3d Modularity & Supergroup Chern-Simons theory

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ICTP, Trieste

Workshop on Black Holes, BPS and Quantum Information

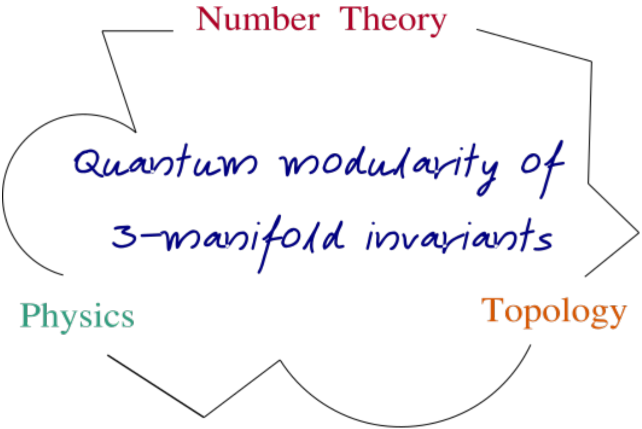
The talk is based on

• **" Supergroups, q-series and 3-manifolds"**, [arXiv:2009.14196] with P. Putrov

• **"Three-Manifold Quantum Invariants and Mock Theta Functions"**, [arXiv:1912.07997] with M. C. N. Cheng, G. Sgroi

• **"3d Modularity"**, [arXiv:1809.10148] with M. C. N. Cheng, S. Chun, S. Gukov, S. M. Harrison

• [arXiv:20XX.XXXXX] with M. C. N. Cheng, S. Chun, B. Feigin, S. Gukov, S. M. Harrison


Number Theory
Quantum modularity of
3-manifold invariants
Physics
Topology

Knots & 3-manifolds

Witten-Reshetikhin-Turaev invariant

[Witten (1988)], [Reshetikhin-Turaev (1990)]

Consider the 3d $SU(2)$ Chern-Simons theory, whose partition function is

$$Z_{\mathrm{CS}}(M_3;k) = \int_{\mathcal{A}} \mathcal{D}A\, e^{\frac{i(k-2)}{4\pi}\int_{M_3}\mathrm{Tr}(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)}$$

where $k \in \mathbb{Z}$ denotes the (shifted) Chern-Simons level.

Later a combinatorial definition based on the rapresentation of $U_q(\mathfrak{sl}_2)$ was found, and led to the extension of the above definition to

$$Z_{\mathrm{CS}}(M_3;k) : \mathbb{Q} \to \mathbb{C}$$

This was proven to be a topological invariant, known as the Witten-Reshetikhin-Turaev invariant (WRT).

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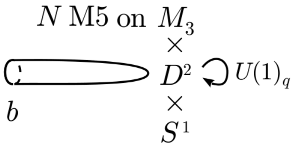
Chern-Simons theory from branes

M-theory on $T^*M_3 \times TN \times S^1$



This system is closely connected to **analytically continued Chern-Simons theory**

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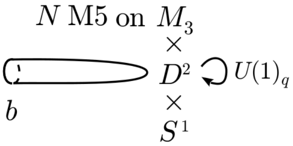

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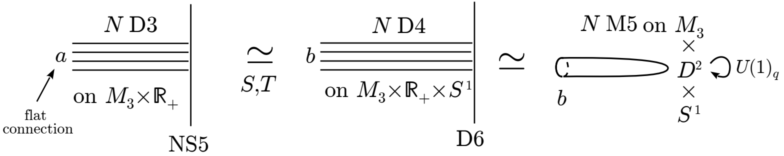
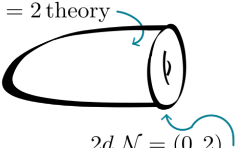


6d $\mathcal{N} = (2,0)$ SCFT

[Gukov, Putrov, Vafa (2016)], [Gukov, Pei, Putrov, Vafa (2017)]

A_1 $6d$ $\mathcal{N} = (2,0)$
on $M_3 \times D^2 \times_q S^1$

Supersymmetric three-dimensional gauge theory $T[M_3]$



Topological Quantum Field Theory

$\longrightarrow$ Three-dimensional Chern-Simons theory on M_3

The supersymmetric index of $T[M_3]$ is the **homological block**

$$\widehat{Z}_a^{\mathfrak{sl}(2)}(M_3;\tau) = Z(D^2 \times_q S^1; \mathcal{B}_a) = \sum_{\substack{i\in\mathbb{Z}+\Delta_a \\ j\in\mathbb{Z}}} (-1)^j q^i \dim\mathcal{H}_a^{i,j}$$

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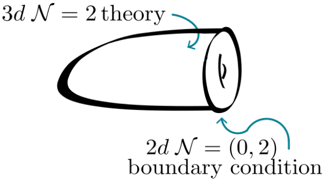

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[Vafa (2001)], [Gaiotto, Witten (2008)], [Mikhaylov, Witten (2014)]

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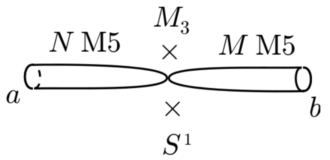

A **supergroup analog** of analytically continued CS theory

$\longrightarrow$ The CS path integral for supergroups does not immediately define an invariant

The homological block is the supersymmetric index of a (rather exotic) three-dimensional quantum field theory

$$\widehat{Z}^{\mathfrak{sl}(m|n)}_{a,b}(M_3;\tau) := \mathrm{Tr}_{\mathcal{H}_{a,b}}(-1)^F q^{L_0}$$

[Ferrari, Putrov (2020)]

