Gagliardo-Nirenberg-Sobolev Inequalities

and their counterparts on bounded domains

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RB, C. Vallejos and H. Van Den Bosch, *Gagliardo–Nirenberg–Sobolev Inequalities for convex domains in* $\mathbb{R}^d$, Mathematics Research Letters **26** (2019) 1291–1312.

RB, C. Vallejos and H. Van Den Bosch, *Existence and non–existence of minimizers for Poincaré Sobolev inequalities*, Calculus of Variations and Partial Differential Equations **59** (2020) 1–19.

RB, Soledad Benguria, *In preparation*, (2021).

Summary

i) Gagliardo-Nirenberg-Sobolev Inequalities, an overview.

ii) Related inequalities on bounded domains with Dirichlet Boundary conditions. The Brezis-Nirenberg problem.

iii) Related inequalities on bounded domains with Neumann Boundary conditions.

Gagliardo-Nirenberg-Sobolev Inequalities, an overview.

Sobolev Inequality

$$\|u\|_q \leq C\|\nabla u\|_p$$

with $q = np/(n-p)$, $1 < p < n$.
S. L. Sobolev, *On a theorem of functional analysis*, Math. Sb. **4** (1938).

The best constants were determined independently by Aubin and Talenti in 1976.

$$C = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{(p-1)/p} \left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right]^{1/n},$$

and the minimizers are given by

$$u(x) = \left(a + b|x - x_0|^{p/(p-1)}\right)^{1-n/p}.$$

In the particular case $p = 2$, one has,

$$\frac{1}{C^2} = \pi\, n(n-2)\left(\Gamma(n/2)/\Gamma(n)\right)^{2/n}.$$

Gagliardo–Nirenberg Inequalities

$$\|u\|_q \leq C\|\nabla u\|_p^{\alpha}\|u\|_r^{1-\alpha}.$$

Here,

$$\frac{1}{q} = \left(\frac{1}{p} - \frac{1}{n}\right)\alpha + \frac{1-\alpha}{r}$$

and $1 < p < n$, $0 \leq \alpha \leq 1$

When $\alpha = 1$ the GN inequalities reduce to the Sobolev inequality.

There is a recent reference with a simple proof of the GN inequalities and very interesting historical remarks:

A. Firenza, *et al*, *Detailed proof of the classical GN interpolation inequalities with historical remarks*, Zeitschrift für Analysis und ihre Anwendungen (2021).

Gagliardo Nirenberg (GN) inequalities

The Gagliardo-Nirenberg Inequalities were found independently by Emilio Gagliardo (1930-2008) and Louis Nirenberg (1925-2020) in 1958. They were presented as unpublished short communications (section III) at the 1958 ICM in Edinburgh:

E. Gagliardo, *Propriétés de certaines classes de fonctions de n variables*, and

L. Nirenberg, *Inequalities for derivatives.*

The first time the GN inequalities appeared in print was in,

E. Gagliardo, *Ulteriore proprietá di alcune classi di funzioni in più variabili*, Ricerche Mat. **8** (1951), 24–51, and,

L. Nirenberg, *On Elliptic Partial Differential Equations*, Ann. Scuola Norm. Sup. Pisa **13** (1959), 115-162.

In one dimension there were previous inequalities of this type by J. Hadamard (1897) and B. Nagy (1941).

Ladýzhenskaya's inequalities

The particular case $q = 4$, $p = 2$, $n = 2$, $\alpha = 1/2$, i.e.,

$$\|u\|_4 \leq C\|\nabla u\|_2^{1/2}\|u\|_2^{1/2},$$

($n = 2$), was proved in (1958) by Olga Ladýzhenskaya (1922-2004), on her article *Solution in the large to the boundary value problem for the Navier-Stokes equations in* 2 *space variables*, Soviet Phys. Dokl. **123** (1958), 1128–1131.

The cases

$$\|u\|_4 \leq C\|\nabla u\|_2^{3/4}\|u\|_2^{1/4},$$

in dimension 3, and

$$\|u\|_{2s} \leq C\|\nabla u\|_s^{1/2}\|u\|_s^{1/2},$$

in dimension 2 are also known as Ladýzhenskaya's inequalities.

Best constants for the Gagliardo Nirenberg (GN) inequalities

The best constants for the GN inequalities are only known in three cases:

i) In $n = 1$, with $q = 6$, $p = 2$ and $r = 2$, and $\alpha = 1/3$ we have,

$$\frac{\pi^2}{4} \int_{\mathbb{R}} u^6 \, dx \leq \int_{\mathbb{R}} u'^2 \, dx \left(\int_{\mathbb{R}} u^2 \, dx \right)^2 .$$

ii) For $n \geq 3$, and $\alpha = 1$, (i.e., in the Sobolev case).

iii) For $n \geq 3$, with $p = 2$, $q = 2t$, $r = t + 1$, $1 < t \leq n/(n-2)$.

M. Del Pino and J. Dolbeault, *Best constants for GN inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. **81** (2002), 847-875.

