Asymptotic properties of modular type objects

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European Research Council
Established by the European Commission

Outline

1. Modular forms

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Modular forms

"Modular forms are functions on the complex plane that are inordinately symmetric.

They satisfy so many internal symmetries that their mere existence seem like accidents.

But they do exist."
-Mazur

B. Mazur

Modularity

Definition:

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight $k \in \mathbb{Z}$ if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

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Extends to $k \in \mathbb{Z} + \frac{1}{2}$, multipliers...

Weakly holomorphic: f holomorphic, linear exponential growth at cusps
Notation: $M_k^!$

Fourier expansion

Holomorphic: bounded at cusps
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Cusp forms: vanish at cusps
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Fourier expansion ($q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$)

$$f(\tau) = \sum_{n\in\mathbb{Z}} c_f(n) q^n$$

Examples

- Dedekind η-function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1 - q^n)$$

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Modularity:

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- Theta function:

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$$

Identity of Gauss

Write

$$\Theta(\tau)^3 =: \sum_{n\geq 0} r(n) q^n.$$

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with the Hurwitz class numbers

$$H(n) := \#\{\text{equivalence classes of integral binary quadratic forms of discriminant } n\}.$$

Partitions

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Generating function: (Euler)

$$P(q) := \sum_{n \geq 0} p(n) q^n = \prod_{n \geq 1} \frac{1}{1 - q^n}$$

L. Euler

Example

Example: Partitions of 4

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

so $p(4) = 5$.

Further values:

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$$p(10) = 42$$

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$$p(50) = 204226$$

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so $p(4) = 5$.

Further values:

$$p(10) = 42$$

$$p(50) = 204226$$

$$p(100) = 190569292$$

Fibonacci numbers

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$$F_0 = 0 \qquad F_1 = 1$$

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$$1 \quad 1 \quad 2 \quad 3 \quad 5$$

but

$$p(5) = 7 \qquad\qquad F_6 = 8$$

L. Fibonacci

Pentagonal Number Theorem

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$$\prod_{n\geq 1}(1-q^n)=\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}}$$

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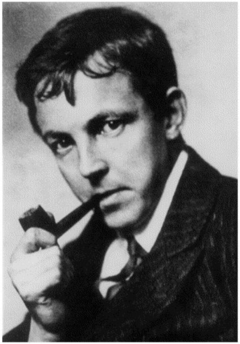
Recursion:

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)$$
$$+p(n-12)+p(n-15)-p(n-22)-\dots$$

Asymptotic behavior

Asymptotic behavior: (Hardy–Ramanujan)

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \qquad (n \to \infty)$$



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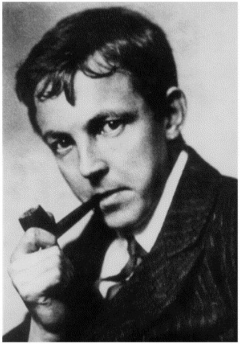
$$A_k(n) := \sum_{h \pmod{k}^*} \omega_{h,k} e^{-\frac{2\pi i n h}{k}}$$



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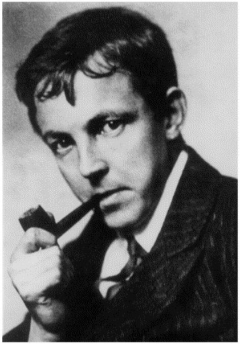

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Exact formula

Bessel function of order α:

$$I_\alpha(x) := \sum_{m\geq 0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

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Rademacher formula:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

H. Rademacher

Idea of proof

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$$a(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{A(q)}{q^{n+1}} dq$$

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$\mathcal{C}$ path inside unit circle, surrounding 0 counterclockwise.
If $A(q)$ is modular can approximate it near roots of unity.

General statement

Kloosterman sums:

$$K(m,n) = \sum_{d \pmod{c}^*} e^{\frac{2\pi i}{c}\left(m\overline{d}+nd\right)}$$

with $d\overline{d} \equiv 1 \pmod{c}$

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Theorem (Rademacher–Zuckerman)

Let $f(\tau) = \sum_n c_f(n)q^n \in M_k^! \ (k \in -2\mathbb{N}_0)$. Then for $n \in \mathbb{N}$

$$c_f(n) = 2\pi(-1)^{\frac{k}{2}} \sum_{m\leq -1} c_f(m) \left(\frac{|m|}{n}\right)^{\frac{1-k}{2}} \sum_{c\geq 1} \frac{K(m,n;c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{|m|n}}{c}\right).$$

Asymptotic main term

Corollary

$$c_f(n) \sim \frac{|n_0|^{\frac{1}{4}-\frac{k}{2}}}{2\sqrt{2}\pi} n^{\frac{k}{2}-\frac{3}{4}} e^{4\pi\sqrt{|n_0|n}},$$

where $n_0 < 0$ *is minimal with* $c_f(n_0) \neq 0$.

