>>> Topological Higher Gauge Theory - from $2BF$ to $3BF$ theory

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>>> A sketch of the talk

- 3-group and 3-gauge theory
  - ↪ based on *R. Picken and J. Faria Martins*, Diff. Geom. Appl. 29, 179 (2011), arXiv:0907.2566.
- $3BF$ action with constraints
  - ↪ Models with relevant dynamics *T. Radenković and M. Vojinović*, J. High Energy Phys.10, 222 (2019), arXiv:1904.07566.
- Quantization of the topological $3BF$ theory
  - ↪ the state sum Z is an example of Porter's TQFT for $d = 4$ and $n = 3$ *T. Porter*, J. Lond. Math. Soc. (2)58, No. 3, 723 (1998), MR 1678163.
- Pachner move invariance - sketch of the proof
  - ↪ This is a generalization of the state sum based on the classical $2BF$ action with the underlying 2-group structure *F. Girelli, H. Pfeiffer and E. M. Popescu*, Jour. Math. Phys. 49, 032503 (2008), arXiv:0708.3051.
- Conclusions

2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{\_, \_\}_{\mathrm{p}})$

* Groups G, H, and L;
* maps ∂ and δ ($\partial\delta = 1_G$);
* an action $\triangleright$ of the group G on all three groups;
* a map $\{\_,\_\}_{\mathrm{p}}$ called the *Peiffer lifting*:

$$\{\_,\_\}_{\mathrm{p}} : H \times H \to L \,.$$

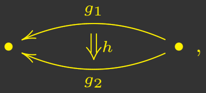
Certain axioms hold true among all these maps:

1. $\delta(\{h_1, h_2\}_{\mathrm{p}}) = \langle h_1, h_2\rangle_{\mathrm{p}}\,, \qquad \forall h_1, h_2 \in H\,,$
2. $[l_1, l_2] = \{\delta(l_1), \delta(l_2)\}_{\mathrm{p}}\,, \qquad \forall l_1, l_2 \in L$. Here, the notation $[l,k] = lkl^{-1}k^{-1}$ is used;
3. $\{h_1 h_2, h_3\}_{\mathrm{p}} = \{h_1, h_2 h_3 h_2^{-1}\}_{\mathrm{p}} \partial(h_1) \triangleright \{h_2, h_3\}_{\mathrm{p}}\,, \qquad \forall h_1, h_2, h_3 \in H\,;$
4. $\{h_1, h_2 h_3\}_{\mathrm{p}} = \{h_1, h_2\}_{\mathrm{p}} \{h_1, h_3\}_{\mathrm{p}} \{\langle h_1, h_3\rangle_{\mathrm{p}}^{-1}, \partial(h_1) \triangleright h_2\}_{\mathrm{p}}\,, \qquad \forall h_1, h_2, h_3 \in H\,;$
5. $\{\delta(l), h\}_{\mathrm{p}} \{h, \delta(l)\}_{\mathrm{p}} = l(\partial(h) \triangleright l^{-1})\,, \qquad \forall h \in H\,, \quad \forall l \in L\,.$

>>> 3-gauge theory

* Curves are labeled with the elements of G, and the elements are composed as

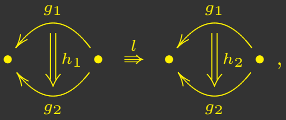
* Surfaces are labeled with the elements $h \in H$. We split the boundary into two curves, the source curve $g_1 \in G$ and the target curve $g_2 \in G$,



so that the surface $h \in H$ satisfies:

$$\boxed{\partial(h) = g_2 g_1^{-1} \,.}$$

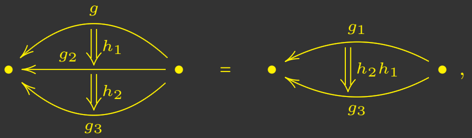
* Volumes are labeled with the elements $l \in L$. We split the boundary into the source surface $\partial_3^-(l) = h_1$ and the target surface $\partial_3^+(l) = h_2$, and the common boundary of h_1 and h_2 we split into the source curve $\partial_2^-(l) = g_1$ and the target curve $\partial_2^+(l) = g_2$,



$$\boxed{\delta(l) = h_2 h_1^{-1} \,.}$$

>>> 3-gauge theory

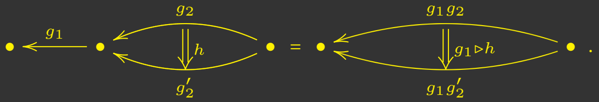
* *Vertical composition of 2-morphisms.* One can compose 2-morphisms (g_1, h_1) and (g_2, h_2) vertically, when they are compatible, when $\partial_2^+(h_1) = \partial_2^-(h_2)$,



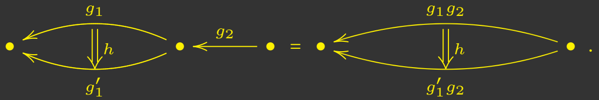
results in a 2-morphism (g_1, h_2h_1),

$$(g_2, h_2)\#_2(g_1, h_1) = (g_1, h_2h_1)\,. \tag{1}$$

* *Whiskering.* One can whisker a 2-morphism h with a morphism g_1 by attaching the whisker g_1 to the surface h from the left, such that $\partial_1^-(g_1) = \partial_1^+(h)$,

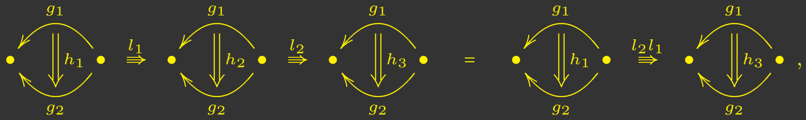


One can whisker g_2 to a surface h from the right, such that $\partial_1^-(h) = \partial_1^+(g_2)$,



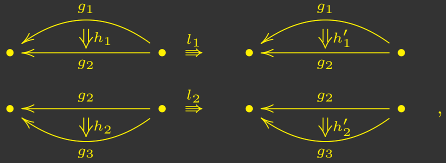
>>> 3-gauge theory

* *Upward composition*. The upward composition of 3-morphisms (g_1, h_1, l_1) and (g_1, h_2, l_2), when they are compatible, when $\partial_3^+(l_1) = \partial_3^-(l_2)$,

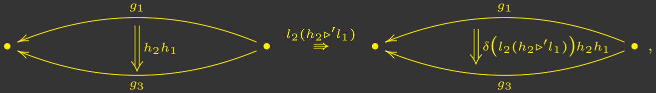


$$(g_1, h_2, l_2)\#_3(g_1, h_1, l_1) = (g_1, h_1, l_2 l_1)\,. \tag{2}$$

* *Vertical composition*. The vertical composition of two 3-morphisms (g_1, h_1, l_1) and (g_2, h_2, l_2), when they are compatible, when $\partial_2^+(l_1) = \partial_2^-(l_2)$,



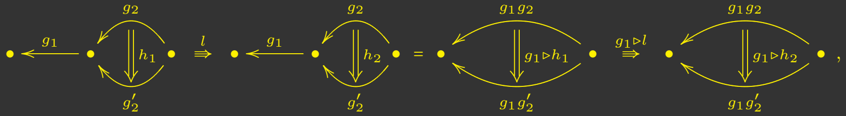
results in a 3-morphism $(g_1, h_2 h_1, l_2(h_2 \triangleright' l_1))$,



$$(g_2, h_2, l_2)\#_2(g_1, h_1, l_1) = (g_1, h_2 h_1, l_2(h_2 \triangleright' l_1))\,. \tag{3}$$

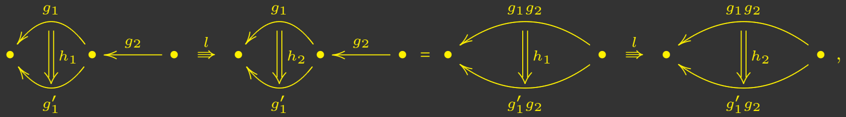
>>> 3-gauge theory

* *Whiskering of the 3-morphisms with morphisms.* Whiskering of a 3-morphism by a morphism from the left is the composition of a volume $l \in L$ and curve $g_1 \in G$ from the left, when they are compatible, when $\partial_1^+(l) = \partial_1^-(g_1)$,



$$g_1 \#_1 (g_2, h_1, l) = (g_1 g_2, g_1 \triangleright h, g_1 \triangleright l)\,. \tag{4}$$