In this case the optimizers are given by

$$u_{a,x_0} = (1 + |a(x - x_0)|^2)^{-1/(t-1)}.$$

Indeed, case iii) includes case ii) which corresponds to $t = n/(n-2)$.

A particular class of Gagliardo–Nirenberg Inequalities

In what follows we will be particularly interested in the subclass of GN inequalities given by

$$\|u\|_{\rho+2} \leq k(\rho, n)\|\nabla u\|_2^{\alpha}\|u\|_2^{1-\alpha}.$$

i.e., in the GN inequalities with $q = \rho + 2$, and $p = r = 2$. Then, we have

$$\alpha = \frac{n}{2}\frac{\rho}{\rho + 2}.$$

These GN inequalities hold for any $\rho \in (0, \rho_0]$, where $\rho_0 = 4/(n-2)$, if $n \geq 3$, and $\rho = \infty$ if $n = 1, 2$.

These are a particular case of GN inequalities that characterize the embedding of $H^1(\mathbb{R}^n)$ in $L_{\rho+2}(\mathbb{R}^n)$. More specifically we focus on the case $\rho = 4/n$.

As mentioned above, sharp constants are only known in the cases i) $n = 1$, $\rho = 4$, and ii) $n \geq 3$, with $\rho = \rho_0$. Otherwise, the best estimates to date for $k(\rho, n)$ are the ones obtained by S. Nasibov (1989).

A subclass of GNS inequalities.

For reasons that are explained later we consider here the particular case

$$\rho = \frac{4}{n}.$$

Thus, for $n \geq 1$ define

$$G(n) = \inf \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \left(\int_{\mathbb{R}^n} u^2 \, dx \right)^{2/n}}{\int_{\mathbb{R}^n} |u|^{2+4/n} \, dx}, \tag{1}$$

where the infimum is taken over functions $u \in H^1(\mathbb{R}^n)$.

As mentioned above $G(1) = \pi^2/4$, whereas the case $n = 2$ is precisely the case considered by Olga Ladýzhenskaya (1958). The sharp constants for $n \geq 2$ are not known.

Nasibov Estimates, 1989.

In his Ph.D. thesis, Michael Weinstein found a beautiful connection between the GN optimizers and the long time behavior of a NLS equation (CMP 1983). This connection was used by S. Nasibov to obtain the best bounds to date.

$$k(\rho, n) \le k_N(\rho, n) \equiv \frac{1}{\chi}\left(\frac{|\mathbb{S}^{n-1}|\, B(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha})}{2}\right)^{\alpha/n} k_{BB}\left(\frac{\rho+2}{\rho+1}\right).$$

Here,

$$\chi = \sqrt{\alpha^\alpha\,(1-\alpha)^{1-\alpha}},$$

and $B(x, y)$ is the Euler Beta function, i.e., $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Moreover,

$$k_{BB}(p) = \left(\left(\frac{p}{2\pi}\right)^{1/p} \Big/ \left(\frac{p'}{2\pi}\right)^{1/p'}\right)^{n/2},$$

for $1 < p < \infty$ and $1/p + 1/p' = 1$, is the optimal constant for the Hausdorff–Young inequality, as it was proven by Babenko (1961) and Beckner (1975). It follows from the previous discussion that (1) holds for any $u \in H^1(\mathbb{R}^n)$, for $n \ge 1$ and

$$G(n) \ge G_N(n) \equiv \frac{1}{k_N(4/n, n)^{2+4/n}}.$$

More recently, (1) has been also proven using a projection of the Fourier transform of u into high and low energy components, a method inspired on Rumin's techniques (2011). Using these techniques one can prove that

$$G(n) \geq G'(n) \equiv \frac{(2\pi)^2 n^{2+2/n} |\mathbb{S}^{n-1}|^{-2/n}}{(n+2)(n+4)}.$$

A detailed proof and further comments and references can be found as Theorem 4.14 in the recent lecture notes Lundholm (2017) [This is based on previous work by Rupert Frank, and by D. Lundholm and J.P. Solovej]

The optimal constant in (1) satisfies $G(1) = \pi^2/4$, $G(2) = S_{2,4}$ and $G(n) \geq S_n$ for all $n \geq 3$. Here $S_n = n(n-2)|\mathbb{S}^n|^{2/n}/4$ is the optimal constant in Sobolev's inequality

$$\int_{\mathbb{R}^n} (\nabla u)^2 \, dx \geq S_n \|u\|^2_{2n/(n-2)},$$

which holds for all $u \in H^1(\mathbb{R}^n)$, and $n \geq 3$, while $S_{2,4}$ is the optimal constant of the inequality

$$\int_{\mathbb{R}^2} (\nabla u)^2 \, dx \geq S_{2,4} \|u\|_2^{-2} \|u\|_4^4.$$

The value of $S_{2,4}$ is not known, but there are well known lower bounds.