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Proof Use $I_\ell(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ (as $x \to \infty$).

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Harmonic Maass forms

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$F : \mathbb{H} \to \mathbb{C}$ real-analytic is a weight k harmonic Maass form if it is modular of weight k and

J. Bruinier

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$$\Delta_k(F)=0$$

with ($\tau=\tau_1+i\tau_2$)

$$\Delta_k:=-\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)+ik\tau_2\left(\frac{\partial}{\partial\tau_1}+i\frac{\partial}{\partial\tau_2}\right)$$

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Notation: $H_k^!$

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Examples

- weight 2 Eisenstein series:

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi\tau_2}$$

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- Class number generating function:

$$\widehat{\mathcal{H}}(\tau) := \sum_{\substack{n\geq 0 \\ n\equiv 0,3 \pmod 4}} H(n)q^n + \frac{i}{8\sqrt{2}\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(w)}{(-i(\tau+w))^{\frac{3}{2}}} dw$$

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shadow ↓

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$$F \quad = \quad \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{part} \\ \text{(mock modular form)}}}{F^+} \quad + \quad \underset{\substack{\uparrow \\ \text{non-holomorphic} \\ \text{part}}}{F^-}$$

Natural splitting

$F \in H_k$

$$F = \underbrace{F^+}_{\substack{\uparrow \\ \text{holomorphic part} \\ \text{(mock modular form)}}} + \underbrace{F^-}_{\substack{\uparrow \\ \text{non-holomorphic part}}}$$

with

$$F^+(\tau) := \sum_{n \gg -\infty} c_F^+(n) q^n$$

$$F^-(\tau) := \sum_{n \geq 1} c_F^-(n) \Gamma(k-1; 4\pi |n| \tau_2) q^n$$

Natural splitting

$F \in H_k$

$$F = F^+ + F^-$$

F^+ ↑ holomorphic part (mock modular form)

F^- ↑ non-holomorphic part

with

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↑ incomplete gamma function

Alternative representation

The non-holomorphic part has the shape

$$\int_{-\bar{\tau}}^{i\infty} f(w)(\tau + w)^{2-k} dw.$$

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ξ-operator: $\xi_k := 2i\tau_2^k \overline{\frac{\partial}{\partial\overline{\tau}}}$ $\qquad H_k \to M_{2-k}$

Ramanujan's last letter

"I am extremely sorry for not writing you a single letter up to now. I recently discovered very interesting functions which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions they enter into mathematics as beautifully as the theta functions. I am sending you with this letter some examples."

S. Ramanujan

Mock theta functions

These mock theta functions are 22 peculiar q-series.

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Example:

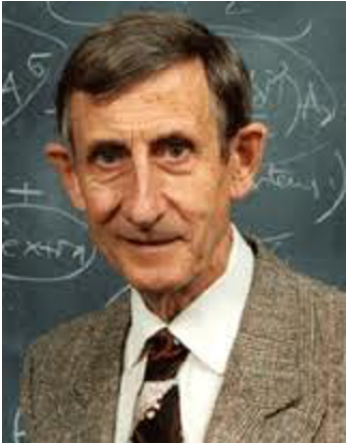
$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = \sum_{n\geq 0} \alpha(n) q^n$$

with

$$(a;q)_n = (a)_n := \prod_{m=0}^{n-1} (1 - aq^m)$$

Dyson's challenge for the future

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future..."



F. Dyson

Mock modularity of $f(q)$

Theorem (Zwegers)

The function $f(q)$ is a mock modular form.

