Symp(starshaped domain)

based on work with

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The problem

Take $K \subset (\mathbb{R}^{2n}, \omega_0)$, compact, simply connected, ∂K smooth

$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$

Symp(K) $:= \{\varphi \colon K \to K \text{ diffeo}, \varphi \text{ extends to } U \supset K,\ \varphi^*\omega_0 = \omega_0\}$

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Problem 1 Is Symp(K) connected ?

Problem 2 Understand the topology of Symp K

(e.g. homotopy type)

Motivation for Problem 1:

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Example

Assume that $K = \mathrm{C}^{2n}(1) := \times_n \overline{D}(1)$
or a suitable $K \supset C^{2n}(1)$ with ∂K smooth.

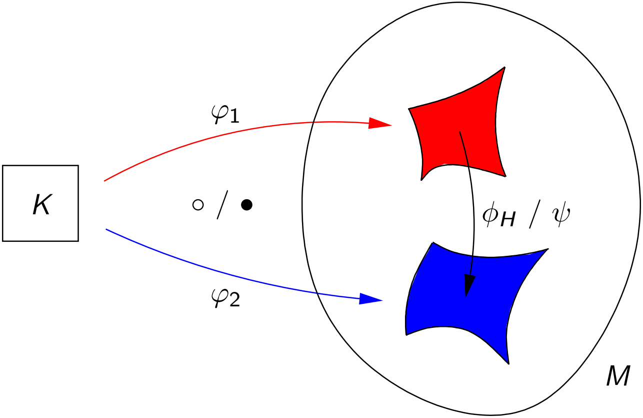
Take $(M, \omega) = \mathrm{B}^{2n}(n+1)$ or $\mathbb{C}\mathrm{P}^n(n+1)$

or $\mathrm{C}^{2n}(2)$ or $\times_n S^2(2)$.

Then there exists a sequence $(\varphi_j)_{j \geq 1} \colon K \overset{s}{\hookrightarrow} (M, \omega)$ such that:
$\exists$ symplectomorphism ψ of (M, ω) with

$$\varphi_j = \psi \circ \varphi_i \qquad (*)$$

only if $i = j$.



cf. **Gutt–Usher**

If the answer to Problem 1 is **yes**, then $(*)$ can be upgraded to

$$\operatorname{im}\varphi_j = \operatorname{im}(\psi \circ \varphi_i)$$

Strategy to analyse topology of $\mathrm{Symp}(K)$:

Split problem into two:

$\varphi \in \mathrm{Symp}(K)$ "splits as"

$\varphi|_{\partial K}$ "$\in$" $\mathrm{SCont}(\partial K)$ and $\varphi_c \in \mathrm{Symp}_c(K^\circ)$

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(1) $\mathrm{Symp}_c(K^\circ)$ is in general hard to understand.

BUT (**Gromov**):

If K^4 is starshaped, then $\mathrm{Symp}_c(K^\circ)$ is contractible.

In this case, $\mathrm{Symp}(K) \simeq \mathrm{SCont}(K)$ ...

(2) $\mathrm{SCont}(\partial K)$ can sometimes be understood.

It is "generically small" (contractible)

Technically, one tries to set up a **fibration**:

First trial: Look at restriction

$$\begin{aligned} \rho\colon \ \mathrm{Symp}(K) &\to \mathrm{Diff}^+_{\omega_\partial}(\partial K) \\ \varphi &\mapsto \varphi|_{\partial K} \end{aligned}$$

Is a fibration, can be useful if ∂K is fibred by circles (**Lalonde–Pinsonnault**)

In general, wish a smaller base, with more geometry.

Set $\lambda = \sum x_i \, dy_i - y_i \, dx_i$

Have: $\varphi^* d\lambda \; = \; d\lambda$

Since $H^1(\partial K; \mathbb{R}) = 0$: $\varphi^* \lambda \; = \; \lambda + dh$

With $\alpha := \lambda_\partial$: $\varphi_\partial^* \alpha \; = \; \alpha + dh_\partial$: $\varphi_\partial \notin \mathrm{Cont}(\partial K)$

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But one can get rid of dh_∂ in a good way (**Abbondandolo–Majer**):

Want to deform $\alpha + dh_\partial$ to α by the Moser method:

Search vector field X_t such that its flow μ_t solves

$$\mu_t^*(\alpha + t \, dh_\partial) = \alpha \quad \forall \, t \in [0, 1].$$

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Which Moser vector field X_t ?