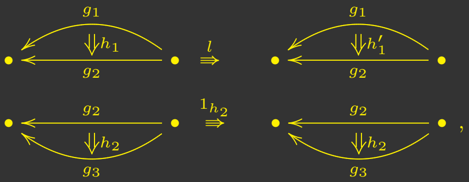
One can whisker a 3-morphism by a morphism from the right, when they are compatible, $\partial_1^-(l) = \partial_1^+(g_2)$,



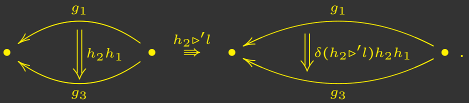
$$(g_1, h_1, l) \#_1 g_2 = (g_1 g_2, h_1, l)\,. \tag{5}$$

>>> 3-gauge theory

* *Whiskering of 3-morphisms with 2-morphisms.* Whiskering of a 3-morphism with a 2-morphisms from below, when they are compatible, $\partial_2^+(l) = \partial_2^-(h_2)$, is formed as a vertical composition of 3-morphisms (g_1, h_1, l) and $(g_2, h_2, 1_{h_2})$,



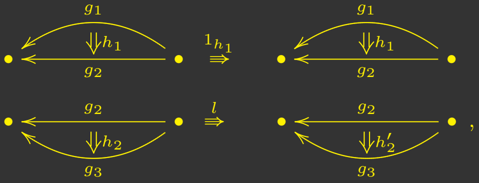
which results in a 3-morphism



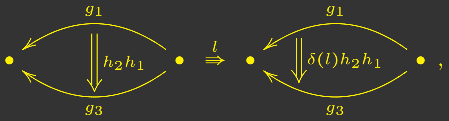
$$(g_1, h_1, l) \#_2 (g_2, h_2) = (g_1, h_2 h_1, h_2 \triangleright' l)\,. \qquad (6)$$

>>> 3-gauge theory

* Whiskering a 3-morphism by 2-morphism from above, when they are compatible, when $\partial_2^-(l) = \partial_2^+(h_1)$,



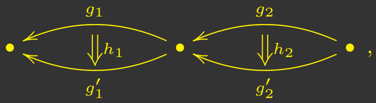
results in a 3-morphism,



$$(g_1, h_1) \#_2 (g_2, h_2, l) = (g_1, h_2 h_1, l)\,. \tag{7}$$

>>> 3-gauge theory

* *The interchanging 3-arrow*. The horizontal composition of two 2-morphisms h_1 and h_2, when they are compatible, when $\partial_1^-(h_1) = \partial_1^+(h_2)$,

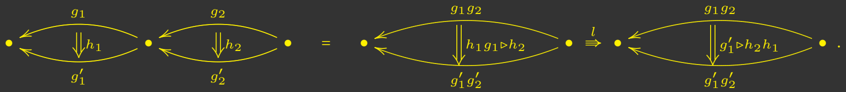


that results in a 3-morphism l, with source surface

$$\partial_3^-(l) = \big((g_1, h_1)\#_1 g_2'\big)\#_2\big(g_1\#_1(g_2, h_2)\big),$$

and target surface

$$\partial_3^+(l) = \big(g_1'\#_1(g_2, h_2)\big)\#_2\big((g_1, h_1)\#_1 g_2\big),$$



One obtains,

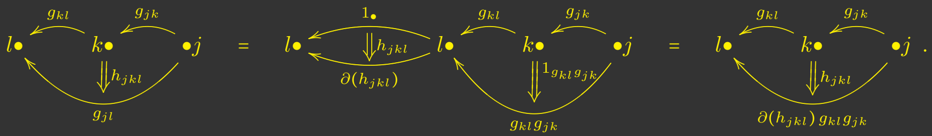
$$(g_1, h_1)\#_1(g_2, h_2) = (g_1g_2, h_1 g_1 \triangleright h_2, l)\,, \tag{8}$$

where the 3-morphism l is Peiffer lifting $\{h_1, g_1 \triangleright h_2\}_{\mathrm{p}}^{-1}$.

>>> 3-gauge theory

Lemma

Let us consider a triangle, $(jk\ell)$. The edges $(jk), j < k$, are labeled by group elements $g_{jk} \in G$ and the triangle $(jk\ell), j < k < \ell$, by element $h_{jk\ell} \in H$.



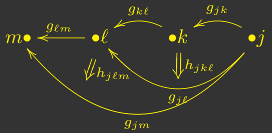
(9)

The curve $\gamma_1 = g_{k\ell}g_{jk}$ is the source and the curve $\gamma_2 = g_{j\ell}$ is the target of the surface morphism $\Sigma : \gamma_1 \to \gamma_2$, labeled by the group element $h_{jk\ell}$,

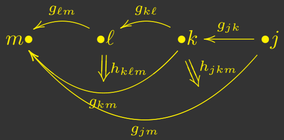
$$g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk} \,. \tag{10}$$

Lemma

Let us consider a tetrahedron, $(jk\ell m)$.



$$= (g_{\ell m} g_{j\ell}, h_{j\ell m}) \#_2 \big(g_{\ell m} \#_1 (g_{k\ell} g_{jk}, h_{jk\ell}) \big) = \big(g_{\ell m} g_{k\ell} g_{jk}, h_{j\ell m} (g_{\ell m} \triangleright h_{jk\ell}) \big) . \tag{11}$$



$$= (g_{km} g_{jk}, h_{jkm}) \#_2 \big((g_{\ell m} g_{k\ell}, h_{k\ell m}) \#_1 g_{jk} \big) = (g_{\ell m} g_{k\ell} g_{jk}, h_{jkm} h_{k\ell m}) . \tag{12}$$

Moving from surface shown on the diagram (11) to the surface shown on the diagram (12) is determined by the group element $l_{jk\ell m}$,

$$h_{jkm} h_{k\ell m} = \delta(l_{jk\ell m}) h_{j\ell m} (g_{\ell m} \triangleright h_{jk\ell}) . \tag{13}$$

>>> 3-gauge theory

Lemma (δ_L)

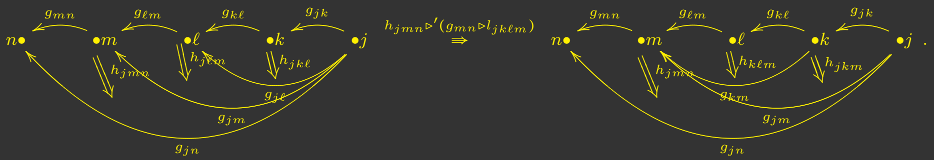
We consider a 4-simplex, $(jk\ell mn)$. We cut the 4-simplex volume along the surface $h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$.

Step 1.

* We move the surface from $h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}$ to $h_{jkm}h_{k\ell m}$ with the 3-arrow $l_{jk\ell m}$.
* To compose the resulting 3-morphism with surface h_{jmn} one must first whisker it from the left with g_{mn}.
* The obtained 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), g_{mn} \triangleright l_{jk\ell m})$ can be whiskered from below with the 2-morphism $(g_{mn}g_{jm}, h_{jmn})$.
* The resulting 3-morphism is

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$ and $\Sigma_2 = h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m})$.



Lemma (δ_L)

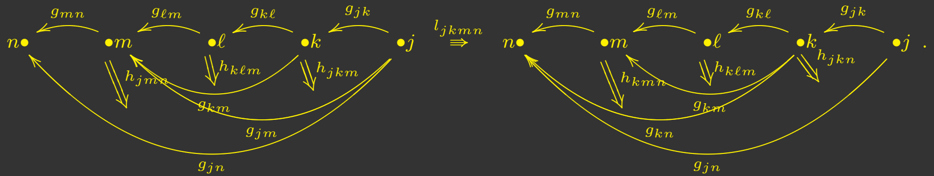
Step 2.

Let us move the surface to $h_{jkn}h_{kmn}g_{m\ell} \triangleright h_{k\ell m}$.

* We consider the 3-morphism $(g_{mn}g_{km}g_{jk}, h_{jmn}g_{mn} \triangleright h_{jkm}, l_{jkmn})$ with the source surface $h_{jmn}g_{mn} \triangleright h_{jkm}$ and target surface $h_{jkn}h_{kmn}$.
* This 3-morphism can be whiskered from above with the 2-morphism $(g_{mn}g_{\ell l}g_{k\ell}g_{jk}, g_{mn} \triangleright h_{k\ell m})$.
* The obtained 3-morphism is

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m}), l_{jkmn})}$$

$$\Sigma_1 \to \Sigma_2\,, \;\; \Sigma_1 = h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m}) \text{ and } \Sigma_2 = h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}\,.$$



(15)

Lemma (δ_L)

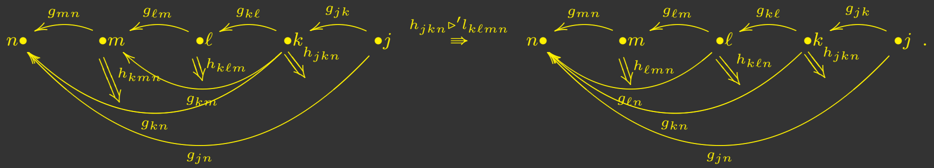
Step 3.