S. Zwegers

Asymptotics for $f(q)$

Ramanujan's claim:

$$\alpha(n) \sim \frac{(-1)^{n+1}}{2\sqrt{n}} e^{\pi\sqrt{\frac{n}{6}}} \qquad (n \to \infty)$$

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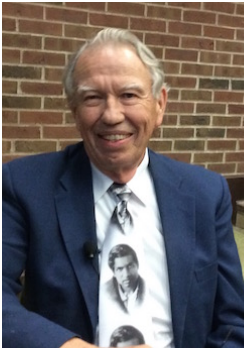

G. Andrews

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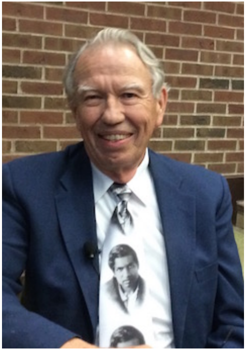

G. Andrews

Key: Bound

$$J(\alpha) := \int_0^\infty \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}} dt \quad (\mathrm{Re}(\alpha) > 0).$$

Andrews–Dragonette Conjecture:

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k \geq 1} \frac{(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor}}{k} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right) \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12n}\right)$$

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K. Ono

Theorem 1 (B.–Ono)

The Andrews-Dragonette Conjecture is true.

A general formula

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Idea of proof: For simplicity assume $k \in -2\mathbb{N}_0$. Let $F \in H_k$.

A general formula

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The Rademacher–Zuckerman exact formula also holds for H_k ($k \in -\frac{1}{2}\mathbb{N}_0$).

Idea of proof: For simplicity assume $k \in -2\mathbb{N}_0$. Let $F \in H_k$.

Poincaré series: $m \in -\mathbb{N}$

$$F_{k,m} :\doteq \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \phi_{k,m}\big|_k \gamma,$$

Idea of proof

where

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

$$f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d} \right),$$

$$\phi_{k,m}(\tau) := \tau_2^{-\frac{k}{2}} M_{-\frac{k}{2}, \frac{1-k}{2}}(4\pi|m|\tau_2) e^{2\pi i m \tau_1}$$

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$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

$$f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d} \right),$$

$$\phi_{k,m}(\tau) := \tau_2^{-\frac{k}{2}} M_{-\frac{k}{2}, \frac{1-k}{2}}(4\pi|m|\tau_2) e^{2\pi i m \tau_1}$$

↑
M–Whittaker function

Proof cont.

Let

$$G(\tau) := \sum_{m<0} c_F^+(m) F_{k,m}(\tau).$$

Proof cont.

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Fourier expansion:

$$F_{k,m}(\tau) = \left(1 - \frac{\Gamma(1-k; 4\pi|m|\tau_2)}{\Gamma(1-k)}\right) q^m + \sum_{n\geq 0} b_m^+(n) q^n + \sum_{n\geq 1} b_m^-(n)\Gamma(1-k; 4\pi n\tau_2) q^{-n}$$

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In particular for $m \in \mathbb{N}$

$$b_m^+(n) = 2\pi(-1)^{1-\frac{k}{2}} \left|\frac{m}{n}\right|^{\frac{1-k}{2}} \sum_{c\geq 1} \frac{K(m,n;c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Outline

1. Modular forms
2. Mock modular forms
3. (Mixed) False theta functions
4. Non-modular forms
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6. Meromorphic modular forms

Rogers false theta functions

Wrong sgn-factors prevent modularity.

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Wrong sgn-factors prevent modularity.

Example:

$$\sum_{n\in\mathbb{Z}} (-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\left(n+\frac{1}{2}\right)^2}$$

Unimodal sequences

Definition:

Sequence $\{a_j\}_{j=1}^{s}$ with

$$a_1 \leq a_2 \leq \ldots \leq \underset{\substack{\uparrow \\ \text{peak}}}{a_k} \geq a_{k+1} \geq \ldots \geq a_s$$

and $a_1 + \ldots + a_s = n$ is a unimodal sequence (stack).

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Let

$$u(n) := \#\text{unimodal sequences of size } n.$$

Generating function

Generating function:

$$U(q) := \sum_{n\geq 0} u(n)q^n = \frac{1}{(q;q)_\infty^2} \sum_{n\geq 1} (-1)^{n+1} q^{\frac{n(n+1)}{2}}$$

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Define

$$\psi(\tau) := i \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}.$$

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Note:

$$U(q) = \frac{i}{2} q^{-\frac{1}{2}} \frac{\psi(\tau)}{\eta(\tau)^2} + \frac{q^{\frac{1}{12}}}{\eta(\tau)^2}$$

Completion

Idea:

"Complete" ψ to obtain a function transforming like a modular form.

