Coherent states for quantum groups

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Research Visitor at IST, March 17-April 16, 2014

Instituto Superior Técnico, Lisboa, April 1, 2014

Outline

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Work of Todorov et al.

- Weak (quasi)Hopf algebras are allowed as true algebras of symmetries in 2d CFTs (Mack, Schomerus 1989)
- In particular variants of hidden quantum group symmetry appear in (gauged) WZNW models
- Todorov et al. (1991) build a preHilbert space of the theory in covariant way from reps. of quantum groups in Hamiltonian approach
- Fields are related to q-coherent states which were given ad hoc

Outline

Setting for classical Perelomov coherent states

- $G^{\mathbb{C}}$ – complex connected ss Lie group with compact real form G, $B \subset G^{\mathbb{C}}$ Borel subgroup
- $\chi : B \to \mathbb{C}$ a character of B; $\mathbb{C}_\chi$ corr. 1-d B-module.
- $p : G^{\mathbb{C}} \to G^{\mathbb{C}}/B$ a principal B-bundle
- associated line bundle $L_\chi = G^{\mathbb{C}} \times_\chi \mathbb{C}_\chi$
- projection $p_L : L_\chi \to G^{\mathbb{C}}/B$.
- The left action of G on $G^{\mathbb{C}}$ induces an action of G on L_χ
- $G^{\mathbb{C}}$ acts on the space $V_\chi = \Gamma L_\chi$ of holomorphic (horizontal!) sections of L_χ by $(g_* s)(x) = gs(g^{-1}x)$. By BOREL-WEIL theorem V_χ is an irreducible unitarizable G-module.
- An invariant unitary product on ΓL_χ, antilinear in 1^{st} and linear in 2^{nd} argument, is denoted $\langle | \rangle$.

Given a (holomorphic) section $s \in \Gamma L_\chi$ and a nonzero point q in some fiber ${p_L}^{-1}(x)$

$$s(x) = s(p_L(q)) = l_q(s) q,$$

for some number $l_q(s)$. The correspondence

$$s \mapsto l_q(s), \qquad l_q : \Gamma L_\chi \to \mathbb{C},$$

is a continuous linear functional. By Riesz's theorem, there is a unique element

$$e_q \in \Gamma L_\chi \text{ such that } \; l_q(s) = \langle e_q | s \rangle.$$

The vectors (sections) of the form $e_q \in V_\chi = \Gamma L_\chi$ are called **coherent vectors.** Their projective classes are called **coherent states.** All this much more general (Rawnsley).

Proposition. (Rawnsley)

- (i) $e_{gq} = g_* e_q$ for all $g \in G^{\mathbb{C}}$.
- (ii) $e_{cq} = \bar{c}^{-1} e_q$ for all $c \in \mathbb{C}$.
- (iii) Coherent states i.e. the projective classes of all coherent vectors belong to the same projective orbit.
- (iv) The set of all e_q where $q \in (p_L)^{-1}(1_G B)$ agrees with the set (ray) of all heighest weight vectors in V_χ for fixed B.
- (v) The set of all e_q where $q \in (p_L)^{-1}(u)$ for fixed $u \in G^{\mathbb{C}}/B$ is the heighest weight space for some subgroup of $G^{\mathbb{C}}$ conjugated to B.

Corollary. Fix open $\mathrm{U} \subset \mathrm{G}^{\mathbb{C}}/\mathrm{B}$, $\mathrm{q} \in \mathrm{L}_\chi$ and $\mathrm{t} : \mathrm{U} \to \mathrm{G}^{\mathbb{C}}$ a section of $\mathrm{p}^{-1}(\mathrm{U}) \to \mathrm{U}$ (principal B-bundle).
For each $\mathrm{g} \in \mathrm{p}^{-1}(\mathrm{U})$ $\exists!\mathrm{b} \in \mathrm{B}$ such that $\mathrm{g} = \mathrm{t}(\mathrm{gB})\mathrm{b}$.
The “homogeneity” formula holds:

$$\mathrm{g}\mathrm{e}_\mathrm{q} = \chi^{-1}(\mathrm{b})\mathrm{e}_{\mathrm{t}(\mathrm{gB})\mathrm{q}} \tag{1}$$