Next, we want to move the surface $h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}$ to surface $h_{jkn}h_{k\ell n}h_{\ell mn}$.

* We whisker the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}, h_{kmn}g_{mn} \triangleright h_{k\ell m}, l_{k\ell mn})$, with the source surface $h_{kmn}g_{mn} \triangleright h_{k\ell m}$ and target surface $h_{k\ell n}h_{\ell mn}$, with the morphism g_{jk} from the right.
* The obtained the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{kmn}g_{mn} \triangleright h_{k\ell m}, l_{k\ell mn})$ we whisker with the 2-morphism $(g_{kn}g_{jk}, h_{jkn})$ from below.
* We obtain the 3-morphism

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}, h_{jkn} \triangleright' l_{k\ell mn})}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}$ and $\Sigma_2 = h_{jkn}h_{k\ell n}h_{\ell mn}$.



(16)

Lemma (δ_L)

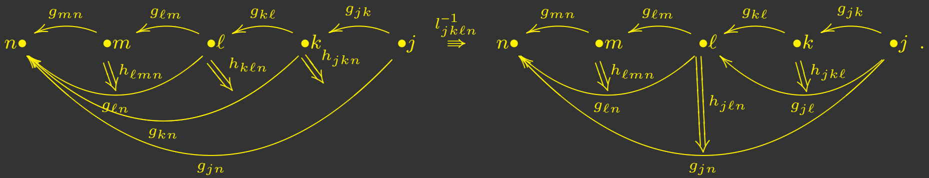
Step 4.

We map the surface $h_{jkn}h_{k\ell n}h_{\ell mn}$ to the surface $h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}$.

* The 3-morphism with the appropriate source and target is constructed by whiskering the 3-morphism $(g_{\ell n}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}, l^{-1}_{jk\ell n})$ with 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{\ell mn})$ from above.
* The obtained 3-morphism is

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})}$$

$$\Sigma_1 \to \Sigma_2\,, \quad \Sigma_1 = h_{jkn}h_{k\ell n}h_{\ell mn} \text{ and } \Sigma_2 = h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}\,.$$



(17)

Lemma (δ_L)

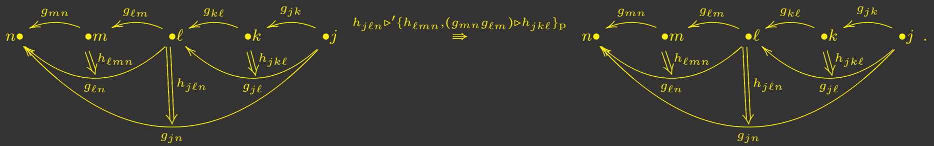
Step 5.

Next we map the surface $h_{j\ell n} g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}$ to the surface $h_{j\ell n} h_{\ell mn} (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}$.

* We use the inverse interchanging 2-arrow composition to map the surface $g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}$ to the surface $h_{\ell mn} (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}$, resulting in the 3-morphism $(g_{mn} g_{\ell m} g_{k\ell} g_{jk}, g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}, \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})$.
* Next, we whisker the obtained 3-morphism with the 2-morphism $(g_{\ell n} g_{j\ell}, h_{j\ell n})$ from below.
* The obtained 3-morphism with the appropriate source and target surfaces is

$$\boxed{(g_{mn} g_{\ell m} g_{k\ell} g_{jk}, h_{j\ell n} g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{j\ell n} g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}$ and $\Sigma_2 = h_{j\ell n} h_{\ell mn} (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}$.



(18)

Lemma

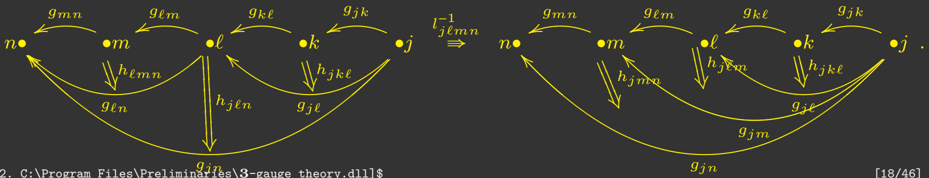
Step 6.

Finally, we construct the 3-morphism that maps the surface $h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}$ to the starting surface $h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$.

* To obtain the 3-morphism with the appropriate source and target surfaces we first move the surface $h_{j\ell n}h_{\ell mn}$ to the surface $h_{jmn}g_{mn} \triangleright h_{j\ell m}$ with the 3-arrow $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$.
* Next, we whisker the 3-morphism $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$ with the 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell})$ from above.
* The obtained 3-morphism

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}, l^{-1}_{j\ell mn})}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}$ and $\Sigma_2 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$.



>>> 3-gauge theory

Lemma (δ_L)

After the upward composition of the 3-morphisms given by the diagrams (14)-(19), the obtained 3-morphism is:

$$
\begin{aligned}
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}, l^{-1}_{j\ell mn})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{m\ell} \triangleright h_{k\ell m}, h_{jkn} \triangleright' l_{jkmn})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m}), l_{jkmn})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))\\
=\ &(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\\
&\quad l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn})l_{jkmn}h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))\,.
\end{aligned}
\tag{20}
$$

The obtained 3-morphism is the identity morphism with source and target surface $\mathcal{V}_1 = \mathcal{V}_2 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$,

$$
l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn})l_{jkmn}h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}) = e\,.
\tag{21}
$$

>>> The $3BF$ theory

One can now generalize the notion of parallel transport from curves to surfaces and volumes.

* Given a 2-crossed module, one can define a <u>3-*connection*</u>, an ordered triple (α, β, γ), where α, β, and γ are algebra-valued differential forms,

$$\begin{aligned} \alpha &= \alpha^{\alpha}{}_{\mu}\, \tau_{\alpha}\, \mathrm{d}x^{\mu}\,, & \alpha &\in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})\,, \\ \beta &= \beta^{a}{}_{\mu\nu}\, t_a\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}\,, & \beta &\in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h})\,, \\ \gamma &= \gamma^{A}{}_{\mu\nu\rho}\, T_A\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho}\,, & \gamma &\in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l})\,. \end{aligned} \tag{22}$$

* Then introduce the <u>*line, surface and volume holonomies*</u>,

$$g = \mathcal{P}\mathrm{exp} \int_{\gamma} \alpha\,, \quad h = \mathcal{P}\mathrm{exp} \int_{S} \beta\,, \quad l = \mathcal{P}\mathrm{exp} \int_{V} \gamma\,. \tag{23}$$

* The corresponding <u>*fake 3-curvature*</u> $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined as:

$$\begin{gathered} \mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}_{\mathrm{pf}}\,. \end{gathered} \tag{24}$$

>>> The $3BF$ theory

At this point one can construct the so-called $3BF$ theory.

* For a manifold $\mathcal{M}_4$ and the 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright, \{\_\,,\_\}_{\mathrm{pf}})$, that gives rise to 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, one defines the $3BF$ action as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,. \tag{25}$$

  * $3BF$ theory is a topological gauge theory,
  * it is based on the 3-group structure,
  * it is a generalization of an ordinary BF theory for a given Lie group G.
* The physical interpretation of the Lagrange multipliers C and D:
  * the $\mathfrak{h}$-valued 1-form C can be interpreted as the tetrad field if if $H = \mathbb{R}^4$ is the spacetime translation group,

$$C \to e = e^a{}_\mu(x) t_a \mathrm{d}x^\mu\,, \tag{26}$$

  * the $\mathfrak{l}$-valued 0-form D can be interpreted as the set of real-valued matter fields, given some Lie group L:

$$D \to \phi = \phi^A(x) T_A\,. \tag{27}$$

>>> Constrained $3BF$ action

* Physically relevant models - The constrained $2BF$ actions describing the *Yang-Mills field* and *Einstein-Cartan gravity*, and constrained $3BF$ actions describing the *Klein-Gordon*, *Dirac*, *Weyl* and *Majorana fields* coupled to Yang-Mills fields and gravity in the standard way are formulated.

