entropy characterized by another parameter 'q', which depends on the non-extensiveness of a mammogram. In previous studies, 'q' was calculated using the histogram distribution, which can lead to erroneous results when pectoral muscles are included. In this study, we have used a type II fuzzy index to find the optimal value of 'q'. The proposed approach has been tested on several mammograms. The results suggest that the proposed Tsallis entropy approach outperforms the two-dimensional non-fuzzy approach and the conventional Shannon entropy partition approach. Moreover, our thresholding technique is completely automatic, unlike the methods of previous related works. Without Tsallis entropy enhancement, detection of microcalcifications is meager: 80.21% Tps (true positives) with 8.1 Fps (false positives), whereas upon introduction of the Tsallis entropy, the results surge to 96.55% Tps with 0.4 Fps.
Efficient classification of COVID-19 CT scans by using q-transform model for feature extraction

Razi J. Al-Azawi¹, Nadia M.G. Al-Saidi², Hamid A. Jalab³, Hasan Kahtan⁴ and Rabha W. Ibrahim⁵

¹ Department of Laser and Optoelectronics Engineering, University of Technology, University of Technology, Baghdad, Iraq, Iraq
² Department of Applied Sciences, University of Technology, University of Technology, Baghdad, Iraq, Iraq
³ Department of Computer System and Technology, Faculty of Computer Science and Information Technology, University of Malaya, Kuala Lumpur, Malaysia
⁴ Department of Software Engineering, Faculty of Computer Science and Information Technology, University of Malaya, Kuala Lumpur, Malaysia
⁵ IEEE: 94086547, Kuala Lumpur, Malaysia

Original scans of infected lungs (fibrosis)

scans after q-enhanced processing (q=0.5)
Experimental Validation of a Nonextensive Scaling Law in Confined Granular Media

The velocity distribution of sheared granular media shows unexpected similarities with turbulent fluid flows.

Gaël Combe, Vincent Richefeu, Marta Stasiak, and Allbens P.F. Atman

Combe, Richefeu, Stasiak and Atman
PRL 115, 238301 (2015)
\[ \langle x^2 \rangle \propto t^\alpha \]

Combe, Richefeu, Stasiak and Atman
PRL 115, 238301 (2015)

FIG. 4. Verification of the Tsallis-Bukman scaling law for different regimes of diffusion. (Top) Evolution of the measured diffusion exponent \( \alpha \) as a function of \( 1/\sqrt{\Delta \gamma} \). The dashed line is a direct application of the scaling law from the fit of the values shown in Fig. 3, \( \alpha(1/\sqrt{\Delta \gamma}) = 2/[3 - q(1/\sqrt{\Delta \gamma})] \). (Inset) A typical diffusion curve showing the mean square displacement fluctuations, \( \langle x^2 \rangle \), in function of the shear strain, \( \gamma \); it allows the assessment of the diffusion exponent, \( \alpha \), for each strain window tested. In the case shown, it corresponds to the smallest strain window, the rightmost point in the curve at the main panel. Note that for a constant strain rate, \( \gamma \) is proportional to time. (Bottom) Measure of the deviation of the data relative to the scaling law prediction, as a function of \( 1/\sqrt{\Delta \gamma} \), showing an agreement on the order of \( \pm 2\% \).

\[ \alpha = \frac{2}{3 - q} \]

CT and DJ Bukman, PRE 54 (1996) R2197
LHC (Large Hadron Collider)

CMS, ALICE, ATLAS and LHCb detectors

~ 4000 scientists/engineers from ~ 200 institutions of ~ 50 countries
SIMPLE APPROACH: TWO-DIMENSIONAL SINGLE RELATIVISTIC FREE PARTICLE

C.Y. Wong, G. Wilk, L.J.L. Cirto and C. T.,
EPJ Web of Conferences 90, 04002 (2015), and PRD 91, 114027 (2015)

\[
\frac{1}{2\pi} \frac{dN}{dp_T dydp_T} \bigg|_{\eta=0} = A e^{-E_T/T} q
\]

\[ [A] = \text{GeV}^{-2} c^3 \]

\[ [T] = \text{GeV} \]

\[ E_T = \sqrt{m^2 c^4 + p_T^2 c^2} \]

\begin{align*}
A / 10^1 & \\
A / 10^2 & \\
A / 10^3 & \\
A / 10^4 & \\
A / 10^5 & \\
A / 10^6 & \\

(A, q, T) = (38, 1.150, 0.13) & \\
(A, q, T) = (43, 1.151, 0.13) & \\
(A, q, T) = (30, 1.127, 0.13) & \\
(A, q, T) = (32, 1.125, 0.13) & \\
(A, q, T) = (27, 1.124, 0.13) & \\
CMS \sqrt{s} = 7 \text{ TeV} & \\
ATLAS \sqrt{s} = 7 \text{ TeV} & \\
CMS \sqrt{s} = 0.9 \text{ TeV} & \\
ATLAS \sqrt{s} = 0.9 \text{ TeV} & \\
ALICE \sqrt{s} = 0.9 \text{ TeV} & \\
\end{align*}
<table>
<thead>
<tr>
<th>Entropy</th>
<th>BOLTZMANN-GIBBS STATISTICAL MECHANICS</th>
<th>NONEXTENSIVE STATISTICAL MECHANICS</th>
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<tbody>
<tr>
<td></td>
<td>$S_{BG} = -k \sum_i p_i \ln p_i$</td>
<td>$S_q = k \frac{1 - \sum_i p_i^q}{q-1}$</td>
</tr>
<tr>
<td></td>
<td>$= k \sum_i p_i \ln \frac{1}{p_i}$</td>
<td>$= k \sum_i p_i \ln \frac{1}{p_i}$</td>
</tr>
<tr>
<td></td>
<td>(additive)</td>
<td>(nonadditive)</td>
</tr>
<tr>
<td>Distribution of velocities $v$</td>
<td>$p(v) \propto e^{-mv^2/2kT}$ (Maxwellian)</td>
<td>$p(v) \propto e_{q}^{-mv^2/2kT_q}$ (q-Gaussian)</td>
</tr>
<tr>
<td>Central Limit Theorem</td>
<td>q-Central Limit Theorem</td>
<td></td>
</tr>
<tr>
<td>Distribution of energies $E_i$</td>
<td>$p_i \propto e^{-E_i/kT}$ (BG weight)</td>
<td>$p_i \propto e_{q}^{-E_i/kT_q}$ (q-weight)</td>
</tr>
<tr>
<td>Large Deviation Theory</td>
<td>$q$-Large Deviation Theory</td>
<td></td>
</tr>
</tbody>
</table>

$$
\ln_q z \equiv \frac{z^{1-q} - 1}{1-q} \quad (\ln_1 z = \ln z) \quad \text{and} \quad e_q^z \equiv [1 + (1-q)z]^\frac{1}{1-q} \quad (e_1^z = e^z)
$$
The book is devoted to the mathematical foundations of nonextensive statistical mechanics. This is the first book containing the systematic presentation of the mathematical theory and concepts related to nonextensive statistical mechanics, a current generalization of Boltzmann-Gibbs statistical mechanics introduced in 1988 by one of the authors and based on a nonadditive entropic functional extending the usual Boltzmann-Gibbs-von Neumann-Shannon entropy. Main mathematical tools like the $q$-exponential function, $q$-Gaussian distribution, $q$-Fourier transform, $q$-central limit theorems, and other related objects are discussed rigorously with detailed mathematical rational. The book also contains recent results obtained in this direction and challenging open problems. Each chapter is accompanied with additional useful notes including the history of development and related bibliographies for further reading.
THANq