A Morse complex for the Hamiltonian action in cotangent bundles

joint with
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(M, ω) sympl. mfd., φ Hamiltonian diffeo.,
i.e. $\varphi = \Phi_1$ where $\{\Phi_t\}$ is the Ham. flow of
some $H : \mathbb{T} \times M \to \mathbb{R}$, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

still open for $\mathbb{T}^{2n} \times \mathbb{C}P^n$

<u>Arnold conjecture</u>

$$\# \{\text{fixed points of } \varphi\} \geq CL(M) + 1$$

and assuming non-degeneracy

$$\geq \text{Sum of Betti numbers of } M$$

<u>Conley - Zehnder, 1984</u> Yes for $(\mathbb{T}^{2n}, \omega_{std})$.
<u>Fortune, 1985</u> Yes for $(\mathbb{C}P^n, \omega_{FS})$.

Common feature : "Contractible" fixed points $\overset{1:1}{\longleftrightarrow}$ critical points of

$$\mathbb{A}_H(x) = \int_{\mathbb{D}} X^*\omega - \int_0^1 H(t, x(t))\, dt$$

$X : \mathbb{D} \to M$ s.t. $X|_{\partial\mathbb{D}} = x$ (well-defined if $\omega|_{\pi_2(M)} = 0$, otherwise multivalued).

Q. What is the functional setting for $\mathbb{A}_H$?

The right space for $\mathbb{A}_H$ should be the space of loops with regularity $1/2$.

Problem $H^{1/2}(\mathbb{T}, M)$ has no structure of a Hilbert mfd. for general mfd M!

Floer's idea use L^2-gradient of $\mathbb{A}_H$ and interpret gradient flow equation as a PDE $\rightsquigarrow$ Floer theory.

Q. To what extent can we extend the Conley-Zehnder approach to arbitrary mfds?

Back to $(\mathbb{R}^{2m}, \omega_{std})$

$$\mathbb{A}_H(x) = \frac{1}{2} \cdot \langle -J\dot{x}, x \rangle_{L^2} - \int_0^1 H(t, x(t))\, dt$$

integration by parts

$x = (q, p)$

$$= \boxed{\langle \dot{q}, p \rangle_{L^2}} - \int_0^1 H(t, q(t), p(t))\, dt$$

this is well-defined even if q and p do not have the same regularity

$\rightsquigarrow$ replace $H^{1/2}(\mathbb{T}, \mathbb{R}^{2m})$ by

$$H^s(\mathbb{T}, \mathbb{R}^m) \times H^{1-s}(\mathbb{T}, (\mathbb{R}^m)^*)$$

for $s \in (1/2, 1)$.

This works for cotangent bundles T^*Q, Q closed or. mfd.

Fact For $s \in (\frac{1}{2}, 1]$:

(i) $H^s(\mathbb{T}, Q)$ Hilbert manifold modeled on $H^s(\mathbb{T}, \mathbb{R}^m)$.

(ii) $\mathcal{M}_s := \bigsqcup_{\gamma \in H^s(\mathbb{T}, Q)} H^{1-s}(\gamma^* T^* Q)$ is the total space of a Hilbert bundle $\mathcal{M}_s \to H^s(\mathbb{T}, Q)$.

$$\Phi_*: H^s(\mathbb{T}, U) \times H^{1-s}(\mathbb{T}, (\mathbb{R}^m)^*) \longrightarrow \mathcal{M}_s$$
$$(\xi, \eta) \mapsto (\varphi_t(\xi), \; d\varphi_t(\xi)^{-*} \cdot \eta)$$

for any time-dep. local parametrization $\varphi_t : U \to Q$, $t \in \mathbb{T}$.

Set $\mathbb{A}_H : \mathcal{M}_s \to \mathbb{R}$ where
$$H(t, q, p) = \underbrace{\boxed{\frac{1}{2} |p|_q^2}}_{=: g_q(p,p)} + U(t, q)$$
outside a compact set of T^*Q.

Thm. (A.- Stanostka, 2020)

$\mathbb{A}_H : \mathcal{M}_s \to \mathbb{R}$ well-defined and at least of class C^2, $\forall s \in (\frac{1}{2}, 1]$. Moreover, $\forall s \in (\frac{1}{2}, 1)$ $\mathbb{A}_H$ satisfies the Palais-Smale condition.

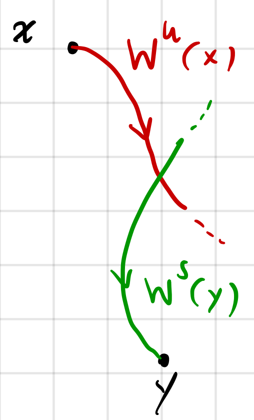
Byproduct / Thm. (Hofer-Viterbo, 1989 - A.-Stanostka, 2020)

Weinstein conjecture holds for compact contact type hypersurfaces $\Sigma \subseteq (T^*Q, \omega_{std})$ "containing" the zero-section in the interior.