* *Gravity and $SU(N)$ Yang-Mills theory*
  * A crossed-module $(H \stackrel{\partial}{\to} G\,,\triangleright)$:
    * $G = SO(3,1)\times SU(N)\,, \qquad H = \mathbb{R}^4\,,$
    * $M_{ab} \triangleright P_c = [M_{ab}, P_c]\,, \quad \tau_I \triangleright P_a = 0\,,$
    * $\partial(\tau_I) = 0\,.$
  * The 2-connection (α,β): $\alpha = \omega^{ab} M_{ab} + A^I \tau_I\,, \qquad \beta = \beta^a P_a\,.$
  * The 2-curvature $(\mathcal{F},\mathcal{G})$: $\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I\,, \quad \mathcal{G} = \nabla\beta P_a\,.$
  *

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla\beta^a\,.$$

  * The constrained action:

$$S = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + B^I\wedge F_I + e_a\wedge\nabla\beta^a - \lambda_{ab}\wedge\left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c\wedge e_d\right)$$
$$+\lambda^I\wedge\left(B_I - \frac{12}{g}M_{abI}e^a\wedge e^b\right) + \zeta^{abI}\left(M_{abI}\varepsilon_{cdef}e^c\wedge e^d\wedge e^e\wedge e^f - g_{IJ}F^J\wedge e_a\wedge e_b\right).$$

>>> Constrained $3BF$ action

* *Real Klein-Gordon field* $\boxed{D = \phi \mathbb{I}}$
  * A 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G \,, \triangleright \,, \{\_\,,\_\})$:
    * $G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,,$
    * $M_{ab} \triangleright P_c = [M_{ab}, P_c]\,, \qquad M_{ab} \triangleright T_A = 0\,,$
    * $\partial(P_a) = 0\,, \qquad \delta(T_A) = 0\,, \qquad \{P_a\,, P_b\} = 0\,.$
  * The 3-connection $(\alpha\,,\beta\,,\gamma)$: $\boxed{\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,.}$
  * The 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$: $\boxed{\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,.}$
  * $$\boxed{S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma\,.}$$
  * The constrained action:

$$\boxed{\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) + \Lambda^{ab} \wedge \left(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned}}$$

>>> Constrained $3BF$ action

* *Weyl spinor fields* $D = \psi_\alpha P^\alpha + \bar{\psi}^{\dot{\alpha}} P_{\dot{\alpha}}$
  * A 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \rhd, \{\_\,,\_\})$:
    * $G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,,$
    * $M_{ab} \rhd P_c = [M_{ab}, P_c]\,, \qquad M_{ab} \rhd P^\alpha = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \qquad M_{ab} \rhd P_{\dot{\alpha}} = \frac{1}{2}(\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} P_{\dot{\beta}}\,,$
    * $\partial(P_a) = 0\,, \qquad \delta(T_A) = 0\,, \qquad \{P_a\,, P_b\} = 0\,.$
  * The 3-connection $(\alpha\,,\beta\,,\gamma)$:

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_\alpha P^\alpha + \bar{\gamma}^{\dot{\alpha}} P_{\dot{\alpha}}\,.$$

  *

$$\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,,$$

$$\mathcal{H} = \left(\mathrm{d}\gamma_\alpha + \frac{1}{2}\omega^{ab}(\sigma^{ab})^\beta{}_\alpha \gamma_\beta\right) P^\alpha + \left(\mathrm{d}\bar{\gamma}^{\dot{\alpha}} + \frac{1}{2}\omega_{ab}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\gamma}^{\dot{\beta}}\right) P_{\dot{\alpha}} \equiv (\overset{\rightarrow}{\nabla}\gamma)_\alpha P^\alpha + (\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}} P_{\dot{\alpha}}\,.$$

  *

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \psi^\alpha \wedge (\overset{\rightarrow}{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}}\,.$$

We construct the constrained $3BF$ action corresponding to the full Standard Model coupled to Einstein-Cartan gravity.

>>> Quantization of the topological $3BF$ theory

We want to construct a *state sum model* from the classical S_{3BF} action by the usual *spinfoam quantization procedure*.

$$Z = \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\, \mathcal{D}B\, \mathcal{D}C\, \mathcal{D}D\, \exp\left(i \int_{M_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \right). \tag{28}$$

↪ The formal integration over the Lagrange multipliers B, C, and D leads to:

$$Z = \mathcal{N} \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\;\; \delta(\mathcal{F})\delta(\mathcal{G})\delta(\mathcal{H}). \tag{29}$$

↪ Discretization of the 3-connection:

- $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g}) \mapsto g_\epsilon \in G$ coloring the edges $\epsilon = (jk) \in \Lambda_1$,
- $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h}) \mapsto h_\Delta \in H$ coloring the triangles $\Delta = (jk\ell) \in \Lambda_2$,
- $\gamma \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l}) \mapsto l_\tau \in L$ coloring the tetrahedrons $\tau = (jk\ell m) \in \Lambda_3$.

$$\left.\begin{array}{lcl} \int \mathcal{D}\alpha & \mapsto & \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \\ \int \mathcal{D}\beta & \mapsto & \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \\ \int \mathcal{D}\gamma & \mapsto & \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \end{array}\right\} \longrightarrow \text{The disretization of path integral measures.}$$

>>> Quantization of the toplogical $3BF$ theory

↪ The condition $\delta(\mathcal{F})$ is disretized as

$$\delta(\mathcal{F}) = \prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\,, \qquad \delta_G(g_{jk\ell}) = \delta_G\Big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\Big)\,. \tag{30}$$

↪ The condition $\delta(\mathcal{G})$ on the fake curvature 3-form reads

$$\delta(\mathcal{G}) = \prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\,, \tag{31}$$

$$\delta_H(h_{jk\ell m}) = \delta_H\Big(\delta(l_{jk\ell m}) h_{j\ell m}\,(g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\Big)\,. \tag{32}$$

↪ The condition $\delta(\mathcal{H})$ is disretized as

$$\delta(\mathcal{H}) = \prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\,, \tag{33}$$

$$\delta_L(l_{jk\ell mn}) = \delta_L\Big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\Big)\,. \tag{34}$$

...all off this $\Longrightarrow$

$$Z = \mathcal{N} \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \Bigg(\prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\Bigg)\Bigg(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\Bigg)\,. \tag{35}$$

This expression can be made independent of the triangulation if one appropriately chooses the constant factor $\mathcal{N}$.

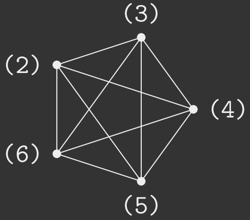
Definition

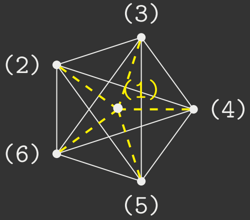
Let $\mathcal{M}_4$ be a compact and oriented combinatorial 4-manifold, and $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \,, \triangleright \,, \{\_\,,\_\}_{\mathrm{pf}})$ be a 2-crossed module. The state sum of *topological higher gauge theory* is defined by

$$
\begin{aligned}
Z \;=\; & |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\
& \times\left(\prod_{(jk)\in\Lambda_1}\int_G dg_{jk}\right)\left(\prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell}\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\int_L dl_{jk\ell m}\right) \\
& \times\left(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(\partial(h_{jk\ell})\,g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\big)\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m}\triangleright h_{jk\ell})\,h_{k\ell m}^{-1}\,h_{jkm}^{-1}\big)\right) \\
& \times\left(\prod_{(jk\ell mn)\in\Lambda_4}\delta_L\big(l_{j\ell mn}^{-1}\,h_{j\ell n}\triangleright'\{h_{\ell mn},(g_{mn}g_{\ell m})\triangleright h_{jk\ell}\}_{\mathrm{p}}\,l_{jk\ell n}^{-1}(h_{jkn}\triangleright' l_{k\ell mn})l_{jkmn}h_{jmn}\triangleright'(g_{mn}\triangleright l_{jk\ell m})\big)\right).
\end{aligned}
\tag{36}
$$

Here $|\Lambda_0|$ denotes the number of vertices, $|\Lambda_1|$ edges, $|\Lambda_2|$ triangles, $|\Lambda_3|$ tetrahedrons, and $|\Lambda_4|$ 4-simplices of the triangulation.