Numerical Values.

n	G_N(n)	$G'(n)$	$G(n)$
1	2.2705	0.6580	2.4674
2	5.3014	2.0944	5.850
3	8.6427	3.9067	9.578
4	12.1605	5.9238	13.489
5	15.7941	8.0619	17.483

Table 1: The values of the second column in this table are the bounds of Nasibov, the values on the third column are the bounds obtained using Rumin's techniques. The fourth column contains the known exact value for $n = 1$, i.e., $\pi^2/4$, whereas the values for $n \geq 2$ are obtained by us through numerical integration of the Euler equations associated with (1)

Motivation: Lieb–Thirring Inequalities (1975).

Consider the Schrödinger equation,

$$-\Delta u - Vu = -eu,$$

acting on $L^2(\mathbb{R}^n)$. Here, $V \geq 0$. The Lieb–Thirring inequalities are given by,

$$\sum_k e_k^\gamma \leq L_{\gamma,n} \int_{\mathbb{R}^n} V^{\gamma+n/2}\, dx.$$

For $n = 1$, they hold for $\gamma \geq 1/2$, for $n = 2$, for $\gamma > 0$, and for $n \geq 3$, for $\gamma \geq 0$. The sharp constants are known for $n = 1$, $\gamma = 1/2$ and for any $n \geq 1$ as long as $\gamma \geq 3/2$.

For $\gamma < 3/2$ it has been conjectured that

$$L_{\gamma,n} = L^1_{\gamma,n},$$

the constants one would get if V has only one bound state.

Motivation: Lieb–Thirring Inequalities (continued).

The problem of maximizing the lowest eigenvalue of the one dimensional Schrödinger operator on the line subject to a constraint on integrals of powers of the potential was first considered by Joseph Keller in 1961. He computed explicitly the constants $L^1_{\gamma,1}$.

In higher dimensions, this problem is connected to the GNS inequalities. In fact, one can show that

$$L^1_{\gamma,1} = \frac{\kappa_1(\gamma)}{k(\rho,n)^{2+n/\gamma}}.$$

Here,

$$\kappa_1(\gamma) = \frac{2\gamma}{n}\left(\frac{n}{2\gamma+n}\right)^{1+n/2\gamma},$$

and

$$\rho = \frac{4}{2(\gamma-1)+n}.$$

For the natural case $\gamma = 1$ one has $\rho = 4/n$ which is the choice we did in (1).

Motivation: Lieb–Thirring Inequalities (continued).

Recently, the Lieb-Thirring conjecture was proven to be wrong in general. See,

R. L. Frank, D. Gontier, and M. Lewin, *The nonlinear Schrödinger equation for orthonormal functions: II. Application to Lieb-Thirring inequalities* , to appear in Communications in Mathematical Physics.

It fails at least when

$$\gamma > \max(0, 2 - n/2).$$

Related inequalities on bounded domains with Dirichlet Boundary conditions. The Brezis-Nirenberg problem.

The Brezis-Nirenberg problem (1983)

Consider the Sobolev quotient with $p = 2$, and $q = 2n/(n-2)$, with $n \geq 3$, and define

$$S(n) = \inf \left[\frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\|u\|_q^2} \right]$$

Where the infimum is taken over functions u such that $\nabla u \in L^2(\mathbb{R}^n)$ and $u \in L^q(\mathbb{R}^n)$. We know that $S(n) = \pi \, n(n-2)(\Gamma(n/2)/\Gamma(n))^{2/n}$ and the minimizers are given by

$$u_\epsilon = \left(\epsilon + |(x - x_0)|^2\right)^{(2-n)/2}.$$

For a bounded domain $\Omega \subset \mathbb{R}^n$ consider

$$S(n, \Omega) = \inf \left[\frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |u|^q \, dx)^{2/q}} \right],$$

defined on $u \in H_0^1(\Omega)$.

The Brezis-Nirenberg problem (1983)

By concentration (i.e., taking the family of minimizers and letting $\epsilon \to 0$) we have that

$$S(n,\Omega) \leq S(n).$$

Using the Brezis-Lieb compactness lemma (1982), if $S(n,\Omega) < S(n)$ there exists a minimizer of $S(n,\Omega)$. By the standard tools of the Calculus of Variations the minimizer if it exists it would satisfy

$$-\Delta u = u^{(n+2)/(n-2)},$$

in Ω, with Dirichlet boundary conditions. Using the Rellich-Pohozaev identity (indeed a virial theorem) one shows that if Ω is star shaped, there is no nontrivial solution.

This was first noticed, for $n = 3$ ($q = 6$), by A. Schuster in 1882, in connection with the Lane-Emden equation (which models polytropes in Astrophysics).

The Brezis–Nirenberg problem on $\mathbb{R}^N$

In 1983 Brezis and Nirenberg considered the nonlinear eigenvalue problem,

$$-\Delta u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$. Among other results, they proved that if $n \geq 4$, there is a positive solution of this problem for all $\lambda \in (0, \lambda_1)$ where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω. They also proved that if $n = 3$, there is a $\mu(\Omega) > 0$ such that for any $\lambda \in (\mu, \lambda_1)$, the nonlinear eigenvalue problem has a positive solution. Moreover, if Ω is a ball, $\mu = \lambda_1/4$.