Ansatz:

$$X_t \;=\; f_t R_\alpha$$

where R_α the Reeb field of α: $d\alpha(R_\alpha, \cdot) = 0, \quad \alpha(R_\alpha) = 1$

$$
\begin{aligned}
0 &\overset{!}{=} \frac{d}{dt}\left(\mu_t^*\left(\alpha + t\, dh_\partial\right)\right) \\
&= \mu_t^*\left(\mathcal{L}_{X_t}(\alpha + t\, dh_\partial) + dh_\partial\right) \\
&= \mu_t^*\big(\underbrace{\imath_{X_t} d(\alpha}_{=0} + t\, d h_\partial) + d\imath_{X_t}(\alpha + t\, dh_\partial) + dh_\partial\big).
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Hence need $X_t = f_t\, R_\alpha$ such that

$$d\imath_{X_t}(\alpha + t\, dh_\partial) + dh_\partial \;=\; 0.$$

Enough:

$$\imath_{X_t}(\alpha + t\, dh_\partial) + h_\partial \;=\; 0.$$

Since $\imath_{X_t}(\alpha + t\, dh_\partial) = f_t\left(1 + t\, dh_\partial(R_\alpha)\right)$, can take

$$f_t \;:=\; -\frac{h_\partial}{1 + t\, dh_\partial(R_\alpha)}, \qquad t \in [0,1].$$

Obtain, with $h(\varphi)$ the function with $\varphi^*\alpha = \alpha + dh_\partial$:

$$\begin{array}{rcl} \rho(\mathrm{Symp}(K)) & \to & \mathrm{SCont}(\partial K) \\ \varphi & \mapsto & \varphi \circ \mu(h(\varphi)) \end{array}$$

is a strong deformation retract.

Proposition The following are Hurewicz fibrations.

$$\mathrm{Symp}(K) \xrightarrow{\rho} \rho(\mathrm{Symp}(K)) \xrightarrow{\circ\mu} \mathrm{SCont}(\partial K)$$

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Proof. Need to prove

$$\begin{array}{ccc} S \times \{0\} & \xrightarrow{\varphi} & E \\ \big\downarrow & \overset{\Phi}{\nearrow} & \big\downarrow p \\ S \times [0,1] & \xrightarrow{\psi} & B. \end{array}$$

Easy, since the fibers are groups: Take any section, and correct. $\square$

The fiber is:

$$\begin{array}{rcl} \mathrm{Symp}_{\partial}(K) & := & \{\varphi \in \mathrm{Symp}(K) \mid \rho(\varphi) = \mathtt{id}_{\partial K}\} \\ \downarrow r_1 & & \\ \mathrm{Symp}_{T\partial}(K) & := & \{\varphi \in \mathrm{Symp}_{\partial}(K) \mid T_p\varphi = \mathtt{id} \ \forall p \in \partial K\} \\ \downarrow r_2 & & \\ \mathrm{Symp}_c(K^\circ) & & \end{array}$$

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So get, up to homotopy equivalence, the Hurewicz fibration

$$\mathrm{Symp}_c(K^\circ) \hookrightarrow \mathrm{Symp}(K) \xrightarrow{(\circ\mu)\circ\rho =: M} \mathrm{SCont}(\partial K)$$

For Problem 1 wish to know:

When are $\mathrm{Symp}_c(K^\circ)$ and $\mathrm{SCont}(\partial K)$ connected?