>>> $1 \leftrightarrow 5$ Pachner move





$1 \leftrightarrow 5$

	l.h.s.	r.h.s
M_0		(1)
M_1		(12), (13), (14), (15), (16)
M_2		(123), (124), (125), (126), (134), (135), (136), (145), (146), (156)
M_3		(1234), (1235), (1236), (1245), (1246), (1256), (1345), (1346), (1356), (1456)
M_4	(23456)	(13456), (12456), (12356), (12346), (12345)

>>> $1 \leftrightarrow 5$ Pachner move

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.h.s.	5	10	10	5	1
r.h.s.	6	15	20	15	5

Right side

$$Z^{1\leftrightarrow 5}_{\text{right}} = |G|^{-11}|H|^{-4}|L|^{-1} \int_{G^5} \prod_{(jk)\in M_1} dg_{jk} \int_{H^{10}} \prod_{(jk\ell)\in M_2} dh_{jk\ell} \int_{L^{10}} \prod_{(jklm)\in M_3} dl_{jklm} \cdot \left(\prod_{(jk\ell)\in M_2} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m)\in M_3} \delta_H(h_{jk\ell m}) \right) \left(\prod_{(jk\ell mn)\in M_4} \delta_L(l_{jk\ell mn}) \right) Z_{\text{remainder}} , \quad (37)$$

Left side

$$Z^{1\leftrightarrow 5}_{\text{left}} = |G|^{-5}|H|^{0}|L|^{-1} \delta_L(l_{23456}) Z_{\text{remainder}} . \quad (38)$$

The $Z_{\text{remainder}}$ denotes the part of the state sum that is the same on both sides of the move, and thus irrelevant for the proof of invariance.

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

On the *left hand side of the move* there is the integrand $\delta_L(l_{23456})$:

$$\delta_L(l_{23456}) = \delta_L\Big(l_{2346}{}^{-1}(h_{236} \rhd' l_{3456})l_{2356}h_{256} \rhd' (g_{56} \rhd l_{2345})l_{2456}{}^{-1}h_{246} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_\mathrm{p}\Big). \tag{39}$$

Let us examine the *right hand side of the move*, given by the equation (37).

* First, one integrates out g_{12} using $\delta_G(g_{123})$, g_{13} using $\delta_G(g_{134})$, g_{14} using $\delta_G(g_{145})$, and g_{15} using $\delta_G(g_{156})$.
* One integrates out h_{123} using $\delta_H(h_{1234})$, h_{124} using $\delta_H(h_{1245})$, h_{125} using $\delta_H(h_{1256})$, h_{134} using $\delta_H(h_{1345})$, h_{135} using $\delta_H(h_{1356})$, and h_{145} using $\delta_H(h_{1456})$.
* Next, one integrates out l_{1235} using $\delta_L(l_{12345})$, l_{1236} using $\delta_L(l_{12346})$, l_{1246} using $\delta_L(l_{12456})$, and l_{1346} using $\delta_L(l_{13456})$.

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

* The δ-functions on the group G now read $\delta_G(e)^6$. First, for $\delta_G(g_{124})$ one obtains

$$\begin{aligned}
\delta_G(g_{124}) &= \delta_G\big(\partial(h_{124})\, g_{24}\, g_{12}\, g_{14}^{-1}\big) \\
&= \delta_G\big(\partial(h_{124})\, g_{24}\, g_{23}^{-1}\, \partial(h_{123})^{-1}\, g_{13}\, g_{14}^{-1}\big) \\
&= \delta_G\big(\partial(h_{124})\, g_{24}\, g_{23}^{-1}\, g_{34}^{-1}\, \partial(h_{234})^{-1} \partial(h_{124})^{-1} \partial(h_{134})\, g_{34}\, g_{13}\, g_{14}^{-1}\big) \\
&= \delta_G\big(\partial(h_{124})\, g_{24}\, g_{23}^{-1}\, g_{34}^{-1}\, (g_{34}\, g_{23}^{-1}\, g_{24}^{-1})\, \partial(h_{124})^{-1}\, e\big) \\
&= \delta_G(e)\,,
\end{aligned} \tag{40}$$

Similarly, $\delta_G(g_{125}) = \delta_G(g_{126}) = \delta_G(g_{135}) = \delta_G(g_{136}) = \delta_G(g_{146}) = \delta_G(e)$.

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

* Let us now show that the remaining δ-functions on the group H equal $\delta_H(e)^4$. First, $\delta_H(h_{1235})$ becomes:

$$\begin{aligned}
\delta_H(h_{1235}) =& \quad \delta_H\Big(\delta(l_{1235})h_{135}(g_{35}\triangleright h_{123})h_{235}^{-1}h_{125}^{-1}\Big)\\
=& \quad \delta_H\Big(\delta\big((h_{125}\triangleright' l_{2345})l_{1245}h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1}h_{135}\triangleright'\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_\mathrm{p}\big)h_{135}(g_{35}\triangleright h_{123})h_{235}^{-1}h_{125}^{-1}\Big)\\
=& \quad \delta_H\Big(\big(h_{125}\delta(l_{2345})h_{125}^{-1}\delta(l_{1245})h_{145}(g_{45}\triangleright\delta(l_{1234}))h_{145}^{-1}\delta(l_{1345})^{-1}h_{135}\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_\mathrm{p})h_{135}^{-1}\big)\\
& \quad h_{135}(g_{35}\triangleright h_{123})h_{235}^{-1}h_{125}^{-1}\Big)\\
=& \quad \delta_H\Big(h_{235}h_{345}(g_{45}\triangleright h_{234}^{-1})h_{245}^{-1}h_{125}^{-1}h_{125}h_{245}(g_{45}\triangleright h_{124}^{-1})h_{145}^{-1}h_{145}(g_{45}\triangleright(h_{124}h_{234}(g_{34}\triangleright h_{123}^{-1})h_{134}^{-1}))\\
& \quad h_{145}^{-1}(h_{145}(g_{45}\triangleright h_{134})h_{345}^{-1}h_{135}^{-1})h_{135}\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_\mathrm{p})h_{135}^{-1}h_{135}(g_{35}\triangleright h_{123})h_{235}^{-1}\Big)\\
=& \quad \delta_H(h_{345}((g_{45}g_{34})\triangleright h_{123}^{-1})h_{345}^{-1}\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_\mathrm{p})(g_{35}\triangleright h_{123}).
\end{aligned} \tag{41}$$

Here, one uses the following identity

$$\delta\{h_1\,,h_2\}_\mathrm{p}(\partial(h_1)\triangleright h_2)h_1h_2^{-1}h_1^{-1} = e\,. \tag{42}$$

Substituting $g_{35} = \partial(h_{345})g_{45}g_{34}$, and applying the (42) identity for $h_1 = h_{345}$ and $h_2 = (g_{45}g_{34})\triangleright h_{123}$, one obtains

$$\delta_H(h_{1235}) = \delta_H(e). \tag{43}$$

Similarly, one obtains for $\delta_H(h_{1236}) = \delta_H(h_{1246}) = \delta_H(h_{1346}) = \delta_H(e)$.

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

* The remaining δ-function on the group L $\delta_L(l_{12356})$, after substituting the equations for l_{1235}, l_{1236}, l_{1246}, and l_{1346}, reads:

$$\delta_L(l_{12356}) = \delta_L\Big(h_{136}\triangleright'\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{p}}^{-1}(h_{136}\triangleright' l_{3456})l_{1356}h_{156}\triangleright'(g_{56}\triangleright l_{1345})l_{1456}{}^{-1} h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{p}}h_{146}\triangleright'(g_{46}\triangleright l_{1234})^{-1}h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{p}}^{-1}l_{1456} h_{156}\triangleright'(g_{56}\triangleright l_{1245})^{-1}l_{1256}^{-1}(h_{126}\triangleright' l_{2456})^{-1}(h_{126}\triangleright' l_{2346}{}^{-1})(h_{126}\triangleright' l_{2356})l_{1256} h_{156}\triangleright'(g_{56}\triangleright((h_{125}\triangleright' l_{2345})l_{1245}h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1}h_{135}\triangleright'\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{p}})) l_{1356}{}^{-1}h_{136}\triangleright'\{h_{356},(g_{56}g_{35})\triangleright h_{123}\}_{\mathrm{p}}\Big). \tag{44}$$

