Harmonic Branched Coverings and Uniformization of CAT(κ) Spheres

Christine Breiner, Brown University

joint work with
Chikako Mese, Johns Hopkins

September 2021

Harmonic Maps

Start with a map

$$u : M \to N$$

where M, N are “geometric spaces” (Riemannian manifolds, metric measure spaces, metric spaces, etc.).

The *energy* of the map u is taken by

- Measuring the stretch of the map at each point $p \in M$.
- Integrating this quantity over M.

Harmonic Maps

Definition

For $u : (M, g) \to (N, h)$ (Riemannian manifolds) the *energy* is

$$E(u) := \int_M |du|^2 dx$$

where $\underline{du \in \Gamma(T^*M \otimes f^*TN)}$ is the differential and

$$|du|^2(x) := g^{ij}(x) h_{\alpha\beta}(u(x)) \frac{\partial u^\alpha}{\partial x^i}(x) \frac{\partial u^\beta}{\partial x^j}(x).$$

$$du_p : T_pM \to T_{u(p)}N$$

Harmonic Maps

Definition

For Riemannian manifolds M, N, the map $u : M \to N$ is *harmonic* if it is a critical point for the energy functional E.

Harmonic Maps

Definition

For Riemannian manifolds M, N, the map $u : M \to N$ is *harmonic* if it is a critical point for the energy functional E.

Restricting to Euclidean case, this means for all $v \in C_0(\Omega, \mathbb{R})$ with $E[v] < \infty$:

$$\lim_{t \to 0} \frac{E[u + tv] - E[u]}{t} = 0.$$

More generally, the Euler-Lagrange Equation is:

$$\underline{\Delta_g u^\gamma + g^{ij}(x)\Gamma^\gamma_{\alpha\beta}(u(x))\frac{\partial u^\alpha}{\partial x^i}(x)\frac{\partial u^\beta}{\partial x^j}(x) = 0.}$$

$\Delta u^\gamma = 0$

Harmonic maps

Smooth Examples

- *harmonic functions*
- *geodesics*
- *isometries*
- *totally geodesic maps*
- *minimal surfaces*
- *holomorphic maps between Kähler manifolds*

Today we consider maps

$$u : \Sigma \to (X, d) \text{ where}$$

- Σ is a Riemann surface
- (X, d) is a compact locally CAT(κ) space:

Today we consider maps

$$u : \Sigma \to (X, d) \text{ where}$$

- Σ is a Riemann surface
- (X, d) is a compact locally CAT(κ) space:
    - (X, d) is a geodesic space.

Harmonic maps into CAT(κ) spaces

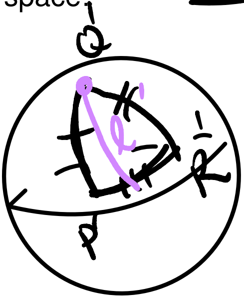
Today we consider maps

$$u : \Sigma \to (X, d) \text{ where}$$

- Σ is a Riemann surface
- (X, d) is a compact locally CAT(κ) space:
  - (X, d) is a geodesic space.
  - (X, d) has curvature $\leq \kappa$.

$\kappa \geq 0$ here

$\ell \leq \ell'$



$S^2\left(\frac{1}{\sqrt{\kappa}}\right)$

Definition (Korevaar-Schoen)

Let $u : \Omega \subset \mathbb{C} \to (X, d)$. For $u \in L^2(\Omega, X)$, we let

$$\boxed{e^u_\epsilon(z)} := \frac{1}{2\pi\epsilon} \int_{\underline{\partial\mathbb{D}_\epsilon(z)}} \frac{d^2(u(z), u(\zeta))}{\epsilon^2} d\theta.$$

Definition (Korevaar-Schoen)

Let $u : \Omega \subset \mathbb{C} \to (X, d)$. For $u \in L^2(\Omega, X)$, we let

$$e^u_\epsilon(z) := \frac{1}{2\pi\epsilon} \int_{\partial \mathbb{D}_\epsilon(z)} \frac{d^2(u(z), u(\zeta))}{\epsilon^2} d\theta.$$

Then the *energy of u* is defined

$$E[u] := \sup_{\substack{\phi \in C_0^\infty(\Omega) \\ \phi \in [0,1]}} \limsup_{\epsilon \to 0} \int_\Omega \phi(z) \boxed{e^u_\epsilon(z)dxdy.} \to de$$

$\exists$ de s.t.

If $E[u] < \infty$ then there exists a function $e^u \in L^1(\Omega, \mathbb{R})$ such that

$$e^u_\epsilon(z)dxdy \rightharpoonup e^u(z)dxdy \text{ (weakly as measures).}$$

Define $|\nabla u|^2(z) := e^u(z)$

If $E[u] < \infty$ then there exists a function $e^u \in L^1(\Omega, \mathbb{R})$ such that

$$e^u_\epsilon(z)dxdy \rightharpoonup e^u(z)dxdy \text{ (weakly as measures).}$$

Definition

A map $u : \Omega \to X$ is *harmonic* if it is locally energy minimizing.

- minimizing for maps $v : \Omega \to X$
- minimizing for diffeomorphisms on Ω.