$\mathrm{Symp}_c(K^\circ)$: Good and bad news:

- If K is starshaped and 4-dimensional:

  $\mathrm{Symp}_c(K^\circ)$ is contractible ("**Gromov**")

  Then $\mathrm{Symp}(K) \xrightarrow[M]{\simeq} \mathrm{SCont}(\partial K)$

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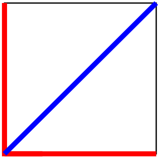
Then $\quad \mathrm{Symp}(K) \xrightarrow[M]{\simeq} \mathrm{SCont}(\partial K)$

Remark This comes from the proof of

$$\mathrm{Symp}_0(S^2 \times S^2) \simeq \mathrm{SO}(3) \times \mathrm{SO}(3)$$



$S^2 \times S^2 \supset D^\circ \times D^\circ \qquad \mathrm{Symp}_c(D^\circ \times D^\circ) \simeq \mathrm{pt}$

$\supset T_1^* S^2 \qquad \mathrm{Symp}_c(T_1^* S^2) = \mathbb{Z} = \langle \tau_{\mathrm{DS}} \rangle$

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  Note: $\pi_0\big(\mathrm{Diff}_c(\mathrm{Int}\, B^n)\big) = \Theta^{n+1}$
  (group of exotic spheres Σ^{n+1})

  It is unknown whether exotic components have symplectic representatives.

  But some have for another ω ! (**Casals–Keating–Smith**)

$\mathrm{SCont}(\partial K)$: Sometimes, this can be understood.

Example 1 (**Eliashberg**, **Casals–Spacil**)

$$\mathrm{SCont}(S^3, \alpha_0) \simeq \mathrm{SU}(2) \qquad (*)$$

Hence $\mathrm{Symp}(\mathrm{B}^4) \simeq \mathrm{SU}(2)$

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This is shown by **Lalonde–Pinsonnault** using "only" Gromov's $\mathrm{Symp}_c(\mathrm{Int}\, B^4) \simeq \mathrm{pt}$

Hence get new proof of $(*)$

Example 2 ???

Assume that K^4 is a starshaped toric domain such that Γ contains no segment of rational slope:

(i) If $a_1 \neq a_2$, then $\mathrm{SCont}(\partial K)$ is connected.

(ii) If K is invariant under $(z_1, z_2) \mapsto (z_2, z_1)$, then $\mathrm{SCont}(\partial K)$ has two components.

Proof. Fix $\varphi \in \mathrm{SCont}(\partial K)$.

Step 1. *φ maps μ-fibers to μ-fibers.*

Proof. ϕ_α^t preserves the fibers $T_p = \mu^{-1}(p)$ and there is a Kronecker (linear flow):

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Proof. ϕ_α^t preserves the fibers $T_p = \mu^{-1}(p)$
and there is a Kronecker (linear flow):

Since φ preserves α, it preserves R_α,
hence takes flow lines to flow lines.

Take an irrational T_p and a flow line.
It is mapped to a flow line, and φ is continuous.
So $\varphi(T_p) = T_{p'}$.
The irrational T_p are dense by assumption. □

Step 2. *If* $\varphi(C_1) = C_1$*, then* $\varphi(T_p) = T_p$*.*

Indeed, φ preserves $\alpha \wedge d\alpha$:

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Indeed, φ preserves $\alpha \wedge d\alpha$:

Step 3. *End of proof of (i):*

Take $G = \{\varphi \in \mathrm{SCont}(\partial K) \mid \varphi(C_1) = C_1\}$

Fix $\varphi \in G$. Look again at irrational torus T_p:

$\varphi(T_p) = T_p$ and $\varphi \circ \phi_\alpha^t = \varphi_\alpha^t \circ \varphi$ for all t.

On the flow line through 0, φ is the translation by $\varphi(0)$.

This line is dense, so φ is translation by $\varphi(0)$.

Irrational tori are dense, so

$$\varphi(\boldsymbol{\mu}, \boldsymbol{\theta}) = (\boldsymbol{\mu}, \boldsymbol{\theta} + F(\boldsymbol{\mu}))$$

$\varphi^* \alpha = \alpha$ becomes $\displaystyle \sum_{i=1}^{2} \mu_i \, dF_i(\boldsymbol{\mu})) = 0$

...

Example 3

Generically, $\mathrm{SCont}_0(S^3, \alpha)$ is contractible. ...
cf. (**Casals–Spacil**)