Using the identity

$$\{h_1h_2,h_3\}_{\mathrm{p}} = (h_1 \triangleright' \{h_2,h_3\}_{\mathrm{p}})\{h_1,\partial(h_2)\triangleright h_3\}_{\mathrm{p}}\,, \tag{45}$$

the delta function $\delta_L(l_{12356})$ becomes:

$$\delta_L(l_{12356}) = \delta_L\Big((h_{136}\triangleright' l_{3456})l_{1356}h_{156}\triangleright'(g_{56}\triangleright l_{1345})l_{1456}{}^{-1} h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{p}}h_{146}\triangleright'(g_{46}\triangleright l_{1234})^{-1}h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{p}}^{-1}l_{1456} \delta(h_{156}\triangleright'(g_{56}\triangleright l_{1245})^{-1})\triangleright'\Big(\big(\delta(l_{1256})^{-1}h_{126}\big)\triangleright'(l_{2456}^{-1}l_{2346}^{-1}l_{2356})h_{156}\triangleright'(g_{56}\triangleright(h_{125}\triangleright' l_{2345}))\Big) h_{156}\triangleright'(g_{56}\triangleright(h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1}))l_{1356}{}^{-1}(h_{136}h_{346})\triangleright'\{h_{346}^{-1}h_{356}g_{56}\triangleright h_{345},(g_{56}g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{p}}\Big). \tag{46}$$

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

One obtains

$$
\begin{aligned}
\delta_L(l_{12356}) = \; & \delta_L\Big((h_{156} \triangleright' (g_{56} \triangleright \delta(l_{1245})^{-1}) \delta(l_{1256})^{-1} h_{126}) \triangleright' \big(l_{2456}^{-1} l_{2346}^{-1} l_{2356} h_{256} \triangleright' (g_{56} \triangleright l_{2345}) \big) \\
& h_{156} \triangleright' \big(g_{56} \triangleright (h_{145} \triangleright' (g_{45} \triangleright l_{1234}) l_{1345}^{-1}) \big) l_{1356}{}^{-1} (h_{136} h_{346}) \triangleright' \{ h_{346}^{-1} h_{356} g_{56} \triangleright h_{345}, (g_{56} g_{45} g_{34}) \triangleright h_{123} \}_{\mathrm{p}} \\
& h_{136} \triangleright' l_{3456} l_{1356} h_{156} \triangleright' (g_{56} \triangleright l_{1345}) (\delta(l_{1456})^{-1} h_{146}) \triangleright' \big(\{ h_{456}, (g_{56} g_{45}) \triangleright h_{134} \}_{\mathrm{p}} \big) \\
& (\delta(l_{1456})^{-1} h_{146}) \triangleright' \big((g_{46} \triangleright l_{1234})^{-1} \big) (\delta(l_{1456})^{-1} h_{146}) \triangleright' \{ h_{456}, (g_{56} g_{45}) \triangleright h_{124} \}_{\mathrm{p}}^{-1} \Big).
\end{aligned}
\tag{47}
$$

The tetrahedron (3456) is part of the integrand on both sides of the move, so using the condition (32) for $\delta_H(h_{3456})$ one can write

$$
h_{346}^{-1} h_{356} g_{56} \triangleright h_{345} = h_{346}^{-1} \triangleright' \delta(l_{3456})^{-1} h_{456}.
$$

Then, using the identity (45) one obtains that

$$
\begin{aligned}
\{ h_{346}^{-1} h_{356} g_{56} \triangleright h_{345}, (g_{56} g_{45} g_{34}) \triangleright h_{123} \}_{\mathrm{p}} = \; & h_{346}^{-1} \triangleright' l_{3456}^{-1} \{ h_{456}, (g_{56} g_{45} g_{34}) \triangleright h_{123} \}_{\mathrm{p}} \\
& \big((g_{46} g_{34}) \triangleright h_{123} h_{346}^{-1} \big) \triangleright' l_{3456} \,,
\end{aligned}
\tag{48}
$$

where in the last row the definition of the action $\triangleright'$ is used. Substituting the equation (48) in the equation (47) one obtains

$$
\begin{aligned}
\delta_L(l_{12356}) = \; & \delta_L\Big((h_{156} \triangleright' (g_{56} \triangleright \delta(l_{1245})^{-1}) \delta(l_{1256})^{-1} h_{126} \delta(l_{2456})^{-1}) \triangleright' \big(l_{2346}^{-1} l_{2356} h_{256} \triangleright' (g_{56} \triangleright l_{2345}) l_{2456}^{-1} \big) \\
& h_{156} \triangleright' (g_{56} \triangleright (h_{145} \triangleright' (g_{45} \triangleright l_{1234}))) (h_{156} \triangleright' (g_{56} \triangleright \delta(l_{1345})^{-1}) \delta(l_{1356})^{-1} h_{136} \delta(l_{3456})^{-1} h_{346}) \triangleright' \\
& (\{ h_{456}, (g_{56} g_{45} g_{34}) \triangleright h_{123} \}_{\mathrm{p}} ((g_{46} g_{34}) \triangleright h_{123}) \triangleright' l_{3456}) (\delta(l_{1456})^{-1} h_{146}) \triangleright' \big(\{ h_{456}, (g_{56} g_{45}) \triangleright h_{134} \}_{\mathrm{p}} \big) \\
& (\delta(l_{1456})^{-1} h_{146}) \triangleright' \big((g_{46} \triangleright l_{1234})^{-1} \big) (\delta(l_{1456})^{-1} h_{146}) \triangleright' \{ h_{456}, (g_{56} g_{45}) \triangleright h_{124} \}_{\mathrm{p}}^{-1} \Big).
\end{aligned}
\tag{49}
$$

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

Commuting the element l_{3456} to the end of the expression, one obtains

$$\begin{aligned}\delta_L(l_{12356}) = \; & \delta_L\Big((h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})\rhd'\big(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}\big)\\ & h_{156}\rhd'(g_{56}\rhd(h_{145}\rhd'(g_{45}\rhd l_{1234})))(h_{156}\rhd'(g_{56}\rhd\delta(l_{1345})^{-1})\delta(l_{1356})^{-1}h_{136}\delta(l_{3456})^{-1}h_{346})\rhd'\\ & \big(\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\big(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}}\big)\\ & (\delta(l_{1456})^{-1}h_{146})\rhd'\big((g_{46}\rhd l_{1234})^{-1}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{p}}^{-1}\\ & (h_{156}g_{56}\rhd h_{145}h_{246}g_{46}\rhd h_{234}h_{346}^{-1})\rhd' l_{3456}\big)\Big).\end{aligned} \tag{50}$$

Acting to the whole expression with
$(h_{156} \rhd' (g_{56} \rhd \delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})^{-1} \rhd'$, one obtains,

$$\begin{aligned}\delta_L(l_{12356}) = \; & \delta_L\big(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}(h_{246}h_{456}(g_{56}g_{45})\rhd h_{124}^{-1})\rhd'\\ & \Big((g_{56}g_{45})\rhd l_{1234}((g_{56}g_{45})\rhd h_{134}h_{456}^{-1})\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}\\ & h_{456}^{-1}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}}h_{456}^{-1}\rhd g_{46}\rhd l_{1234}^{-1}(h_{456}^{-1}g_{46}\rhd h_{124})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}^{-1}\}_{\mathrm{p}}\Big)\\ & (h_{246}g_{46}\rhd h_{234}h_{346}^{-1})\rhd' l_{3456}.\end{aligned} \tag{51}$$

Using the identity

$$\{h_1, h_2h_3\}_{\mathrm{p}} = \{h_1, h_2\}_{\mathrm{p}}(\partial(h_1) \rhd h_2) \rhd' \{h_1, h_3\}_{\mathrm{p}}\,, \tag{52}$$

for $\{h_{456}, (g_{56}g_{45}) \rhd (h_{134}g_{34} \rhd h_{123})\}_{\mathrm{p}}$,

$$\{h_{456}, (g_{56}g_{45}) \rhd (h_{134}g_{34} \rhd h_{123})\}_{\mathrm{p}} = \{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{p}}(g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}\,. \tag{53}$$

>>> Proof of $1 \leftrightarrow 5$ Pachner move invariance

one obtains:

$$\begin{aligned} \delta_L(l_{12356}) = \; & \delta_L\big(l_{2346}^{-1} l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345}) l_{2456}^{-1} \\ & h_{246} \rhd' \Big(\big(h_{456}(g_{56}g_{45}) \rhd h_{124}^{-1}\big) \rhd' \Big((g_{56}g_{45}) \rhd l_{1234} h_{456}^{-1} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{134}g_{34} \rhd h_{123})\}_{\mathrm{p}} \\ & h_{456}^{-1} \rhd g_{46} \rhd l_{1234}^{-1}\Big)\{h_{456}, (g_{56}g_{45}) \rhd h_{124}^{-1}\}_{\mathrm{p}}\Big)(h_{246}g_{46} \rhd h_{234}h_{346}^{-1}) \rhd' l_{3456} \,. \end{aligned} \tag{54}$$