The Brezis–Nirenberg problem on $\mathbb{R}^N$

For positive radial solutions of this problem in a (unit) ball, one is led to an ODE that still makes sense when n is a real number rather than a natural number.

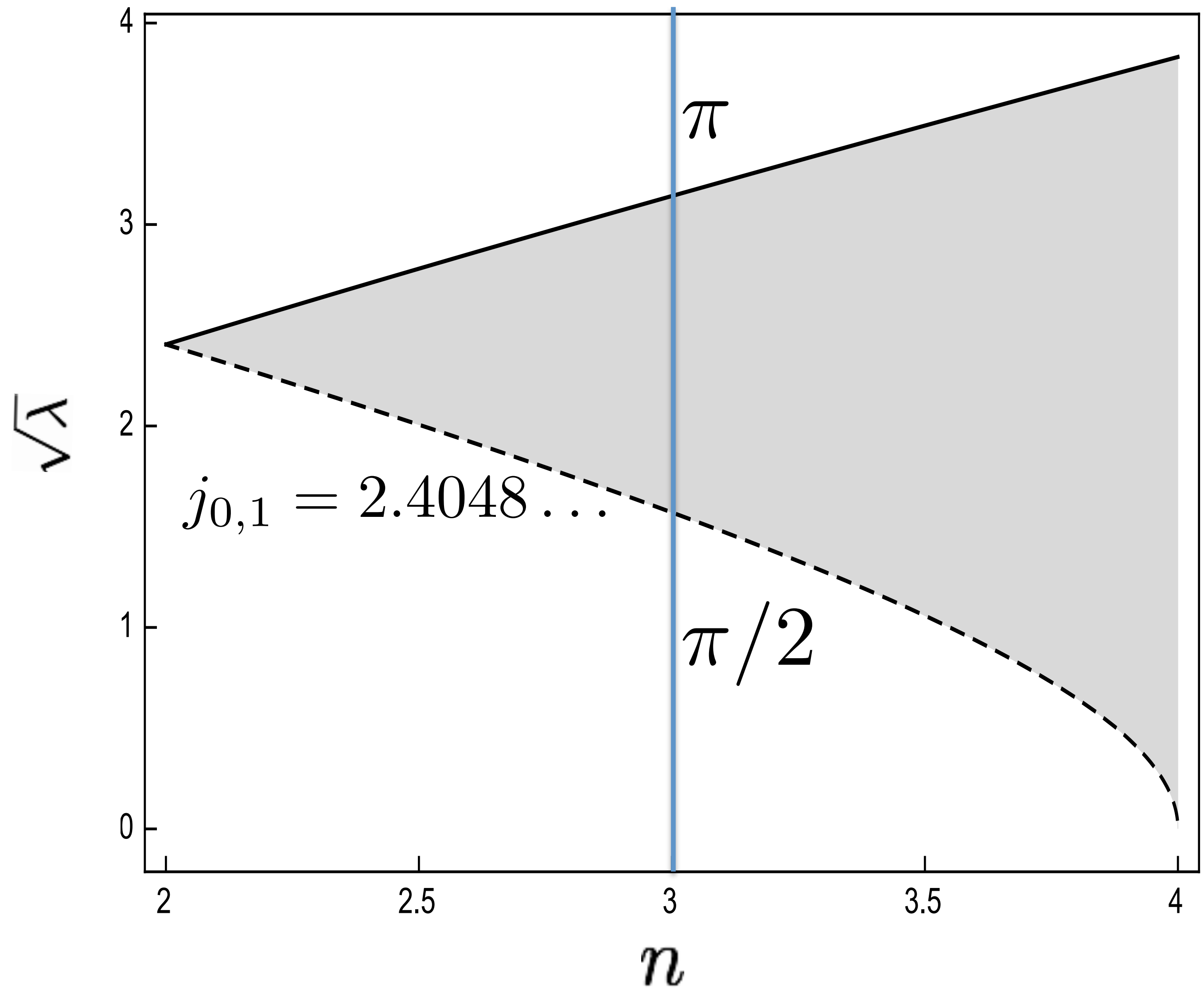
Precisely this problem with $2 \le n \le 4$, was considered by E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations, **156** (1999), pp. 407–426.

Among other things Jannelli proved that this problem has a positive solution if and only if λ is such that

$$j_{-(n-2)/2,1} < \sqrt{\lambda} < j_{+(n-2)/2,1},$$

where $j_{\nu,k}$ denotes the k–th positive zero of the Bessel function J_ν.

The Brezis–Nirenberg problem on $\mathbb{R}^N$



GENERAL HYPERBOLIC CASE

1. $a \in C^3[0, \infty]$;

2. $a''(0) = 0$;

3. $a'(x) > 0$ for all $x > 0$; and

4. $\lim_{x \to 0} \frac{a(x)}{x} = 1$.