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"Complete" ψ to obtain a function transforming like a modular form.

Completion: $w \in \mathbb{H}$

$$\widehat{\psi}(\tau, w) := i \sum_{n\in\mathbb{Z}} \operatorname{erf}\left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}\right)\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}$$

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Note that for $\varepsilon > 0$

$$\lim_{t\to\infty} \widehat{\psi}(\tau, \tau + it + \varepsilon) = \psi(\tau).$$

Modularity

Remark

Motivation for changing sgn into error function taken from false theta functions.

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Theorem 3 (B.–Nazaroglu)

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C. Nazaroglu

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Sketch of proof:

Poisson summation.

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Motivation for changing sgn into error function taken from false theta functions.

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The function $\widehat{\psi}$ transforms like a modular form.

Sketch of proof:
Poisson summation.

Note: Could also view the false theta functions as Eichler integrals.

C. Nazaroglu

Asymptotics: (Auluck, Wright)

$$u(n) \sim \frac{1}{8 \cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{n}{3}}} \qquad (n \to \infty)$$

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Notation: $K_k(n)$, $K_k(n,r)$ Kloosterman sums

An exact formula

Theorem 4 (B.–Nazaroglu)

We have

$$u(n) = \frac{2\pi}{12n-1} \sum_{k\geq 1} \frac{K_k(n)}{k} I_2\left(\frac{\pi}{3k}\sqrt{12n-1}\right)$$

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$$- \frac{\pi}{2^{\frac{3}{4}}\sqrt{3}(24n+1)^{\frac{3}{4}}}\sum_{k\geq 1}\sum_{r \pmod{2k}}\frac{K_k(n,r)}{k^2}$$
$$\times \int_{-1}^{1}\left(1-x^2\right)^{\frac{3}{4}}\cot\left(\frac{\pi}{2k}\left(\frac{x}{\sqrt{6}}-r-\frac{1}{2}\right)\right) I_{\frac{3}{2}}\left(\frac{\pi}{3\sqrt{2}k}\sqrt{\left(1-x^2\right)(24n+1)}\right)dx.$$

Idea of proof

Write

$$U(q) = -q^{-\frac{1}{24}} f(\tau) + q^{\frac{1}{12}} g(\tau)$$

with

$$f(\tau) := -\frac{i}{2} \frac{\psi(\tau)}{\eta(\tau)^2}, \quad g(\tau) := \frac{1}{\eta(\tau)^2}.$$

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Set for $\varrho \in \mathbb{Q}$

$$\mathcal{E}_\varrho(\tau) := \int_\varrho^{\tau+i\infty+\varepsilon} \frac{\eta(z)^3}{\sqrt{i(z-\tau)}} dz,$$

where the integration path avoids the branch-cut.

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where the integration path avoids the branch-cut.

Lemma

We have, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$,

$$f(\tau) = e^{\frac{\pi i}{4}} \nu_\eta(M)^{-1} \sqrt{-i(c\tau+d)} \left(f\left(\tfrac{a\tau+b}{c\tau+d}\right) - \frac{1}{2} g\left(\tfrac{a\tau+b}{c\tau+d}\right) \mathcal{E}_{\frac{a}{c}}\left(\tfrac{a\tau+b}{c\tau+d}\right)\right).$$

Mordel-type integrals

For $\mathrm{Re}(V) > 0$

$$\mathcal{E}_\varrho\left(\varrho + iV\right) = -\frac{i}{\pi}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\left(n+\frac{1}{2}\right)^2\varrho}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{e^{-\pi Vx^2}}{x-\left(n+\frac{1}{2}\right)(1+i\varepsilon)}dx.$$

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Write

$$e^{2\pi dV}\mathcal{E}_{\frac{h'}{k}}\left(\frac{h'}{k}+iV\right) = \mathcal{E}^*_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) + \mathcal{E}^{\mathrm{e}}_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right),$$

Mordel-type integrals (cont.)

where

$$\mathcal{E}^*_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) := -\frac{i}{\pi}e^{2\pi dV}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\left(n+\frac{1}{2}\right)^2\frac{h'}{k}} \times \lim_{\varepsilon\to 0^+}\int_{-\sqrt{2d}}^{\sqrt{2d}}\frac{e^{-\pi Vx^2}}{x-\left(n+\frac{1}{2}\right)(1+i\varepsilon)}dx,$$

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Mordel-type integrals (cont.)

