Smoothability of non normal stable Gorenstein Godeaux surfaces

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* I wish!!

S is a <u>canonical</u> surface : projective, complex, irred,
has finitely many RDPs (rational double points)
i.e. singularities obtained contracting A_n $n \geq 1$ D_n $n \geq 4$
E_6, E_7, E_8 configuration of (-2) curves in a smooth surf $\tilde{S}$

<u>Ex</u> A_n $S = \{xy = z^{n+1}\} \subseteq \mathbb{A}^3$ $\tilde{S}$ = closure of Γ_φ

$$\varphi : S \dashrightarrow (\mathbb{P}^1)^n \qquad \varphi(x,y,z) = \left(\frac{x}{z}, \frac{x}{z^2}, \dots, \frac{x}{z^n}\right) = \left(\frac{z^n}{y}, \dots, \frac{z}{y}\right)$$

S is a (canonical) <u>surface of general type</u> if
$\exists\, S \hookrightarrow \mathbb{P}^N = \mathbb{P}^N_{\mathbb{C}}$ s.t. $\mathcal{O}_{\mathbb{P}}(1)|_S \cong \omega_S^{\otimes k}$ for some $k \geq 1$

ω_S dual sheaf unique l.b. on S s.t. $\omega_S|_{\text{nonsing locus } S_0} \cong \Omega^2_{S_0}$

RDPs are <u>Gorenstein</u> i.e. ω_S is a line bdl.

Studying moduli = studying families.

A <u>family</u> of proj surfaces over a base B is
a flat proj mor $\pi : S_B \longrightarrow B$ s.t. $\forall b \in B$ $\pi^{-1}(b)$ is
a surface.

Flatness is an alg version of continuity; in particular the discrete invariants of a surface of gen type

$p_g(S) := \dim H^0(S, \omega_S) \quad q(S) := \dim H^1(S, \mathcal{O}_S) \quad K_S^2$

are loc constant in the fibers. $p_g \geq 0$, $q \geq 0$, $K_S^2 \geq 1$.

<u>Def</u> A <u>Godeaux surface</u> is a surf of gen type with

$p_g = q = 0$, $K_S^2 = 1$.

<u>Ex</u> Let $f(x,y,z,w)$ be a deg 5 homog poly. $X = Z(f) \subseteq \mathbb{P}^3$

Then $\pi_1(X) = \{1\}$ $K_X^2 = 5$ $q(X) = 0$ $p_g(X) = 4$ gen by x, y, z, w.

Let $G = \mu_5$ acting on $\mathbb{P}^3$ via $t(x,y,z,w) = (tx, t^2y, t^3z, t^4w)$.

If X is smooth, G-invariant and G acts without fix points then $S := X/G$ is a Godeaux surface with $\pi_1(S) = G$.

<u>Thm</u> A Godeaux surface S must have $\pi_1(S) \cong \mathbb{Z}/a\mathbb{Z}$ with $a \in \{1, 2, 3, 4, 5\}$. We can classify case $a \in \{3, 4, 5\}$.

Def (CHEATING) The moduli space M of surfaces of general type with fixed K^2, p_g and q is a quasiproj var together with, for any family of sogt $\pi : S_B \to B$ with given invariants, a morphism $B \xrightarrow{\alpha_\pi} M$ s.t.

(1) it commutes with pullback

$$\begin{tikzcd} S_{\tilde{B}} \arrow[r] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\lrcorner"] & S_B \arrow[d, "\pi"] \\ \tilde{B} \arrow[r, "\varphi"'] & B \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \tilde{B} \arrow[rr, "\varphi"] \arrow[dr, "\alpha_{\tilde{\pi}}"'] & & B \arrow[dl, "\alpha_\pi"] \\ & M & \end{tikzcd} \quad \text{commutes.}$$

(2) The case $B = \text{point}$ induces a map

$$\left\{ \begin{array}{l} \text{isom cl} \\ \text{of sogt with fix inv} \end{array} \right\} \longrightarrow M \quad \text{which is bij.}$$

M is a quasiproj sch, unique up to can. isom.

<u>CHEATING</u> We pretend that $\exists\ S_M \longrightarrow M$ universal family s.t.

for every family $S_B \xrightarrow{\pi} B$ there is unique cartesian diagram

$$\begin{tikzcd} S_B \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & S_M \arrow[d] \\ B \arrow[r, "\alpha_\pi"'] & M \end{tikzcd}$$

True if (i) look at surface with no nontrivial autom.
or (ii) we replace M by an algebraic stack.

<u>Thm</u> For Godeaux surfaces, $M = \coprod_{a=1}^{5} M_a$ where M_a corresponds to surfaces with $\pi_1 = \mathbb{Z}/a\mathbb{Z}$.