Proof. $\mathrm{g}_*\mathrm{e}_\mathrm{q} = \mathrm{t}(\mathrm{gB})_*\mathrm{b}_*\mathrm{e}_\mathrm{q} = \mathrm{t}(\mathrm{gB})_*\chi^{-1}(\mathrm{b})\mathrm{e}_\mathrm{q}$; taking into account that $\chi^{-1}(\mathrm{b})$ is a scalar, this equals to $\chi^{-1}(\mathrm{b})\mathrm{t}(\mathrm{gB})_*\mathrm{e}_\mathrm{q}$, hence by *(i)* of the Proposition, also to $\chi^{-1}(\mathrm{b})\mathrm{e}_{\mathrm{t}(\mathrm{gB})\mathrm{q}}$.

Definition. The **local family of coherent states** corresponding to the triple (U, t, q) is the map

$$C_{(U,t,q)} : U \to V_\chi \equiv \Gamma L_\chi, \qquad C_{(U,t,q)} : [g] \mapsto e_{t([g])q}. \qquad (2)$$

Let W be the Weyl group of G. Gauss decomposition singles out special (U_w, t_w, q) which will have a generalization for quantum groups.
For any $w \in W$, there is a Zariski open subset $G^{\mathbb{C}}_w \subset G^{\mathbb{C}}$ consisting of all $g \in G^{\mathbb{C}}$ for which $\exists!$ decomposition $g = wyb$ where $y \in G^{\mathbb{C}}$ belongs to the unipotent subgroup of the opposite Borel B', and $b \in B$.

$G_w^{\mathbb{C}}$ is also B-invariant, hence a total space of the restricted fibration over a Zariski open subset $U_w = G_w^{\mathbb{C}}/B \subset G^{\mathbb{C}}/B$. Define the local section $t_w : G_w^{\mathbb{C}}/B \to G_w^{\mathbb{C}} \subset G^{\mathbb{C}}$ by $t_w([g]) = wy$ where $g = wyb$ as above. We denote

$$C_w := C_{(w,v_0)} := C_{(G_w^{\mathbb{C}}/B,t_w,q)}$$

where $v_0 = e_q$ is a fixed highest weight vector in V_χ.
Corollary above becomes
Proposition. If $g = wyb$ is the Gauss decomposition in G_w then for all $g \in G$

$$gv_0 = \chi^{-1}(b)C_w(gB), \tag{3}$$

and C_w is the unique element in $\mathcal{O}(G_w^{\mathbb{C}}/B) \otimes V_\chi$ for which this holds.

The collection of maps $\{C_w, w \in W\}$ will be generalized to the quantum group setting below. They can be viewed as $C_w \in \mathcal{O}(G^{\mathbb{C}}_w/B) \otimes V_\chi$ where $\mathcal{O}(G^{\mathbb{C}}_w/B)$ is the complex algebra of all algebraic functions on $G^{\mathbb{C}}_w/B$.

Outline

Basic idea of ncg

- A space and its geometry is determined by a (sufficient) collection of objects which live on the space.
- Objects: functions (observables), bundles, sheaves, stacks...
- Organize into: (operator) algebras, algebras/spaces of cocycles, categories, higher categories
- Space-algebra duality is of spectral nature

Dualities: examples of the idea

- **Gel'fand-Neimark theorem**: The category of compact Hausdorff spaces is antiequivalent to the category of commutative unital C^*-algebras (evaluation functionals, maximal ideals, spectral theory of Banach algebras)
- **Giraud's theorem**: Category satisfying the Giraud's axioms is a category of sheaves on a Grothendieck site the (evaluation presheaves, Yoneda arguments, using generating sets)
- **Gabriel-Rosenberg theorem**: A (quasicompact quasiseparated) algebraic scheme $(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is determined by the abelian category of quasicoherent sheaves of $\mathcal{O}_\mathrm{X}$-modules (abelian sheaves and the spectra of categories, almost minimal topologizing subcategories)
- Serre-Swan, Stone, Tannaka, Bondal-Orlov, Isbell...