Using the identity (52) for $\{h_{456}, (g_{56}g_{45}) \rhd (h_{124}^{-1}\delta(l_{1234})h_{134}g_{34} \rhd h_{123})\}_{\mathrm{p}}$ one obtains the terms featuring l_{1234} cancel,

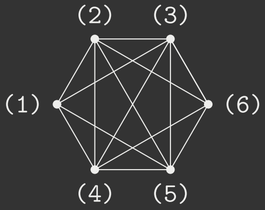
$$\begin{aligned} \delta_L(l_{12356}) = \; & \delta_L\big(l_{2346}^{-1} l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345}) l_{2456}^{-1} \\ & h_{246} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{124}^{-1}\delta(l_{1234})h_{134}g_{34} \rhd h_{123})\}_{\mathrm{p}} (h_{246}g_{46} \rhd h_{234}h_{346}^{-1}) \rhd' l_{3456} \\ = \; & \delta_L\big({l_{2346}}^{-1} l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345}) {l_{2456}}^{-1} h_{246} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}} (\delta(l_{2346})^{-1} h_{236}) \rhd' l_{3456})\big) \\ = \; & \boxed{\delta_L(l_{23456})} \,. \end{aligned} \tag{55}$$

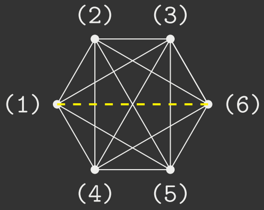
The delta function $\delta_L(l_{12356})$ on the r.h.s. reduces to the delta function $\delta_L(l_{23456})$ of the l.h.s. The integrations over l_{1234}, l_{1245}, l_{1256}, l_{1345}, l_{1356}, and l_{1456} are trivial, and finally one obtains,

$$r.h.s. = \delta_G(e)^6 \delta_H(e)^4 \delta_L(l_{23456}) = |G|^6 |H|^4 \delta_L(l_{23456}) \,. \tag{56}$$

The prefactors $|G|^{-11}|H|^{-4}|L|^{-1}$ on the r.h.s. and $|G|^{-5}|H|^{0}|L|^{-1}$ on the l.h.s., compensate for left-over factors.

>>> $2 \leftrightarrow 4$ Pachner move





$2 \leftrightarrow 4$

	l.h.s.	r.h.s
M_0		
M_1		(16)
M_2		$(126), (136), (146), (156)$
M_3	(2345)	$(1236), (1246), (1256), (1346), (1356), (1456)$
M_4	$(23456), (12345)$	$(12346), (12356), (12456), (13456)$

>>> $2 \leftrightarrow 4$ Pachner move

	$\|\Lambda_0\|$	$\|\Lambda_1\|$	$\|\Lambda_2\|$	$\|\Lambda_3\|$	$\|\Lambda_4\|$
l.h.s.	6	14	16	9	2
r.h.s.	6	15	20	14	4

Right side

$$Z^{2\leftrightarrow 4}_{left} = |G|^{-8}|H|^{-1}|L|^{-1} \int_L dl_{2345}\delta_H(h_{2345})\left(\prod_{(jk\ell mn)\in M_4}\delta_L(l_{jk\ell mn})\right) Z_{\texttt{remainder}}\,, \tag{57}$$

Left side

$$Z^{2\leftrightarrow 4}_{right} = |G|^{-11}|H|^{-3}|L|^{-1}\int_G dg_{16}\int_{H^4} dh_{126}dh_{136}dh_{146}dh_{156}\int_L dl_{1236}dl_{1246}dl_{1256}dl_{1346}dl_{1356}dl_{1456}$$
$$\left(\prod_{(jk\ell)\in M_2}\delta_G(g_{jk\ell})\right)\left(\prod_{(jk\ell m)\in M_3}\delta_H(h_{jk\ell m})\right)\left(\prod_{(jk\ell mn)\in M_4}\delta_L(l_{jk\ell mn})\right) Z_{\texttt{remainder}}. \tag{58}$$

>>> Proof of $2 \leftrightarrow 4$ Pachner move invariance

* On the *left hand side of the move* one has the following integrals and the integrand,

$$\int_L dl_{2345} \delta_H(h_{2345}) \delta_L(l_{23456}) \delta_L(l_{12345}). \tag{59}$$

We integrate out l_{2345} using $\delta_L(l_{12345})$. The δ-function $\delta_H(h_{2345})$ now reads,

$$\delta_H(h_{2345}) = \quad \delta_H(e). \tag{60}$$

The remaining δ-function $\delta_L(l_{23456})$, reads

$$\begin{aligned} \delta_L(l_{23456}) = \quad & \delta_L\Big(l_{2456}{}^{-1} l_{2346}{}^{-1} l_{2356} (h_{256} g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1235} \big(h_{256} g_{56} \rhd h_{125}{}^{-1} g_{56} \rhd h_{135}\big) \rhd' \\ & \Big((g_{35} \rhd h_{123} h_{356}^{-1}) \rhd' l_{3456}) \{g_{56} \rhd h_{345}, (g_{56} g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1} (g_{56} \rhd h_{345} (g_{56} g_{45}) \rhd (h_{123} h_{234}^{-1}) h_{456}^{-1}) \rhd' \\ & \{h_{456}, (g_{56} g_{45}) \rhd h_{234}\}_{\mathrm{p}}\Big) (h_{256} g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1345} \\ & (h_{256} g_{56} \rhd h_{125}{}^{-1} g_{56} \rhd h_{145}) \rhd' ((g_{56} g_{45}) \rhd l_{1234})^{-1} (h_{256} g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1245}^{-1}\Big). \end{aligned} \tag{61}$$

Finally, the l.h.s. reads:

$$\boxed{l.h.s. = \delta_H(e) \delta_L(l_{23456}) = |H| \delta_L(l_{23456})}. \tag{62}$$

>>> Proof of $2 \leftrightarrow 4$ Pachner move invariance

* *On the right hand side* of the move there is the integral

$$\int_G dg_{16} \int_{H^4} dh_{126} dh_{136} dh_{146} dh_{156} \int_L dl_{1236} dl_{1246} dl_{1256} dl_{1346} dl_{1356} dl_{1456} \left(\prod_{(jk\ell) \in M_2} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m) \in M_3} \delta_H(h_{jk\ell m}) \right) \left(\prod_{(jk\ell mn) \in M_4} \delta_L(l_{jk\ell mn}) \right). \tag{63}$$

  * One integrates out g_{16} using $\delta_G(g_{126})$, h_{126} using $\delta_H(h_{1236})$, h_{136} using $\delta_H(h_{1346})$, and h_{146} using $\delta_H(h_{1456})$.
  * One integrates out l_{1236} using $\delta_L(l_{12346})$, l_{1246} using $\delta_L(l_{12456})$, l_{1346} using $\delta_L(l_{13456})$.
  * The remaining δ-functions on the group G reduces to $\delta_G(e)^3$,

$$\delta_G(g_{136}) = \delta_G(g_{146}) = \delta_G(g_{156}) = \delta_G(e).$$

  * One obtains that the remaining δ-functions on H reduce on $\delta_H(e)^3$,

$$\delta_H(h_{1256}) = \delta_H(h_{1356}) = \delta_H(h_{1456}) = \delta_H(e).$$

>>> Proof of $2 \leftrightarrow 4$ Pachner move invariance

* For the remaining δ-function $\delta_L(l_{12356})$,

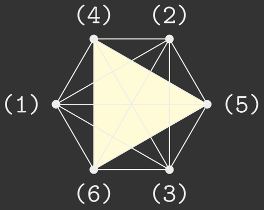
$$\delta_L(l_{12356}) = \delta_L\Big({l_{2456}}^{-1}{l_{2346}}^{-1}l_{2356}(h_{256}g_{56} \triangleright {h_{125}}^{-1}) \triangleright' g_{56} \triangleright l_{1235}(h_{256}g_{56} \triangleright {h_{125}}^{-1}g_{56} \triangleright h_{135}) \triangleright' \Big((g_{35} \triangleright h_{123}h_{356}^{-1}) \triangleright' l_{3456})\{g_{56} \triangleright h_{345}, (g_{56}g_{45}g_{34}) \triangleright h_{123}\}_{\mathrm{p}}^{-1}(g_{56} \triangleright h_{345}(g_{56}g_{45}) \triangleright (h_{123}h_{234}^{-1})h_{456}^{-1}) \triangleright' \{h_{456}, (g_{56}g_{45}) \triangleright h_{234}\}_{\mathrm{p}}\Big)(h_{256}g_{56} \triangleright {h_{125}}^{-1}) \triangleright' g_{56} \triangleright l_{1345} (h_{256}g_{56} \triangleright {h_{125}}^{-1}g_{56} \triangleright h_{145}) \triangleright' ((g_{56}g_{45}) \triangleright l_{1234})^{-1}(h_{256}g_{56} \triangleright {h_{125}}^{-1}) \triangleright' g_{56} \triangleright l_{1245}^{-1}\Big). \tag{64}$$

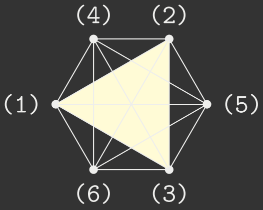
which is precisely the equation (61).
The remaining integration over the element h_{156} H and remaining integrations over the three elements l_{1246}, l_{1256}, and l_{1356}, are trivial, yielding the result of the r.h.s. to:

$$\boxed{r.h.s. = \delta_G(e)^3\, \delta_H(e)^3\, \delta_L(l_{12356}) = |G|^3\,|H|^3\,\delta_L(l_{12356})}. \tag{65}$$

The prefactors are $|G|^{-8}|H|^{-1}|L|^{-1}$ on the l.h.s., and $|G|^{-11}|H|^{-3}|L|^{-1}$ on the r.h.s. compensate for the left-over factors.