Motivation - Uniformization

- Uniformization Theorem For Riemann Surfaces [Koebe, Poincaré]

  Every simply connected Riemann surface is conformally equivalent to the open disk, the complex plane, or the Riemann sphere.

Motivation - Uniformization

- Uniformization Theorem For Riemann Surfaces [Koebe, Poincaré]

  Every simply connected Riemann surface is conformally equivalent to the open disk, the complex plane, or the Riemann sphere.

- A consequence:

  Every smooth Riemannian metric g defined on a closed surface S is conformally equivalent to a metric of constant Gauss curvature.

Non-smooth Uniformization

$H := \sup_z H(z)$, f is H-quasiconformal

- Measurable Riemann Mapping Theorem [Moorey '38, Ahlfors-Bers '60]

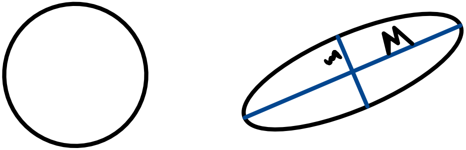
  Let $\mu : \mathbb{C} \to \mathbb{C}$ be an L^∞ function with $||\mu||_{L^\infty} < 1$. Then there exists a unique homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that

$$\partial_{\bar{z}} f(z) = \mu(z) \partial_z f(z).$$

"analytic distortion"

The *dilatation* of f at z is $H(z) := \frac{1+|\mu(z)|}{1-|\mu(z)|}$.

"geometric/metric distortion"



$H = \frac{M}{m}$

Non-smooth Uniformization

Other non-smooth uniformization results:

- Reshetnyak ‘93
- Bonk-Kleiner ‘02
- Rajala ‘17
- Lytchak-Wenger ‘20

Non-smooth Uniformization

Other non-smooth uniformization results:

- Reshetnyak '93
- Bonk-Kleiner '02
- Rajala '17
- Lytchak-Wenger '20

We use global existence and branched covering results to show:

Non-smooth Uniformization

Other non-smooth uniformization results:

- Reshetnyak '93
- Bonk-Kleiner '02
- Rajala '17
- Lytchak-Wenger '20

We use global existence and branched covering results to show:

- For (S, d) a locally CAT(κ) sphere, there exists a harmonic homeomorphism $h : \mathbb{S}^2 \to (S, d)$ which is

  • almost conformal (in K.S. sense)

  • 1-Q.C.

Global Existence

Theorem (B.-Fraser-Huang-Mese-Sargent-Zhang, '20)

Let Σ be a compact Riemann surface and (X, d) be a compact, locally CAT(κ) space. Let $\phi : \Sigma \to X$ be a finite energy, continuous map. Then either:

- *there exists a harmonic map $u : \Sigma \to X$ homotopic to ϕ or*
- *there exists an almost conformal harmonic map $v : \mathbb{S}^2 \to X$.*

Global Existence

Theorem (B.-Fraser-Huang-Mese-Sargent-Zhang, '20)

Let Σ be a compact Riemann surface and (X, d) be a compact, locally CAT(κ) space. Let $\phi : \Sigma \to X$ be a finite energy, continuous map. Then either:

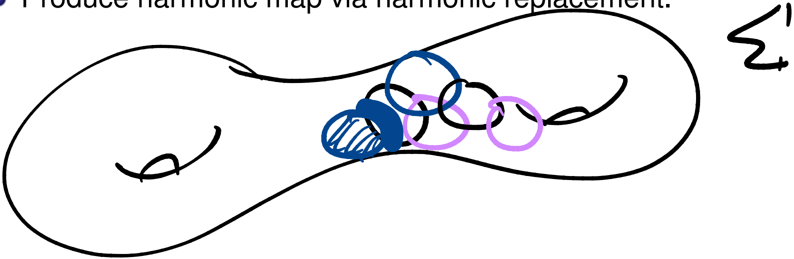
- *there exists a harmonic map $u : \Sigma \to X$ homotopic to ϕ or*
- *there exists an almost conformal harmonic map $v : \mathbb{S}^2 \to X$.*

What's missing for a uniformization theorem?

If $\Sigma = S^2$, then $u : S^2 \to X$ or $v : S^2 \to X$. If ϕ is homeo.

Global Existence

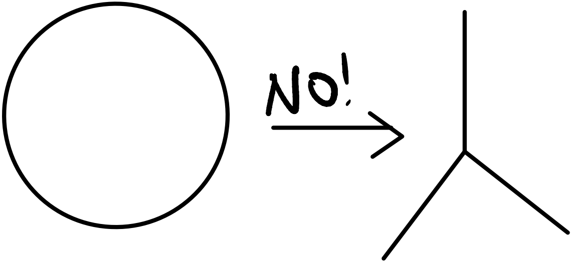
- Generalizes Sacks-Uhlenbeck existence of minimal two spheres.
- No PDE available.
- Exploits local convexity properties of CAT(κ) spaces.
- Existence and regularity of Dirichlet solutions required.
- Produce harmonic map via harmonic replacement.