Error bounds: For $0 \leq d < \frac{1}{8}$

$$\mathcal{E}^{\mathrm{e}}_{\frac{h'}{k},d}\left(\frac{h'}{k} + iV\right) = O\left(\log(k)\right).$$

Mordel-type integrals (cont.)

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Write

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Enter the Circle Method.

Outline

1. Modular forms
2. Mock modular forms
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Unimodal sequences

Recall:

Sequence $\{a_j\}_{j=1}^{s}$ with

$$a_1 \leq a_2 \leq \ldots \leq a_k \geq a_{k+1} \geq \ldots \geq a_s$$

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Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

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Let

$$\mathrm{ss(n)} := \#\text{shifted stacks of size } \mathrm{n}.$$

Shifted stacks

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Thus $\mathrm{ss}(4) = 3$.

Generating function

Auluck:

$$\mathcal{S}_s(q) := \sum_{n\geq 0} \mathrm{ss}(n) q^n = 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q)_{n-1}^2 (1-q^n)}$$

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(log) asymptotic behavior (Wright)

$$\log(\mathrm{ss}(n)) \sim 2\pi\sqrt{\frac{n}{5}} \qquad (n \to \infty)$$

Generating function

Theorem 5 (B.–Mahlburg)

We have

$$\mathrm{ss}(\mathrm{n}) \sim \frac{\phi^{-1}}{2\sqrt{2}5^{\frac{3}{4}}\mathrm{n}} \mathrm{e}^{2\pi\sqrt{\frac{\mathrm{n}}{5}}} \qquad (\mathrm{n} \to \infty)$$

with ϕ the Golden Ratio.

K. Mahlburg

Key idea of the proof

- Embed into the modular world:

$$\mathcal{S}_s(q) = 1 + \mathrm{CT}_{[\zeta]}\left(\sum_{r\geq 0} \frac{\zeta^{-r} q^{\frac{r^2-r}{2}}}{(q)_r} \sum_{m\geq 1} \frac{\zeta^m q^m}{(q)_{m-1}}\right)$$

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  CT is the constant term
- Use Jacobi triple product formula

$$\left(-\zeta^{-1}\right)\infty\left(-\zeta q\right)_\infty (q)_\infty = -q^{-\frac{1}{8}}\zeta^{-\frac{1}{2}}\vartheta\left(z+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right),$$

  where

$$\vartheta(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{\pi i n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)}.$$

Proof continued . . .

- Modular inversion:

$$\vartheta\left(z+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right)=i\sqrt{\frac{2\pi}{\varepsilon}}e^{-\frac{2\pi^2}{\varepsilon}\left(z+\frac{1}{2}\right)^2}\vartheta\left(\frac{2\pi\left(z+\frac{1}{2}\right)}{i\varepsilon};\frac{2\pi i}{\varepsilon}\right)$$

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- Laurent expansion of quantum dilogarithm $\mathrm{Li}_2(\zeta;q):=-\mathrm{Log}((\zeta;q)_\infty)$

$$\mathrm{Li}_2\left(e^{-B\varepsilon}\zeta;e^{-\varepsilon}\right)=\frac{1}{\varepsilon}\mathrm{Li}_2(\zeta)+\left(B-\frac{1}{2}\right)\log(1-\zeta)+O(\varepsilon).$$

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Ranks and cranks

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where

$$o(\lambda) := \#\text{ of 1s in } \lambda,$$

$$\mu(\lambda) := \#\text{parts } > o(\lambda).$$

Crank generating function

Let (basically)

$$M(m,n) := \# \text{ of partitions of } n, \text{ crank } m,$$
$$N(m,n) := \# \text{ of partitions of } n, \text{ rank } m.$$

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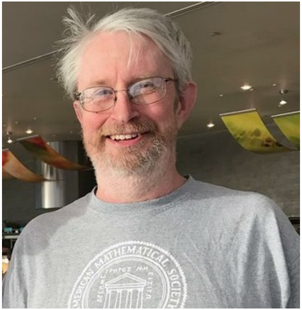
Andrews–Garvan:

$$C(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n)\zeta^m q^n$$
$$= \frac{1-\zeta}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-\zeta q^n}$$