Outline

Multiplication and comultiplication

Functions multiply and add **pointwise**, therefore functions on a space form a commutative algebra. In ncg we give up noncommutativity. All morphisms dualize.
In particular, if a space X is replaced by a function algebra $\mathrm{Fun}(\mathrm{X})$, then a group is a function algebra with additional **comultiplication** Δ, dual to the operation on the group: $\Delta(\mathrm{f})(\mathrm{x}\otimes\mathrm{y}) = \mathrm{f}(\mathrm{x}\cdot\mathrm{y})$; $\mathrm{f}\in\mathrm{Fun}(\mathrm{X}\times\mathrm{X})\cong\mathrm{Fun}(\mathrm{X})\hat{\otimes}\mathrm{Fun}(\mathrm{X})$.

Bialgebras

Bialgebra B – associative algebra $(\mathrm{B}, \mathrm{m}, \eta)$ and a coalgebra: has comultiplication $\Delta : \mathrm{H} \to \mathrm{H} \otimes \mathrm{H}$ which is coassociative

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and counital: $\exists \epsilon : \mathrm{H} \to \mathbb{C}$,

$$(\epsilon \otimes \mathrm{id}) \circ \Delta \cong \mathrm{id} \cong (\mathrm{id} \otimes \epsilon) \circ \Delta.$$

Compatibility: Δ, ϵ homomorphisms of algebras.
In the case of a group $\mathrm{X} = \mathrm{G}$, $\epsilon(\mathrm{f}) = \mathrm{f}(1_{\mathrm{G}})$, $\mathrm{f} \in \mathrm{Fun}(\mathrm{G})$.

Sweedler notation

Sweedler notation: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

$$\sum \sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}$$

so we write simply

$$\sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

“only the order matters”

Hopf algebra of functions

A **Hopf algebra** is a bialgebra $(B, m, \eta, \Delta, \epsilon)$ with an antipode map $S : B \to B^{op}$,

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon = m \circ (id \otimes S) \circ \Delta.$$

For a group G, $(Sf)(g) = f(g^{-1})$, $g \in G$, $f \in Fun(G)$

(Co)modules

Algebras have actions, **modules**: $\nu : A \otimes M \to M$.
Coalgebras have coactions, **comodules**: $\rho : M \to C \otimes M$.
Extend Sweedler to $\rho(m) = \sum m_{(-1)} \otimes m_{(0)}$.
Modules over bialgebras have a **tensor product**:
$\nu(a, m \otimes n) = \sum \nu_M(a_{(1)}, m) \otimes \nu_N(a_{(2)}, n)$; dually comodules over bialgebras have a tensor product. Over Hopf algebras we also have duals (via antipode).
In physics, comultiplication so that the Hilbert space of **multiparticle state** inherits symmetry via tensor product of representations and quantum numbers appropriately "add".

Comodule algebras as quantum spaces

- X space, G group, then an action $\mathrm{G} \times \mathrm{X} \to \mathrm{X}$ dualizes to a coaction $\rho : \mathrm{Fun}(\mathrm{X}) \to \mathrm{Fun}(\mathrm{G} \times \mathrm{X}) \cong \mathrm{Fun}(\mathrm{G})\hat{\otimes}\mathrm{Fun}(\mathrm{X})$.
- Multiplication is moreover an algebra map ! We say that $\mathrm{Fun}(\mathrm{G})$ is a left comodule algebra over $\mathrm{Fun}(\mathrm{G})$.
- Similarly we think of noncommutative left and right comodule algebras over Hopf algebras as **quantum** G**-space**s.

Coinvariants

If a discrete group G acts on a set X from the right then the function on X/G are in obvious 1-1 correspondence with G-invariant functions on X, that is $\mathrm{f}(\mathrm{xg}) = \mathrm{f}(\mathrm{x})$.
In terms of dual coaction, $\rho(\mathrm{x}) = \mathrm{x} \otimes 1$.
x is a ρ-**coinvariant** if this equality holds. Thus we think of coinvariants as functions on the orbits. The coinvariants of a comodule M form a submodule $\mathrm{M}^{\mathrm{coH}}$.