Given $n \in (2, 4)$, we study the existence of positive solutions $u \in H_0^1(\Omega)$ of

$$-u''(x) - (n-1)\frac{a'(x)}{a(x)}u'(x) = \lambda u(x) + u(x)^p \tag{1}$$

with boundary condition $u'(0) = u(R) = 0$. Here, as in the original problem, $p = (n+2)/(n-2)$ is the critical Sobolev exponent.

In the Euclidean case $a(x) = x$, in the hyperbolic case, $a(x) = \sinh(x)$.

GENERAL HYPERBOLIC CASE (Existence)

For any $2 < n < 4$ and $0 < R < \infty$ the boundary value problem

$$-u''(x) - (n-1)\frac{a'}{a}u'(x) = \lambda u(x) + u(x)^{\frac{n+2}{n-2}} \tag{1}$$

with $u \in H_0^1(\Omega)$, $u'(0) = u(R) = 0$, and $x \in [0, R]$ has a positive solution if $\lambda \in (\mu_1, \lambda_1)$.

Here, λ_1 is the first positive eigenvalue of

$$y'' + \frac{a'}{a}y' + \left(\lambda - \alpha^2\left(\frac{a'}{a}\right)^2 + \alpha\frac{a''}{a}\right)y = 0 \tag{2}$$

with boundary conditions $\lim_{x\to 0} y(x)x^{\alpha} = 1$. And μ_1 is the first positive eigenvalue of (1) with boundary conditions $\lim_{x\to 0} y(x)x^{-\alpha} = 1$.

GENERAL HYPERBOLIC CASE (nonexistence)

There is no positive solution to problem (1) if $\lambda \geq \lambda_1$, or if $N^* \leq \lambda \leq \mu_1$, where

$$N^* = \sup\left\{\frac{\alpha^2}{a^2}(a'^2 - 1) - \frac{\alpha a''}{a}\right\}.$$

Moreover, then problem (1) has no solution if $\lambda \leq M^*$, where

$$M^* = \inf\left\{\alpha^2 \frac{a''}{a} - \frac{\alpha}{2}\left(\frac{a'''}{a'} + \frac{a''}{a}\right)\right\}.$$

Notice that in the cases that have already been studied, N^* and M^* coincide. In fact, in the Euclidean case, $N^* = M^* = 0$, in the spherical case $N^* = M^* = -n(n-2)/4$, and in the hyperbolic case, $N^* = M^* = n(n-2)/4$.

General Brezis-Nirenberg Problem

Finally, we obtain a uniqueness result under an additional assumption on a. Namely, we show the following:

Theorem [Uniqueness]. Suppose that a satisfies the condition $a'a'' - aa''' \geq 0$ for all $x \in (0, R)$. Then if $\lambda > N^*$, problem (BN) has at most one positive solution.

Remark. In the previously studied cases (i.e, Euclidean, hyperbolic, spherical), we have that $a'a'' - aa''' = 0$ for all x. so the above condition is trivially satisfied.

General Brezis-Nirenberg Problem: Existence

The solutions to (BN) correspond to minimizers of the quotient

$$Q_\lambda(u) = \frac{\omega_n \int_0^R u'^2 a^{n-1}\, dx - \lambda \omega_n \int_0^R u^2 a^{n-1}\, dx}{\left(\omega_n \int_0^R u^{p+1} a^{n-1}\, dx\right)^{\frac{n-2}{n}}} \tag{1}$$

Here, ω_n represents the surface area of the unit sphere in n-dimensions, and is explicitly given by $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}\right)$.

We will begin by showing that there exists a choice of cutoff function φ, with $\varphi(0) = 1$ and $\varphi'(0) = \varphi(R) = 0$, such that if we define

$$u_\epsilon(x) = \frac{\varphi(x)}{(\epsilon + x^2)^{\frac{n-2}{2}}}, \tag{2}$$

then $Q_\lambda(u_\epsilon) < S_n$ if $\lambda > \mu_1$. Here S_n is the Sobolev constant. To do so, we start by obtaining estimates for each of the three integrals in the quotient $Q_\lambda(u_\epsilon)$. We use a classical argument due to Lieb: if $Q_\lambda(u_\epsilon) < S_n$, and $\lambda < \lambda_1$, then there exists a solution to equation (BN).

General Brezis-Nirenberg Problem: Existence

Lemma. There exists a cutoff function φ with $\varphi(0) = 1$ and $\varphi'(0) = \varphi(R) = 0$, such that if we define

$$u_\epsilon(x) = \frac{\varphi(x)}{(\epsilon + x^2)^{\frac{n-2}{2}}}, \tag{1}$$

then $\lambda \geq \mu_1$ implies that $Q_\lambda(u_\epsilon) < S_n$, where S_n is the Sobolev constant.