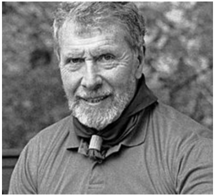
F. Garvan

Rank generating function

Atkin–Swinnerton-Dyer:

$$R(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} N(m,n)\zeta^m q^n$$

$$= \frac{1-\zeta}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-\zeta q^n}$$



A. Atkin

P. Swinnerton-Dyer

Moments

Crank and rank moments ($r \in \mathbb{N}_0$) (Atkin–Garvan)

$$M_r(n) := \sum_{m\in\mathbb{Z}} m^r M(m,n)$$
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Note: $N_{2r+1}(n) = M_{2r+1}(n) = 0$

Asymptotics and inequalities

Theorem 6 (B.–Mahlburg–Rhoades, Garvan)

(1) *As* $n \to \infty$

$$M_{2k}(n) \sim N_{2k}(n).$$

R. Rhoades

Asymptotics and inequalities

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(1) *As* $n \to \infty$

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R. Rhoades

Important tool: modularity (differentiate with respect to Jacobi variable)

Positive moments

Positive crank and rank moments ($r \in \mathbb{N}$) (Andrews–Chan–Kim)

$$M_r^+(n) := \sum_{m \in \mathbb{N}} m^r M(m,n)$$

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Note:

$$M_{2k}^+(n) = \frac{1}{2} M_{2k}(n)$$

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B. Kim

Theorem 7 (B.–Mahlburg, Andrews–Chan–Kim)

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Asymptotics and inequalities

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Idea of proof

Difficulty: Non-modularity

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- Relate generating functions to "false Lerch sums" ($\ell \in \{1,3\}$)

$$\frac{1}{(q)_\infty} \sum_{n\geq 1} \frac{(-1)^{n+1} q^{\frac{\ell n^2}{2}+\frac{rn}{2}}}{(1-q^n)^r}.$$

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- Understand the asymptotic behavior near $q = 1$.

- Circle Method

Non-modular products

Define

$$\sigma(q) := \prod_{j,k\geq 1}\left(1-q^{\frac{jk(j+k)}{2}}\right)^{-1} := \sum_{n\geq 0} r(n)q^n.$$

Counts certain representations of $\mathrm{SU}(2)$.

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Asymptotics (Romik)

$$r(n) \sim \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} - A_2 n^{\frac{3}{10}} - A_3 n^{\frac{1}{5}} - A_4 n^{\frac{1}{10}}\right)$$

for explicit C_0, A_1, A_2, A_3, A_4.

Question by Romik

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What about further terms in the asymptotic expansion?

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Theorem 8 (B.–Frake)

Let $L \in \mathbb{N}_0$. We have, as $n \to \infty$,

$$r(n) = \frac{1}{n^{\frac{3}{5}}} \left(\sum_{j=0}^{L} \frac{C_j}{n^{\frac{j}{10}}} + O_L\left(n^{-\frac{L}{10}-\frac{3}{80}}\right) \right) \times \exp\left(A_1 n^{\frac{2}{5}} - A_2 n^{\frac{3}{10}} - A_3 n^{\frac{1}{5}} - A_4 n^{\frac{1}{10}}\right),$$

where the constants C_j do not depend on L and n and can all be calculated explicitly.

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- Properties of the Dirichlet series

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Outline

1. Modular forms
2. Mock modular forms
3. (Mixed) False theta functions
4. Non-modular forms
5. Mixed mock modular forms
6. Meromorphic modular forms

Mixed mock modular forms

Definition: A mixed harmonic Maass form has the shape

$$H = \sum_j \underbrace{f_j}_{\in M_k^!} \underbrace{F_j}_{\in H_\ell^!} .$$

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Mixed mock modular form: holomorphic part

$$H^+ := \sum_j f_j F_j^+$$

Algebraic geometry: an example

Recall

$$\mathcal{H}(\tau) = \sum_n H(n) q^n.$$

Generating function for Euler numbers:

$$F(\tau) := \sum_{n \geq 0} \beta(n) q^n := \frac{\mathcal{H}\big| U_4(\tau)}{\eta(\tau)^6},$$

where for $f(\tau) = \sum_n a(n) q^n$ the U-operator is

$$f\big| U_\ell(\tau) := \sum_n a(\ell n) q^n$$

Notation

Notation:

For $k \in \mathbb{N}$, $g \in \mathbb{Z}$, $t \in \mathbb{R}$

$$f_{k,g}(t) := \begin{cases} \frac{\pi^2}{\sinh^2\left(\frac{\pi t}{k} - \frac{\pi i g}{2k}\right)} & \text{if } 2k \nmid g, \\ \frac{\pi^2}{\sinh^2\left(\frac{\pi t}{k}\right)} - \frac{k^2}{t^2} & \text{if } 2k \mid g. \end{cases}$$

Notation (cont.)