Outline

Classical principal bundles

- Principal G-bundle is a right G-space for which the map

$$\mathrm{X} \times \mathrm{G} \to \mathrm{X} \times_{\mathrm{X}/\mathrm{G}} \mathrm{X}, \quad (\mathrm{x}, \mathrm{g}) \mapsto (\mathrm{x}, \mathrm{xg})$$

  is a homeomorphism. Its inverse is a translation or division map τ.
- In algebraic case, the local triviality of bundle X (in faithfully flat topology) follows if we assume in addition that X is fathfully flat and locally of finite type over X/G.
- If G is affine algebraic group and X affine G-variety, we can dualize this to Hopf algebra $\mathrm{H} = \mathrm{Fun}(\mathrm{G})$ and $\mathrm{E} = \mathrm{Fun}(\mathrm{X})$.

Hopf-Galois extensions

- For a Hopf algebra H, a Hopf-Galois extension $\mathrm{U} := \mathrm{E}^{\mathrm{coH}} \hookrightarrow \mathrm{E}$ is a right comodule algebra E for which the canonical map

$$\mathrm{E} \otimes_{\mathrm{U}} \mathrm{E} \to \mathrm{E} \otimes \mathrm{H}, \quad \mathrm{e} \otimes \mathrm{e}' \mapsto \sum \mathrm{e}\mathrm{e}'_{(0)} \otimes \mathrm{e}'_{(1)},$$

  is a vector space isomorphism. This is dual to principal bundle condition (affine case).
- $\mathrm{G}^{\mathbb{C}}/\mathrm{B}$ is a **projective** variety so the bundles over it are *not* affine. Therefore we need to **globalize Hopf-Galois extensions** to do bundles over quantum analogues.

Aside: C^*-principal bundles

In approach via operator algebras, every space is determined by a single C^*-algebra of functions, but no good principal bundle theory is developed (unlike vector bundles which are just projective modules). In addition our local trivializations involve unbounded elements, what is technical to deal with. Our bundles will be naturally algebraic so we take advantage of this. Price: the space is not determined by a single algebra but we need to glue local coordinate patches using category theory and localization functors!

Localization of rings

Commutative algebraic geometry: given ring R, the base of Zariski topology on *Spec* R given by principal localizations of the ring. Ring R, $f \in R$, ring of fractions $R[f^{-1}]$ consisting of equivalence classes of pairs (r, f) viewed as fractions r/f. For a noncommutative ring R this works mostly only when we invert very special, (say left) Ore sets S:

$$\forall s \in S', \forall r \in R\, \exists s' \in S\, \exists r' \in R, \;\; s'r = r's$$

Localization functor

Each left Ore localization of a ring $R \hookrightarrow R[S^{-1}]$, induces the extension of scalars, the **localization functor**

$$Q^* = R[S^{-1}] \otimes_R - : {}_R Mod \to {}_{R[S^{-1}]} Mod$$

which is an exact functor having a fully faithful and exact right adjoint Q_* (restriction of scalars, forgetful functor).
In algebraic geometry, modules over a ring are identifies with quasicoherent sheaves of modules over corresponding affine scheme. More general exact functors on abelian categories having fully faithful right adjoints will be viewed as restriction functors for quasicoherent sheaves on noncommutative schemes.

Noncommutative schemes

- Let A be an abelian category and $Q^*_\lambda : A \to A_\lambda$, $\lambda \in \Lambda$ a family of exact localization functors where each Q^*_Λ has a right adjoint functor $Q_{\lambda *} : A_\lambda \to A$; we say that the family is a **cover** if it is **conservative** (morphism f invertible iff each Q^*_λ invertible)
- A is representing an **affine scheme** if it is of the form ${}_R Mod$ for some ring R; thus we identify rings and (spectra of) their categories of modules
- A is $Qcoh_X$ for a **noncommutative scheme** X if it has a cover by localizations $Q^*_\lambda : A \to A_\lambda$ where each A_λ is affine (+ subtle conditions by Rosenberg)

Noncommutative G-schemes

- Q: What is an action of a Hopf algebra H on noncommutative scheme X ?
- A: Action of the corresponding monoidal category (of H-modules) on $Qcoh_X$ which is *admissible* (subtle condition of Z.Š)
- Admissibility follows if it is true over an affine cover: there we can express everything by comodule algebras (not categories).