One gets,

$$Q_\lambda(u_\epsilon) = S_n + \epsilon^{\frac{n-2}{2}} (K_n \omega_n)^{\frac{2-n}{2}} \omega_n T(\varphi) + \mathcal{O}(\epsilon),$$

where

$$T(\varphi) \equiv \int_0^R \frac{\varphi'^2 a^{n-1}}{x^{2n-4}}\, dx + (n-1)(n-2) \int_0^R \frac{\varphi^2 a^{n-1}}{x^{2n-2}} \left(\frac{a'}{a} x - 1 \right) dx - \lambda \int_0^R \frac{\varphi^2 a^{n-1}}{x^{2n-4}}\, dx. \tag{2}$$

Therefore, it suffices to show that there exists a choice of φ for which $T(\varphi)$ is negative when $\lambda > \mu_1$.

General Brezis-Nirenberg Problem: Existence

Making the change of variables $\varphi = y\, x^{n-2} a^{\frac{2-n}{2}}$ we can write the Euler equation associated to $F(\varphi)$ (the first two terms of $T(\varphi)$) as,

$$y'' + \frac{a'}{a} y' + \left(\alpha \frac{a''}{a} - \alpha^2 \left(\frac{a'}{a} \right)^2 + \mu \right) y = 0. \tag{1}$$

Since by hypothesis $\varphi(0) = 1$, and since $\varphi(x) = y(x) x^{n-2} a(x)^{\frac{2-n}{2}}$, it must follow that $\lim_{x\to 0} y(x) x^{-\alpha} = 1$. This is precisely the boundary condition defining μ_1.

General Brezis-Nirenberg Problem: Non Existence

To prove that there are no solutions for $\lambda \geq \lambda_1$ is standard.
To prove that there are no solutions when $N^* \leq \lambda \leq \mu_1$, one uses a Rellich–Pohozaev argument.

Let g be a smooth non-negative function such that $g(0) = g'(0) = 0$ and $g(R) > 0$, and u a positive solution of (BN). Then, one has the following virial theorem:

$$\frac{1}{2} a^{2n-2}(R) u'(R)^2 g(R) = \int_0^R u^2 A \, dx + \int_0^R u^{p+1} B \, dx, \tag{1}$$

General Brezis-Nirenberg Problem: Non Existence

$$\frac{1}{2}a^{2n-2}(R)u'(R)^2g(R) = \int_0^R u^2 A\, dx + \int_0^R u^{p+1} B\, dx, \tag{1}$$

with,

$$A[g] \equiv a^{2n-2}\left[\frac{g'''}{4} + g''\left(\frac{a'}{a}\right)\frac{3(n-1)}{4} + g'\left(\lambda + \frac{(n-1)(2n-3)}{4}\left(\frac{a'}{a}\right)^2 + \frac{(n-1)}{4}\left(\frac{a''}{a}\right)\right) + \lambda(n-1)\left(\frac{a'}{a}\right)g\right]; \tag{2}$$

and

$$B[g] \equiv \frac{1}{2}g'a^{2n-2} + \frac{(ga^{2n-2})'}{p+1}. \tag{3}$$

Our goal is to show there is choice of function g such that $A[g] \equiv 0$, and if $\lambda \le \mu_1$, then $B[g] < 0$. Since the left-hand-side of the first equation is positive, we will obtain a contradiction, thus concluding there are no solutions for values of λ in this range.

General Brezis-Nirenberg Problem: Non-Existence.

The goal is achieved by taking: $g(x) = a^{2-n}(x)y_1(x)y_2(x)$, where y_1 and y_2 are linearly independent solutions to

$$y'' + \left(\frac{a'}{a}\right) y' + ky = 0, \tag{1}$$

where, as before,

$$k = \alpha \left(\frac{a''}{a}\right) - \alpha^2 \left(\frac{a'}{a}\right)^2 + \lambda. \tag{2}$$

Then $A[g] \equiv 0$. and $B[g] < 0$, and we are done.

General Brezis-Nirenberg Problem: Uniqueness

For proving uniqueness when $\mu_1 < \lambda < \lambda_1$ one uses the classical techniques by Kwong an Li (1992).

Related inequalities on bounded domains with Neumann Boundary conditions.

Gagliardo–Nirenberg-Sobolev (GNS), and Poincaré–Sobolev (PS) inequalities.

Define

$$G(n) = \inf \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx \left(\int_{\mathbb{R}^n} u^2\, dx\right)^{2/n}}{\int_{\mathbb{R}^n} |u|^{2+4/n}\, dx}, \tag{1}$$

where the infimum is taken over functions $u \in H^1(\mathbb{R}^n)$. As usual we call the inequality associated to (1) the GNS inequality.

For a bounded domain $\Omega \subset \mathbb{R}^n$, we define

$$G(\Omega, n) = \inf \frac{\int_{\Omega} |\nabla u|^2\, dx \left(\int_{\Omega} u^2\, dx\right)^{2/n}}{\int_{\Omega} |u - u_\Omega|^{2+4/n}\, dx}, \tag{2}$$

where the infimum is taken over functions $u \in W^{1,1}(\Omega)$ and u_Ω is its average $|\Omega|^{-1} \int_\Omega u$. As usual we call the inequality associated to (2) the PS inequality.

Since we are specially interested in the case $\Omega = [0,1]^n$, we will write $G([0,1]^n, n) \equiv G_Q(n)$.