Kloosterman sums:

$$\mathcal{K}_\ell(n,m;k) := \sum_{h \pmod{k}^*} \psi_\ell\left(h,h',k\right) e^{-\frac{2\pi i}{k}\left(hn+\frac{h'n}{4}\right)}$$

with $hh' \equiv -1 \pmod{k}$

Notation (cont.)

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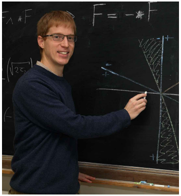
Bessel function integral:

$$\mathcal{I}_{k,g}(n) := \int_{-1}^{1} f_{k,g}\left(\frac{t}{2}\right) I_{\frac{7}{2}}\left(\frac{\pi}{k}\sqrt{(4n-1)\left(1-t^2\right)}\right)\left(1-t^2\right)^{\frac{7}{4}} dt$$

An exact formula

Theorem 9 (B.–Manschot)

$$\beta(n) = -\frac{\pi}{6(4n-1)^{\frac{5}{4}}}\sum_{k\geq 1}\frac{\mathcal{K}_0\left(n,0;k\right)}{k} I_{\frac{5}{2}}\left(\frac{\pi}{k}\sqrt{4n-1}\right)$$



J. Manschot

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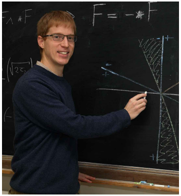
$$+ \frac{1}{\sqrt{2}(4n-1)^{\frac{3}{2}}} \sum_{k\geq 1} \frac{\mathcal{K}_0(n,0;k)}{\sqrt{k}} I_3\left(\frac{\pi}{k}\sqrt{4n-1}\right)$$



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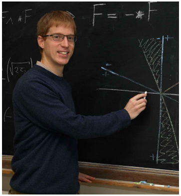
$$- \frac{1}{8\pi(4n-1)^{\frac{7}{4}}} \sum_{k\geq 1} \sum_{\substack{\ell\in\{0,1\} \\ -k<g\leq k \\ g\equiv \ell \pmod 2}} \frac{\mathcal{K}_\ell\left(n,g^2;k\right)}{k^2} \mathcal{I}_{k,g}(n)$$



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Sketch of proof

Circle Method.

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Bound integrals of the shape

$$\mathcal{I}_{k,g,b}(w) := e^{\frac{2\pi b}{kw}} w^{\frac{5}{2}} \int_{-\infty}^{\infty} f_{k,g}(t) e^{-\frac{2\pi t^2}{kw}} dt.$$

Negligibable for $b \leq 0$. "Principal part integrals" for $b > 0$.

Main term

Corollary

We have as $n \to \infty$

$$\beta(n) = \left(\frac{1}{96} n^{-\frac{3}{2}} - \frac{1}{32\pi} n^{-\frac{7}{4}} + O\left(n^{-2}\right)\right) e^{2\pi\sqrt{n}}.$$

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Ramanujan (Bialek)

For $z_2 \gg 1$:

$$\frac{1}{E_4(z)} = \sum_{n\geq 0} \beta_n e^{2\pi i n z}$$

$\frac{1}{E_4}$

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For $z_2 \gg 1$:

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Here

$$\beta_n := (-1)^n \frac{3}{E_6\left(e^{\frac{\pi i}{3}}\right)} \sum_{\lambda} \sum_{(c,d)} \frac{h_{(c,d)}(n)}{\lambda^3} e^{\frac{\pi \sqrt{3} n}{\lambda}},$$

$\frac{1}{E_4}$ (cont.)

where

- $\lambda \in \mathbb{N}$ has the form
  $$\lambda = 3^a \prod_{j=1}^{r} p_j^{a_j} \quad (a \in \{0,1\}, p_j = 6m+1 \text{ prime}, \ a_j \in \mathbb{N}_0),$$