Noncommutative G-torsors

- Q: What is a principal bundle in the setup of noncommutative schemes ?
- A: Very complicated (Z.Š, G. Böhm, Rosenberg, unfinished) but it satisfies the descent along torsors: equivariant sheaves over the total space are sheaves over the base space *and* the base space embeds affinely. This is a categorification of the condition that functions on the quotient X/G correspond to coinvariants in the algebra of functions on G-space X.
- Easy: In any case, it is Hopf-Galois in coordinate patches obtained by localization functors **if the action restricts to those.**

- The notion of equivariant sheaves extends to Hopf algebras and noncomm. geometry (Lunts, Škoda 2002, 2008).
- Locally equivariant sheaves are equivalent to relative Hopf modules over a H-comodule algebra E: left E action and right H-coaction compatible. Then for a faithfully flat Hopf-Galois extension ${}_E\mathcal{M}^H \cong {}_{E^{coH}}\mathcal{M}$ (Schneider's theorem, 1991) – this is a special case of the descent along globalized torsors (G. Böhm, Z. Š) !

Trivial principal bundles

- (Classical topology) If $t : Y \to X$ is a global section of a principal G-bundle $p : X \to Y$ then

$$\gamma_t(f)(y) = f(\tau(t(p(y))), y))$$

  defines a map of $C(G)$-comodule algebras $C(G) \to C(Y)$.
- We say that a H-comodule algebra E is **cleft** if there is comodule map $\gamma : H \to E$, and trivial if γ is in addition an algebra map. These are isomorphic to Hopf algebraic semidirect (smash) product algebras.

Locally trivial principal bundles

- (Classical topology) A principal bundle is locally trivial if there is a cover of the base by the opens over which there are sections.
- An Ore localization of an H-comodule algebra $E \to E[S^{-1}]$ is **coaction compatible** if coaction ρ extends to a comodule algebra action ρ_S

$$\begin{array}{ccc} E & \xrightarrow{\rho} & E \otimes H \\ \downarrow & & \downarrow \\ E[S^{-1}] & \xrightarrow{\rho_S} & E[S^{-1}] \otimes H \end{array}$$

Then we can talk about **localized coinvariants** of ρ in $E[S_i^{-1}]$. Locally trivial if $E[S^{-1}]$ is trivial over $E[S^{-1}]^{coH}$ for some cover by Ore sets S_i.

Gluing modules

Localization functors $Q_\lambda^* : A \to A_\lambda$ analogues of the restrictions to open sets U_λ. When a family of objects in A_λ-s corresponds to one object in A ? This can be answered by descent theory: objects in A_λ and A_μ need to be compatible in both consecutive localizations with the help of transition cocycles. This is the hard part of in noncommutative case which we skip, except that the global sections of an associated line bundle are

$$\Gamma L_\chi \cong \left\{ f = \prod_\lambda f_\lambda \mid f_\lambda \gamma_\lambda(\chi) = f_\mu \gamma_\mu(\chi) \forall \lambda, \mu, \ \textit{in} \ \begin{array}{c} Q_\lambda^* Q_{\mu *} E_\mu \\ \textit{and} \ Q_\mu^* Q_{\lambda *} E_\lambda \end{array} \right\}$$

Outline

Idea

- The orbit of heighest weight vector v_0 is $\{gv_0, g \in G\}$. Dualize actions to coaction to express the orbit in those terms. For this we need to localize to an affine open set in the orbit, hence localize.
- We know how the sections of line bundle look like noncommutatively
- We know the classical case

$$gv_0 = \chi^{-1}(b) C_w(gB),$$

in Gauss decomposition. Therefore we ask to have in some local trivialization by localizations,

$$\rho_w v_\chi = C_w \gamma_w(\chi)$$

where C_w is in $V \otimes \mathcal{G}_w^{co\mathcal{B}}$. Here $\pi : \mathcal{G} \to \mathcal{B}$ surjective epimorphism of Hopf algebras which are viewed as