Poincaré–Sobolev (PS) inequalities.

For a bounded domain $\Omega \subset \mathbb{R}^n$, we define

$$G(\Omega, n) = \inf \frac{\int_\Omega |\nabla u|^2 \, dx \left(\int_\Omega u^2 \, dx\right)^{2/n}}{\int_\Omega |u - u_\Omega|^{2+4/n} \, dx}, \tag{1}$$

where the infimum is taken over functions $u \in W^{1,1}(\Omega)$ and u_Ω is its average $|\Omega|^{-1} \int_\Omega u$.

Concerning the PS inequalities, our main results include the following.

- **Existence** of minimizers for C^3-smooth domains in $\mathbb{R}^n$ for $n \geq 2$.
- **Existence** of minimizers in elongated rectangles.
- **Existence** of minimizers in hypercubes in $\mathbb{R}^n$ for $n \geq 10$.
- **Non-existence** of minimizers in the isosceles rectangular triangle.

Bound on the Kinetic Energy with the semiclassical constant, with a gradient correction.

P.-T. Nam, *Lieb–Thirring inequality with semiclassical constant and gradient error term* J. Functional Analysis **274**, 1739–1746 (2017)

$$T_\psi \geq (1-\varepsilon) K_d^{\mathrm{sc}} \int_{\mathbb{R}^d} \rho_\psi(x)^{1+2/d} - \frac{C_d}{\varepsilon^{3+4/d}} \int_{\mathbb{R}^d} \left(\nabla \sqrt{\rho_\psi(x)} \right)^2 dx. \tag{1}$$

Here,

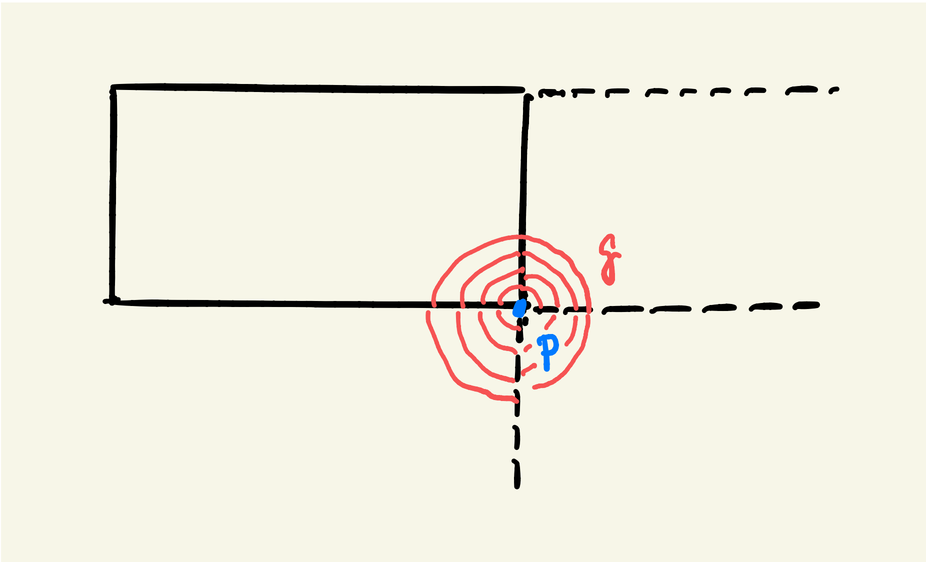
$$K_d^{\mathrm{sc}} = \frac{d}{d+2} \frac{4\pi^2}{(q\omega_d)^{2/d}}, \tag{2}$$

where ω_d is the volume of the unit ball in d dimensions.

In particular, if $d = 3$ and $q = 1$

$$K_3^{\mathrm{sc}} = \frac{3}{5}(6\pi^2)^{2/3} \approx 9.116\ldots$$

Lemma. For all $n \geq 1$, we have $G_Q(n) \leq G(n)/4$.



In the one-dimensional case, we obtain the corresponding lower bound as well.

Theorem. In one dimension, we have

$$G_Q(1) = \frac{G(1)}{4} = \frac{\pi^2}{16},$$

and the infimum is not attained.

Finally, for dimensions $n \geq 1$, we obtain the following dichotomy.