>>> $3 \leftrightarrow 3$ Pachner move





$3 \leftrightarrow 3$

	l.h.s.	r.h.s
M_0		
M_1		
M_2	(456)	(123)
M_3	$(1456), (2456), (3456)$	$(1234), (1235), (1236)$
M_4	$(23456), (13456), (12456)$	$(12356), (12346), (12345)$.

>>> $\mathbf{3 \leftrightarrow 3}$ Pachner move

Left side

$$Z^{3\leftrightarrow 3}_{left} = \int_H dh_{456} \int_{L^3} dl_{1456} dl_{2456} dl_{3456} \delta_G(g_{456}) \, \delta_H(h_{3456}) \delta_H(h_{2456}) \delta_H(h_{1456}) \delta_L(l_{23456}) \delta_L(l_{13456}) \delta_L(l_{12456}) Z_{\text{remainder}} \,, \tag{66}$$

Right side

$$Z^{3\leftrightarrow 3}_{right} = \int_H dh_{123} \int_{L^3} dl_{1234} dl_{1235} dl_{1236} \delta_G(g_{123}) \, \delta_H(h_{1234}) \delta_H(h_{1235}) \delta_H(h_{1236}) \delta_L(l_{12356}) \delta_L(l_{12346}) \delta_L(l_{12345}) Z_{\text{remainder}} \,. \tag{67}$$

* Let us first investigate the *r.h.s. of the move*:

$$\int_H dh_{123} \int_{L^3} dl_{1234} dl_{1235} dl_{1236} \delta_G(g_{123})\, \delta_H(h_{1234}) \delta_H(h_{1235}) \delta_H(h_{1236}) \delta_L(l_{12356}) \delta_L(l_{12346}) \delta_L(l_{12345}). \tag{68}$$

  * First, one integrates out the l_{1235}, using $\delta_L(l_{12345})$, one integrates out l_{1236}, using $\delta_L(l_{12356})$, and one integrates out h_{123}, using $\delta_H(l_{1234})$.
  * Similarly, one obtains that $\delta_H(h_{1235}) = \delta_H(h_{1236}) = \delta_H(e)$.
  * The remaining δ-function $\delta_L(l_{12346})$ reads

$$\begin{aligned} \delta_L(l_{12346}) = \quad & \delta_L\big((h_{146}g_{46} \triangleright h_{134}) \triangleright' \{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright (h_{134}^{-1} h_{124} h_{234})\}_{\mathrm{p}}^{-1} l_{1346}^{-1} l_{1356} \\ & h_{156} \triangleright' (g_{56} \triangleright l_{1345}) h_{156} \triangleright' (g_{56} \triangleright l_{1245})^{-1} (h_{156} g_{56} \triangleright h_{125}) \triangleright' (g_{56} \triangleright l_{2345}^{-1}) l_{1256}^{-1} h_{126} \triangleright' l_{2356}^{-1} (h_{126} \triangleright' l_{2346}) l_{1246}\big). \end{aligned} \tag{69}$$

One obtains that the integration over l_{1234} is trivial, and the r.h.s. of the move finally reads

$$\begin{aligned} r.h.s. = \quad & \delta_G(e) \delta_H(e)^2 \delta_L\big(h_{156} \triangleright' (g_{56} \triangleright l_{1245})^{-1}\, h_{156} \triangleright' (g_{56} \triangleright (h_{125} \triangleright' l_{2345}))^{-1}\, l_{1256}^{-1}\, h_{126} \triangleright' l_{2356}^{-1} (h_{126} \triangleright' l_{2346}) \\ & l_{1246} (h_{146} g_{46} \triangleright h_{134}) \triangleright' \{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56} g_{45}) \triangleright (h_{134}^{-1} h_{124} h_{234})\}_{\mathrm{p}}^{-1} l_{1346}^{-1} l_{1356} h_{156} \triangleright' (g_{56} \triangleright l_{1345})\big). \end{aligned} \tag{70}$$

>>> Proof of $3 \leftrightarrow 3$ Pachner move invariance

* The *integral of the l.h.s.* reads

$$\int_H dh_{456} \int_{L^3} dl_{1456} dl_{2456} dl_{3456} \delta_G(g_{456})\, \delta_H(h_{3456}) \delta_H(h_{2456}) \delta_H(h_{1456}) \delta_L(l_{23456}) \delta_L(l_{13456}) \delta_L(l_{12456})\,. \tag{71}$$

  * One integrates out the l_{1456}, exploiting $\delta_L(l_{13456})$, one one integrates out the l_{2456}, exploiting $\delta_L(l_{23456})$, and one integrates out h_{456}, exploiting $\delta_H(h_{3456})$.
  * Using this we obtain
$$\delta_G(g_{456}) = \delta_G(e)\,. \tag{72}$$
  * Similarly as done for the right-hand side of the move, one shows
$$\delta_H(h_{1456}) = \delta_H(h_{2456}) = \delta_H(e).$$
  * The remaining $\delta_L(l_{12456})$ now reads
$$\begin{aligned} \delta_L(l_{12456}) = \quad & \delta_L\big(l_{1246}^{-1} h_{126} \rhd' l_{2346}{}^{-1} h_{126} \rhd' l_{2356} (h_{126} h_{256}) \rhd' (g_{56} \rhd l_{2345})) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) \\ & h_{156} \rhd' (g_{56} \rhd l_{1345})^{-1} l_{1356}^{-1} l_{1346} (h_{146} g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56} g_{45}) \rhd (h_{134}^{-1} h_{124} h_{234})\}_{\mathrm{p}}\big)\,. \end{aligned} \tag{73}$$

One obtains that the integral over l_{3456} is now trivial and l.h.s. of the move finally reduces to:

$$\begin{aligned} l.h.s. = \quad & \delta_G(e) \delta_H(e)^2 \delta_L\big(h_{126} \rhd' l_{2346} l_{1246} (h_{146} g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56} g_{45}) \rhd (h_{134}^{-1} h_{124} h_{234})\}_{\mathrm{p}}^{-1} l_{1346}^{-1} \\ & l_{1356}\, h_{156} \rhd' (g_{56} \rhd l_{1345})\, h_{156} \rhd' (g_{56} \rhd l_{1245})^{-1} (h_{156} g_{56} \rhd h_{125}) \rhd' (g_{56} \rhd l_{2345})^{-1}\, l_{1256}^{-1}\, h_{126} \rhd' l_{2356}^{-1}\big)\,. \end{aligned} \tag{74}$$

The expressions (70) and (74) are the same, which proves the invariance of the state sum (10) under the Pachner move $3 \leftrightarrow 3$. The numbers of k-simplices agree on both sides of the $3 \leftrightarrow 3$ move for all k, and the prefactors play no role in this case.

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* Physically relevant models - The constrained $2BF$ actions describing the *Yang-Mills field* and *Einstein-Cartan gravity*, and constrained $3BF$ actions describing the *Klein-Gordon*, *Dirac*, *Weyl* and *Majorana fields* coupled to Yang-Mills fields and gravity in the standard way.
* Starting from the notion of Lie 3-groups, we generalize the integral picture of gauge theory to a 3-gauge theory that involves curves, surfaces, and volumes labeled with elements of non-Abelian groups.
* The definition of the discrete state sum model of topological higher gauge theory in dimension d=4.
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Thank you for your attention!