Details

• (D1) A surjective map of Hopf $*$-algebras $\pi : \mathcal{G} \to \mathcal{B}$.
• (D2) A group–like element $\chi \in \mathcal{B}$ ($\Delta(\chi) = \chi \otimes \chi$). Define $V = V_\chi = Ind_{\mathcal{B}}^{\mathcal{G}}\mathbb{C} = \mathbb{C}_\chi \Box^{\mathcal{B}} \mathcal{G}$ (cotensor product, think here of holomorphically induced representation)
• (D3) A coinvariant inner product $\langle|\rangle$ on V_χ, i.e.

$$\langle v|z\rangle 1_H = \sum \langle v_{(0)}|z_{(0)}\rangle z_{(1)} v_{(1)}^*$$

• (D4) A weight covector $v_\chi \in V_\chi$ ($(id \otimes \pi)\rho v_\chi = v_\chi \otimes \chi$) with norm 1.
• (D5) A Zariski local trivialization $\{w = (\iota_w, \mathcal{G}[S_w^{-1}], \gamma_w)\}_{w \in W}$ (W some index set, in our examples Weyl group) of $\mathcal{G}$ as a right $\mathcal{B}$-comodule algebra.

Haar integral

A **left-invariant integral** (= left Haar integral) on a Hopf algebra H is a linear functional $\int$ on H such that

$$\langle h \otimes \int, \Delta(f)\rangle = \langle h, 1\rangle\langle \int, f\rangle, \qquad \forall h \in H^*.$$

A left Haar integral $\int$ is **normalized** if $\langle \int, 1\rangle = 1$.
Theorem. *Let $\int$ be a left integral on a Hopf $*$-algebra H, and $(V, \rho, \langle,\rangle)$ a simple unitary right H-comodule. Fix a vector $w \in V$. Define the operator $A : V \to V$ by*

$$A|v\rangle = \sum \langle w_{(0)}|v\rangle w_{(0)'} \int w^*_{(1)} w_{(1)'}$$

Then A is a scalar operator.

Resolution of unity

Define the vertical measure by $d\mu_\chi = \gamma_\lambda(\chi)(\gamma_\lambda(\chi))^*$.
Theorem. $C_\lambda d\mu_\lambda(\chi) C_\lambda^*$ does not depend on λ hence it defines a well-defined element in $\mathcal{G} \otimes End\, V_\chi$. The Haar integral

$$\alpha = \int_{\mathcal{G}} C_\lambda d\mu_\lambda(\chi) C_\lambda^*$$

is a scalar operator. Therefore, for $\alpha \neq 0$ and every linear operator $\mathcal{H}$,

$$\mathcal{H}|v\rangle = \alpha^{-1} \int_{\mathcal{G}} H|C_\lambda\rangle d\mu_\lambda(\chi) \langle C_\lambda | v\rangle.$$

Outline

Ordinary linear groups

$M(n, \mathbb{C})$ of $n \times n$ matrices with (commutative) entries in a field $\mathbb{C}$ is isomorphic to $\mathbb{C}^{n^2}$ as a $\mathbb{C}$-vector space. This isomorphism induces a structure of affine $\mathbb{C}$-variety on $M(n, \mathbb{C})$. The regular functions on that variety are polynomials in matrix entries. Introduce n^2 *regular* functions

$$t^i_j : M(n, \mathbb{C}) \to \mathbb{C}, \qquad t^i_j(a) = a^i_j, \quad a \in M(n, \mathbb{C}), \;\; i, j = 1, \ldots n.$$

Then $Fun(M(n, \mathbb{C})) \cong \mathbb{C}[t^1_1, t^1_2, \ldots, t^n_n]$ is the ring of *global regular functions* on $M(n, \mathbb{C})$. If we divide this by the ideal $\langle det\, T - 1 \rangle$ we get $Fun(SL(n, \mathbb{C}))$.

Matrix bialgebras

Let $\mathcal{G}$ be a bialgebra, possibly noncommutative, over a field $\mathbb{C}$ and $G = (g^i_j)^{i=1,\ldots,n}_{j=1,\ldots,n}$ an $n \times n$-matrix over $\mathcal{G}$. $\mathcal{G}$ is a **matrix bialgebra** with basis G if the set of entries of G generates $\mathcal{G}$ and if the comultiplication Δ and counit ϵ satisfy

$$\Delta G = G \otimes G \quad \text{i.e.} \quad \Delta g^i_j = \sum_{k=1}^n g^i_k \otimes g^k_j$$

$$\epsilon G = \mathbf{1} \quad \text{i.e.} \quad \epsilon(g^i_j) = \delta^i_j$$

A **matrix Hopf algebra** $\mathcal{G}$ with basis $T = (t^i_j)$ is a Hopf algebra which possess with matrix subbialgebra B with basis T such that the Hopf envelope map $H(id) : H(B) \to \mathcal{G}$ is onto.