Theorem. For $n \geq 2$, if a minimizer for $G_Q(n)$ does not exist, then

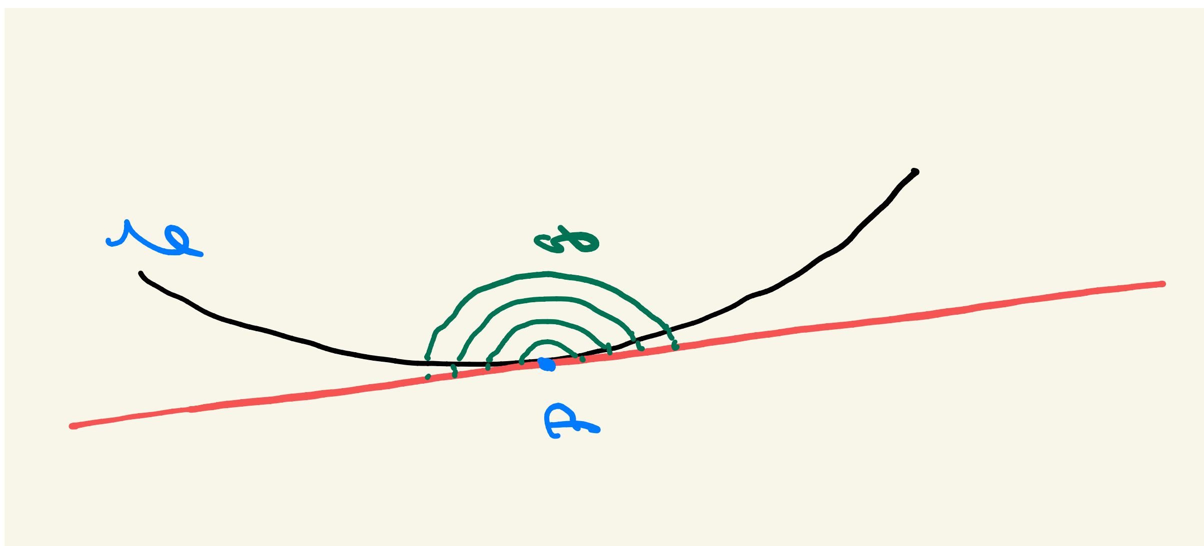
$$G_Q(n) = G(n)/4.$$

Existence of Minimizers for Smooth Domains.

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a C^3-domain and $n \geq 2$. Then a minimizer for $G(\Omega, n)$ exists and

$$G(\Omega, n) < G(\mathbb{R}^n_+, n) = G(n)/2^{2/n},$$

where $\mathbb{R}^n_+$ is the halfspace $\mathbb{R}_+ \times \mathbb{R}^{n-1}$.

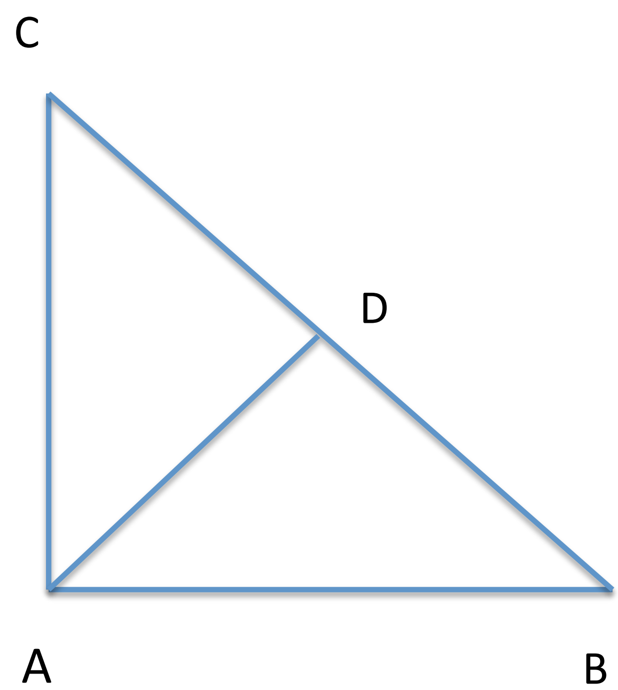


Non Existence of Minimizers for the isosceles right angle triangle.

Theorem.

Let $\Omega \subset \mathbb{R}^2$ be the isosceles rectangular triangle. There exist no minimizers for the Poincaré–Sobolev inequality in Ω and

$$G(\Omega, 2) = G(2)/8 \sim 0.732...$$



$$u = u_s + u_a$$

Rectangles and (Hyper)–Cubes.

Theorem. Let $\Omega_b \subset \mathbb{R}^2$ be the rectangle $[0, b^{-1}] \times [0, b]$. There exist b_c satisfying $b_c \leq 2.12$ such that, For all $b > b_c$, minimizers for the Poincaré–Sobolev inequality on Ω_b exist.

Theorem. Let Ω_n be the n-dimensional hypercube. Minimizers for for the Poincaré–Sobolev inequality on Ω_n exist for $n \geq 10$.

Related inequalities on bounded domains with Dirichlet Boundary conditions.

Gagliardo–Nirenberg-Sobolev (GNS), inequalities, on bounded domains with Dirichlet boundary conditions.

Define

$$G(n) = \inf \frac{\int_\Omega |\nabla u|^2 \, dx \left(\int_\Omega u^2 \, dx \right)^{2/n}}{\int_\Omega |u|^{2+4/n} \, dx}, \tag{1}$$

where the infimum is taken over functions $u \in H_0^1(\Omega)$. Here Ω is a bounded domain in $\mathbb{R}^n$.

Using the standard Rellich–Pohozaev technique we proved,

Theorem. If Ω is star–shaped, there is no minimizer of (1).

THANK YOU !!