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- (c,d) distinct solution to $\lambda = c^2 - cd + d^2$,
- $h_{(1,0)}(n) := 1, \ h_{(2,1)}(n) := (-1)^n$, for $\lambda \geq 7$

$$h_{(c,d)}(n) := 2\cos\left((ad + bc - 2ac - 2bd + \lambda)\frac{\pi n}{\lambda} - 6\arctan\left(\frac{\sqrt{3}c}{2d-c}\right)\right).$$

Asymptotic main term

Asymptotic main term:

$$\beta_n \sim \frac{3(-1)^n}{E_6\left(e^{\frac{\pi i}{3}}\right)} e^{\pi\sqrt{3}n} \qquad (n \to \infty)$$

Very rapid growth!

Building blocks

Define

$$f_{k,j,r}(z) := \sum_{m\geq 0} \sum_{\mathfrak{b}\subseteq \mathbb{Z}[i]}^{*} \frac{c_{4k}(\mathfrak{b}, m)}{N(\mathfrak{b})^{\frac{k}{2}-j}} (4\pi m)^r e^{\frac{2\pi m}{N(\mathfrak{b})}} e^{2\pi imz},$$

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Meromorphic cusp forms: decay like cusp form

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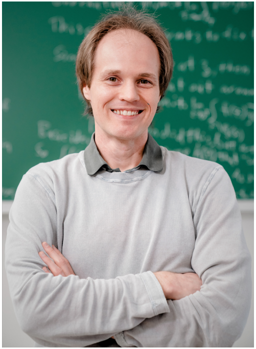
Notation: $\mathbb{S}_k$

General theorem

Theorem 10 (B.–Kane)

Let $f \in \mathbb{S}_{2-2k}$ with $k > 0$ with only pole in $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ in i. Then

$$f(z) \doteq \sum_{n \geq 0} a_n \sum_{j=0}^{n} \frac{(2k+n-1)!}{(2k+n-1-j)!} \times \binom{n}{j} f_{2k+2n,j,n-j}(z).$$



B. Kane

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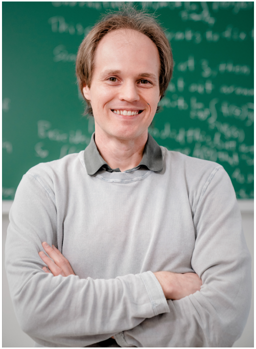

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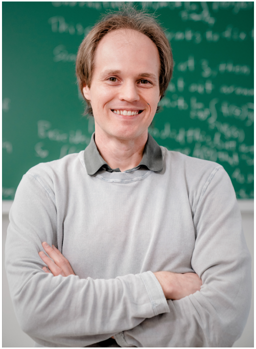

B. Kane

Remark

Similar for poles at other points.

Polar harmonic Maass forms

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Notation: $\mathbb{H}_k$

Building blocks

Let

$$H_{2k}(\mathfrak{z},z) := \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \frac{1}{1-e^{2\pi i(z-\mathfrak{z})}}\Big|_{2k,\mathfrak{z}}\gamma,$$

$$\widehat{H}_{2k}(\mathfrak{z},z) := H_{2k}(\mathfrak{z},z) + \sum_{r=0}^{2k-2} \frac{(2iz_2)^r}{r!}\frac{\partial^r}{\partial\bar{z}^r}H_{2k}(\mathfrak{z},\bar{z}).$$

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Lemma 11

(1) $\mathfrak{z} \mapsto \widehat{H}_{2k}(\mathfrak{z}, z) \in \mathbb{S}_{2k}$

(2) $z \mapsto \widehat{H}_{2k}(\mathfrak{z}, z) \in \mathbb{H}_{2-2k}$

A key lemma

Lemma 12

Let $F \in \mathbb{H}_{2-2k}, k < 0$ *with* $\xi_{2-2k}(F) \in S_k$. *Then*

$$F(z) = \sum_{n=0}^{n_\ell} a_n \left[R^n_{2k,\mathfrak{z}} \left(\mathcal{H}_{2k}(\mathfrak{z}, z) \right) \right]_{\mathfrak{z}=i},$$

where

$$R_{\kappa,\mathfrak{z}} := 2i \frac{\partial}{\partial \mathfrak{z}} + \frac{\kappa}{\mathfrak{z}}.$$

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Thank
You