Quantum matrix bialgebra

Let $q \in \mathbb{C}, q \neq 0$. The quantum matrix bialgebra $\mathcal{M}_q(n, \mathbb{C}) = \mathcal{O}(M_q(n, \mathbb{C}))$ is the free matrix bialgebra $\mathcal{NM}(n, \mathbb{C})$ with basis $T = (t^\alpha_\beta)$ modulo the smallest biideal I such that the following relations hold in quotient:

$\alpha = \beta, \ \gamma < \delta$ (same row)	$t^\alpha_\gamma t^\alpha_\delta = q t^\alpha_\delta t^\alpha_\gamma$
$\alpha < \beta, \ \gamma = \delta$ (same column)	$t^\alpha_\gamma t^\beta_\gamma = q t^\beta_\gamma t^\alpha_\gamma$
$\alpha < \beta$ and $\gamma < \delta$	$t^\alpha_\gamma t^\beta_\delta - t^\beta_\delta t^\alpha_\gamma = (q - q^{-1}) t^\beta_\gamma t^\alpha_\delta$
$\alpha < \beta$ and $\delta < \gamma$	$t^\alpha_\gamma t^\beta_\delta = t^\beta_\delta t^\alpha_\gamma$

Quantum determinant

The **quantum determinant** $D \in \mathcal{M}_q(n, \mathbb{C})$ is defined by any of the formulas

$$D = \sum_{\sigma \in \Sigma(n)} (-q)^{l(\sigma)-l(\tau)} t^{\tau(1)}_{\sigma(1)} t^{\tau(2)}_{\sigma(2)} \cdots t^{\tau(n)}_{\sigma(n)}$$

and is central element. Analogously quantum minors (not central).
Quantum special linear group:

$$\mathcal{G} = SL_q(n) = M_q(n)/\langle D - 1 \rangle$$

Quantum Borel subgroup

$$\mathcal{B} = SL_q(n)/\langle t^i_j, i < j \rangle$$

Let $b^i_j = t^i_j + I$ be the generators of $\mathcal{B}$. Both are matrix Hopf algebras.

Outline

Quantum Borel subgroup coacts on $SL_q(n)$ from the right

$$\rho_{\mathcal{B}} : SL_q(n) \to SL_q(n) \otimes \mathcal{B}$$

For every permutation w permute permute the rows of T and look at the multiplicative set S_w generated by the principal (right lower corner) quantum minors.

Theorem. (hard) S_w is an Ore set for every w.
(easy) The corresponding localization is compatible with natural coaction $\rho_{\mathcal{B}}$.
(easy for some q) These Ore localizations cover $\mathcal{G}$.
(easy) There is a unique quantum Gauss decomposition $T = wU_wA_w$ where w is a permutation matrix, U_w is a matrix whose entries are localized coinvariants of $\mathcal{G}[S_w^{-1}]$. Elements of U_w and A_w generate $\mathcal{G}[S_w^{-1}]$.
(hard) $\gamma_w : b^i_j \mapsto (A_w)^i_j$ is a map of $\mathcal{B}$-comodule algebras $\mathcal{B} \to \mathcal{G}[S_w^{-1}]$. Hence we have the locally trivial principal $\mathcal{B}$-bundle.