M_3, M_4, M_5 have been explicitely described by M-Reid they are irred and smooth of exp dim 8.

<u>Not known</u>. Are M_1, M_2 smooth? Are they irred?

We know loc. dim at each point ≥ 8, but don't know it.

<u>Thm</u> (Kollár-Shepherd Barron; Alexeev) each mod space M has a natural (proj) compactification $\overline{M}$ corresponding to families of stable surfaces.

(Franciosi - Pardini - Soehnke) Classify non-normal Gorenstein stable Godeaux surfaces.

Q (Referee of FPS) are these surfaces in the closure of M inside $\overline{M}$? I.e. are they smoothable?

Def A stable surface S is smoothable if $\exists$ $\pi: S_B \longrightarrow B$ family with B nonsing connected curve and $b_0 \in B$ s.t. $S_{b_0} \cong S$.

Q (FFP) How does $\overline{M}$ look like near such surfaces?

Theorem Assume S is a quasismooth nonnormal Gorenstein stable Godeaux surface. Then

(1) S is smoothable

(2) $\overline{M}$ is nonsingular of dim 8 at $[S]$.

Rem This is good, but also bad ☺

Def A surface S is <u>quasismooth</u> if its only singularities are (1) a double curve Y (looks like $\{xy=0\} \subseteq \mathbb{A}^3$)
(2) pinch points ($uv^2 = w^2 \subseteq \mathbb{A}^3$) analytically or étale locally

Such S has nonsingular normalization $\tilde{S} \xrightarrow{\varepsilon} S$ and $\tilde{Y} = \varepsilon^{-1}(Y)$ is a nonsing curve in $\tilde{S}$
$\tilde{Y} \longrightarrow Y$ is a double cover branched over pinch pts.

Rem Conversely, given $\tilde{S}$ nonsing surf, $\tilde{Y} \subseteq \tilde{S}$ nonsing curve, $i : \tilde{Y} \longrightarrow \tilde{Y}$ involution, $Y := \tilde{Y}/\langle i \rangle$. We can glue to get a pushout diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, hook] \arrow[d] & \tilde{S} \arrow[d, "\varepsilon"] \\ Y \arrow[r, hook] & S \end{tikzcd}$$

with S quasismooth
ε normalization
Y double curve.

Exercise $\tilde{A} = \mathbb{C}[x,y]$ $\quad \tilde{B} = \mathbb{C}[x] = \tilde{A}/(y)$ $\quad i : \tilde{B} \longrightarrow \tilde{B}$, $i(x) = -x$

$B = (\tilde{B})^{inv} = \mathbb{C}[u] \subseteq \tilde{B}$ Define $A := \pi^{-1}(B) \subseteq \tilde{A}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ \tilde{A} \arrow[r, "\pi"'] & \tilde{B} \end{tikzcd} \qquad u = x^2$$

Show that $A = \mathbb{C}[u,v,w] \big/ uv^2 = w^2$

where $v = y$ $w = xy$.

<u>Sketch of proof</u>

(1) We compute T_S, T^1_S for a stable Gor God S in terms of the gluing data where $T^1_S := \mathcal{E}xt^1(\Omega_S, \mathcal{O}_S)$ is an inv sheaf on $\mathrm{Sing}(S)$.
(in general for all quasismooth varieties)

(2) Show that for a s.G.G. S, T^1_S is gen by global sections, $H^1(T^1_S) = H^2(T_S) = 0$.

(3) use inf def th $\Rightarrow$ $\overline{M}$ is smooth of exp dim at $[S]$.
8 at $[S]$.

(4) use [Tziolas] to show S is formally smoothable.
i.e. for $B_n := \operatorname{Spec} \mathbb{C}[t]/t^{n+1}$ there exist families

$$\begin{tikzcd} S = S_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S_1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S_2 \arrow[r] \arrow[d] & \cdots \\ \operatorname{Spec}\mathbb{C} = B_0 \arrow[r, hook] & B_1 \arrow[r, hook] & B_2 \arrow[r] & \cdots \end{tikzcd} \qquad (\dagger)$$

such that if there is a family $S_B \xrightarrow{\pi} B$ with B a smooth curve, $b_0 \in B$, t loc. coord for B at b_0 inducing $(\dagger)$ with $B_n = \mathcal{O}_{B,b_0}/(t^{n+1})$ then the general fiber of π is nonsingular.

(5) use [Nobile] to show that since S is Gorenstein and stable such a smooth curve B exists.

Possible future work

(i) For each S, find $a \in \{1, \dots, 5\}$ s.t. $[S] \in$ closure of M_a.
(unique because $\overline{M}$ is normal at $[S]$).

(ii) extend result to all other stable Gor Godeaux in [FPS]. Work in progress.
(meaningful case: gluing $\mathbb{F}$ nodal curve)