Local analysis

Definition

We will say a harmonic map $u : \Sigma \to (X, d)$ from a Riemann surface into a locally CAT(κ) space is *non-degenerate* if, at every point, infinitesimal circles map to infinitesimal ellipses. (That is, tangent maps of u do not collapse along any ray.)



Local analysis

Theorem (B.-Mese '20)

A proper, non-degenerate harmonic map from a Riemann surface to a locally CAT(κ) surface is a branched cover.

harmonic + non-degenerate $\Longrightarrow$ map is discrete

+

characterization of Alexandrov tangent maps $\Longrightarrow$ map is open

$\Downarrow$ Väisälä

local homeo away from set B, w/ topological dim 0.

We will show B is discrete.

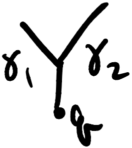
Definition

Given a geodesic space (X, d), the Alexandrov Tangent Cone of X at q is the cone over the space of directions $\mathcal{E}_q$ given by

$$T_qX := [0, \infty) \times \mathcal{E}_q / \sim$$

with metric

$$\delta((s, [\gamma_1]), (t, [\gamma_2])) := t^2 + s^2 - 2st \cos([\gamma_1], [\gamma_2]).$$



Alexandrov Tangent Maps

Definition

Let $u : \mathbb{D} \to X$ be a harmonic map into a CAT(κ) space (X, d). Let

$$\log_\sigma : (X, d_\sigma) \to (T_q X, \delta)$$

such that $\log_\sigma(q') := (d_\sigma(q, q'), [\gamma_{q'}])$. *Then for maps u_σ which converge to a tangent map of u, the maps*

u_*

$$\boxed{\log_\sigma \circ u_\sigma} : \mathbb{D} \to T_q X$$

converge to what is called an Alexandrov tangent map of u.

v_*

Key Points

- In general, tangent cones need not be well behaved. We prove: If (S,d) is a CAT(κ) surface then T_qS is a metric cone over a finite length simple closed curve.
- In general, Alexandrov tangent maps need not be harmonic. We prove: If $u:\Sigma \to (X,d)$ harmonic & (X,d) is locally CAT(κ) manifold, then its Alex. tangent maps are harmonic.

Key points

Kuwert classified homogeneous harmonic maps from $\mathbb{C}$ into an NPC cone $(\mathbb{C}, ds^2)$ where

$$ds^2 = \beta^2 |z|^{2(1-\beta)} dz^2$$

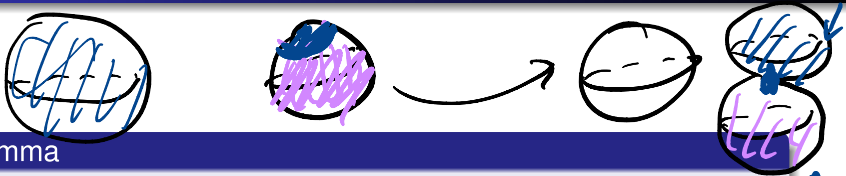
For a non-degenerate, harmonic u, tangent maps are thus of the form

$$v_*(z) = \begin{cases} cz^{\alpha/\beta} \text{ with } \alpha/\beta \in \mathbb{N}, & \text{if } k = 0, \\ c\left(\frac{1}{2}\left(k^{-\frac{1}{2}} z^\alpha + k^{\frac{1}{2}} \bar{z}^\alpha\right)\right)^{1/\beta}, & \text{if } 0 < k < 1. \end{cases}$$

- α is order of u at 0
- β gives curvature + metric
- k "stretch function"

If $k=1$, u is degenerate.

Application: Almost conformal harmonic maps



Lemma

A non-trivial almost conformal harmonic map $u : \Sigma \to (S, d)$ from a Riemann surface to a locally CAT(κ) surface is non-degenerate.

Reminder:

- Global existence $\Rightarrow$ if $\exists\ \phi : \mathbb{S}^2 \to (S,d)$ w/ finite energy then $\exists$ an almost conformal harmonic $u: \mathbb{S}^2 \to (S,d)$.
- Lemma $\Rightarrow$ u is non-degenerate
- Th'm $\Rightarrow$ u is a branched cover

Application: Uniformization

Theorem (B.-Mese '20)

If (S, d) is a locally CAT(κ) sphere, then there exists a map $h : \mathbb{S}^2 \to (S, d)$ such that

- *h is an almost conformal harmonic homeomorphism.*
- *h and h^{-1} are 1-quasiconformal.*
- *h is unique up to a Möbius transformation.*
- *the energy of h is twice the Hausdorff 2-dimensional measure of (S, d).*

- There exists a finite energy map.

convex geometry

- Use global existence and local analysis to find almost conformal, harmonic branched cover u.
- Use u to define an equivalence relation on $\mathbb{S}^2$ and a complex structure on the quotient space Q.

$$\begin{tikzcd} \mathbb{S}^2 \arrow[d, "\pi"'] \arrow[dr, "u"] & \\ Q \arrow[r, "id"] & (\mathbb{S}^2, d) \end{tikzcd}$$

- Set $p \sim q$ if $u(p) = u(q)$.
- $id \circ \pi = u$
- id is homeomorphism
- B is discrete
- Removable sing thm for harmonic maps