Outline

$SU_q(2)$

$SL_q(2)$ is a noncommutative Hopf algebra over $\mathbb{C}$ with 4 generators a, b, c, d, usually assembled in a matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with relations

$$ab = qba,\ ac = qca,\ bc = cb,\ bd = qdb,\ cd = qdc,$$

$$ad - da = (q - q^{-1})bc,\ det_q T := ad - qbc = 1.$$

$SU_q(2)$ is a real form of $SL_q(2)$ determined by formulas

$$a^* = d,\ \ b^* = -qc,\ \ c^* = -q^{-1}b,\ \ d^* = a.$$

Haar integral on $SU_q(2)$

A vector space basis of $SL_q(2)$ is

$$\{a^k b^r c^s\}_{k>0,r,s\geq 0} \cup \{b^r c^s d^t\}_{r,s,t\geq 0}.$$

$SL_q(2)$ splits into a direct sum $\mathbb{C}[\zeta] \oplus compl(\zeta)$ where $\mathbb{C}[\zeta]$ is the span of the basis elements of the form $(bc)^r$ and $compl(\zeta)$ the span of the rest of basis.
$SU_q(2)$ posses a unique Haar functional $\int$, found by WORONOWICZ. With respect to the direct sum decomposition above, $\int$ is nontrivial only on $\mathbb{C}[\zeta]$ where it is given by formulas involving JACKSON's q-integral, or equivalently

$$\int \zeta^r = \frac{1-q^{-2}}{1-q^{-2(r+1)}}, \quad r = 0, 1, 2, \ldots$$

Manin plane

Manin plane

$\mathbb{C}^2_q$ is an algebra with two generators x, y and a single relation $xy = qyx$. Elements of the form $x^r y^s$ form a basis of $\mathbb{C}^2_q$. The latter is a right $SL_q(2)$-comodule algebra via

$$\rho(x^r y^s) = (x \otimes a + y \otimes c)^r (x \otimes b + y \otimes d)^s.$$

$\mathbb{C}^2_q$ splits into the homogeneous components $V_n = \oplus_{r+s=n} \mathbb{C} x^r y^s$ of dimension $n+1$, which are irreducible and unitary. We will find CS there by decomposing the coaction at heighest weight $r = 0$. The Weyl group of SL_2 has two elements corresponding in q-case to the 2 charts or localizations $SL_q(2)[b^{-1}]$ and $SL_q(2)[d^{-1}]$.

The character is $\chi = \lambda^{-n}$ where $\lambda = a + I$ in Borel. From the Gauss decomposition, $\gamma_d(\lambda) = a - bd^{-1}c$, $\gamma_d(\lambda^{-1}) = d, \gamma_d(c + I) = c$. Recall the condition

$$(\rho_{\mathcal{B}})_d v_\chi = C_d \gamma_d(\chi)$$

In the chart $SL_q(2)[d^{-1}]$,

$$C_d := \sum_{i=0}^{n} q^{-\binom{i}{2}} \sqrt{\begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}}} v_i^n \otimes u^i,$$

where $v_i^n = \sqrt{\begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}}} x^i y^{n-i}$ are orthonormal and $u = bd^{-1}$ is the coordinate in (in fact the generator of) $SL_q(2)[d^{-1}]^{co\mathcal{B}}$.

Resolution of unity

Using calculations with basis in $SU_q(2)$, and the expression for Ramanujan's q-beta function

$$\int_0^1 x^\alpha \frac{(qx;q)_\infty}{(q^\beta x;q)_\infty} d_q x = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

one obtains

$$\int_{SU_q(2)} u^i d^n (u^j d^n)^* = \begin{cases} 0, & i \neq j \\ \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}}^{-1} q^n q^{2\binom{i}{2}} [n+1]_q^{-1}, & i = j \end{cases}$$

This implies the normalized resolution of unity formula

$$Id = q^{-n}[n+1]_q \int_{SU_q(2)} |C\rangle d\mu(\chi) \langle C|.$$

Outline

- Covariant minimal uncertainty relations? Cf. Delbourgo; connection to quantum moment map (and Spera's work).
- Extend the examples to Woronowicz quantum groups (the Hopf algebras here are dense subalgebras).
- For noncompact quantum groups, probably we need operator algebraic framework to introduce the resolution of the unity (or other replacement for Haar integral).
- Apply back WZNW setup. How to deal with root of unity problems ?
- The connections to the approach by Jurčo and Števiček (avoiding quantum coset space).
- Concrete computations: $SU_q(n)$-resolution of unity
- Quasiclassical precursors in Poisson geometry (deformation of usual coherent states along given r-matrix)?

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