Morse complex in finite dim. setting

$f : W \to \mathbb{R}$ Morse function, g Riem. metric on W

$\dim W < +\infty$.



x crit. point of Morse index $\mu(x)$ ①

$\rightsquigarrow \dim W^u(x) = \mu(x)$

y " " $\mu(y)$ ①

$\rightsquigarrow \operatorname{codim} W^s(y) = \mu(y)$

Morse-Smale condition $\rightsquigarrow$ $W^u(x) \cap W^s(y)$ pre-compact ③ mfd of dim $\mu(x) - \mu(y)$ ②.

④ Sard-Smale theorem

Rmk. For $\mathcal{A}_H$ the Morse index is always infinite

$\rightsquigarrow$ need some additional structure to make comparisons.

Notice :

$$\mathbb{A}_H(q,p) = \int_0^1 (q,p)^*\lambda - \int_0^1 H(t,q(t),p(t))\,dt$$

$$\rightsquigarrow \quad \mathbb{A}_H \circ \Psi_*(\xi,\eta) = \boxed{\int_0^1 \eta[\dot{\xi}]\,dt} + \underbrace{K(t,\xi,\eta)}_{\hookrightarrow \text{ has compact differential}}$$

$$= \frac{1}{2} \langle L(\xi,\eta), (\xi,\eta) \rangle_{\mathbb{H}}$$

where $\mathbb{H} = H^s(\mathbb{T},\mathbb{R}^n) \times H^{1-s}(\mathbb{T},(\mathbb{R}^n)^*)$

and $\langle \cdot , \cdot \rangle_{\mathbb{H}}$ standard Hilbert product

$L : \mathbb{H} \to \mathbb{H}$, $\quad L(\xi,\eta) = (\Delta^{-s}\dot{\eta}, -\Delta^{s-1}\dot{\xi})$

where Δ is the Laplace operator.

<u>Idee</u> : set $\mathcal{E}_\Psi \subseteq T(\operatorname{Im} \Psi_*)$ as the push-forward of $\mathbb{H}^-$, the neg. eigenspace of L.

$\rightsquigarrow$ $\mathcal{E} = \{ \mathcal{E}_\Psi \,/\, \Psi_* \text{ local param.} \}$ collection of local subbundles.

Define relative Morse index of $x \in \operatorname{crit} \mathbb{A}_H$ by

$$\mu(x;\mathcal{E}) := \dim\left(V^-(d^2\mathbb{A}_H(x)), \mathcal{E}_\Psi\right) \in \mathbb{Z}$$

where $x \in \operatorname{Im} \Psi_*$.

Why does this not depend on Ψ_* ?

Thm. (Abbondandolo - A. - Stanostka, 2021)

$\forall\ \tau_*$ transition map:

(i) $d\tau_*(\mathbb{H}^-)$ compact perturbation of $\mathbb{H}^-$,

(ii) $\dim(d\tau_*(\mathbb{H}^-), \mathbb{H}^-) = 0$.

$\Longrightarrow$ $\mu(x;\mathcal{E})$ is well-defined. ∎

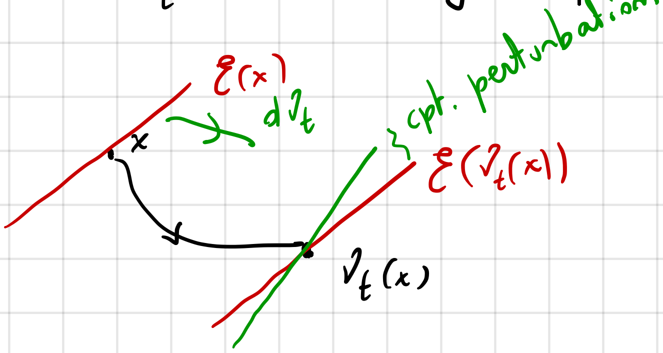
$\rightsquigarrow$ $\dim(V^-(d^2\mathbb{A}_H(x)), \mathcal{E}_\psi)$

$$= \dim(V^-(d^2\mathbb{A}_H(x)), \mathcal{E}_{\psi'}) + \underbrace{\dim(\mathcal{E}_\psi, \mathcal{E}_{\psi'})}_{=0}$$

Finite dimensionality of $W^u(x) \cap W^s(y)$

$\mathcal{E} = \{\mathcal{E}_\psi\}$ invariant under a suitably defined neg. gradient flow ϑ_t for $\mathbb{A}_H$

$d\vartheta_t(x)[\mathcal{E}(x)]$ cpt. pert. $\mathcal{E}(\vartheta_t(x))$



(i) $p \in W^u(x) \Longrightarrow T_pW^u(x)$ compact perturbation of $\mathcal{E}(p)$ with $\dim(T_pW^u(x), \mathcal{E}(p)) = \mu(x;\mathcal{E})$

(ii) $p \in W^s(y) \Rightarrow (T_pW^s(y), \mathcal{E}(p))$ Fredholm pair with $\operatorname{ind}(T_pW^s(y), \mathcal{E}(p)) = -\mu(y;\mathcal{E})$

$p \in W^u(x) \cap W^s(y)$

$\overset{\checkmark}{\Longrightarrow} \left(T_p W^u(x) \,,\, T_p W^s(y) \right)$ Fredholm pair

with index $\mu(x;\mathcal{E}) - \mu(y;\mathcal{E})$.

For <u>pre-compactness</u> need more information on $\mathcal{E}$.