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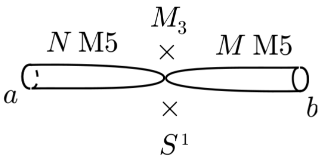

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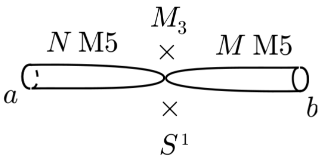

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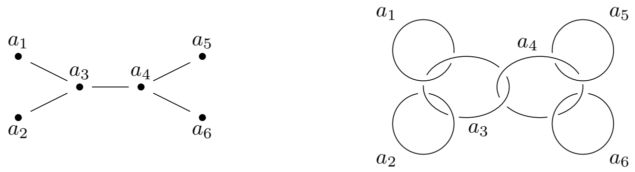
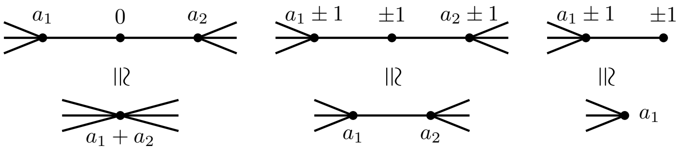
Plumbed 3-manifolds

A plumbed 3-manifold $M_3(\mathcal{G})$ is determined by a **weighted simple graph** (V, E, a_v), which represents Dehn surgery on the framed link $\mathcal{L}(\mathcal{G})$.



Dehn surgeries on different $\mathcal{L}(\mathcal{G})$ can produce homeomorphic manifolds: a topological invariant must be invariant under the following Kirby moves



Plumbed 3-manifolds

$M_3(\mathcal{G})$ is equivalently described by a **linking matrix** M, a square matrix of size $L := |V|$ with entries

$$M_{vv'} = \begin{cases} a_v & \text{if} \quad v = v' \\ 1 & \text{if} \quad (v, v') \in E \\ 0 & \text{otherwise} \end{cases}$$

Given M, $H_1(M_3, \mathbb{Z}) = \mathbb{Z}^L / M\mathbb{Z}^L$. If M is non-degenerate ($b_1(M_3) = 0$), it induces a non-degenerate symmetric bilinear pairing on $H_1(M_3, \mathbb{Z})$

$$\left(H_1(M_3, \mathbb{Z})\,,\ x \mapsto -(x, M^{-1}x) + \mathbb{Z}\right)$$

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$SU(2)$ Homological Blocks

[Gukov, Pei, Putrov, Vafa (2017)], [Gukov, Manolescu (2019)]

For a weakly negative plumbed three-manifold M_3 and gauge group $SU(2)$ the **homological block** $\widehat{Z}_a^{\mathfrak{sl}(2)}(M_3;\tau)$ is

$$\widehat{Z}_a^{\mathfrak{sl}(2)}(M_3;\tau) := (-1)^\pi q^{\frac{3\sigma - \mathrm{Tr} M}{4}} \times \\ \times \, \mathrm{vp} \prod_{v\in V} \oint_{|y_v|=1} \frac{dy_v}{2\pi i y_v} \left(y_v - {y_v}^{-1}\right)^{2-\delta_v} \Theta_a^M(\tau,\mathbf{z})$$

where $q = e^{2\pi i\tau}$, $y_v = e^{2\pi i z_v}$, $\delta_v = \deg(v)$, and the label a can be identified with elements of the set of $\mathrm{Spin}^c(M_3)/\mathbb{Z}_2$. The theta function reads

$$\Theta_a^M(\tau,\mathbf{z}) = \sum_{\mathbf{n}\in 2M\mathbb{Z}^L + a} q^{-\frac{\mathbf{n}^T M^{-1}\mathbf{n}}{4}} \, e^{2\pi i \mathbf{z}^T \mathbf{n}}.$$

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$SU(2)$ Homological blocks

[Gukov, Pei, Putrov, Vafa (2017)]

The radial limit, taking $\tau \in \mathcal{H}$ to the boundary of the upper-half plane, relates the homological blocks to the **WRT invariants**

$$Z_{\text{CS}}(M_3;k) = (i\sqrt{2k})^{-1} \sum_{a,b\in(\mathbb{Z}^L/M\mathbb{Z}^L)/\mathbb{Z}_2} e^{2\pi i k\lambda(a,a)} X_{ab} \lim_{\tau\to\frac{1}{k}} \widehat{Z}_b^{\mathfrak{sl}(2)}(M_3;\tau),$$

where $\lambda(a,b)$ is the linking pairing on $H_1(M_3,\mathbb{Z})$ and the matrix X has as elements

$$X_{ab} = \frac{e^{2\pi i\lambda(a,b)} + e^{-2\pi i\lambda(a,b)}}{|\mathcal{W}_a|\sqrt{|\text{Det}M|}}.$$

where $\mathcal{W}_a = \text{Stab}_{\mathbb{Z}_2}(a)$.

$SU(2|1)$ Homological Blocks

[Ferrari, Putrov (2020)]

The homological block associated to $M_3(\mathcal{G})$ with gauge group $SU(2|1)$ is

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where $a, b \in \mathbb{Z}^L/M\mathbb{Z}^L$, the contour Ω has to be carefully chosen.
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The homological blocks turn out to be a nice q-series when there exists a "good" expansion chamber.

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Modular forms

Definition

A **modular form** $\varphi(\tau)$ of weight k, multiplier system χ with respect to the group Γ is a holomorphic function $\varphi : \mathcal{H} \to \mathbb{C}$ which satisfies

$$\varphi|_{k,\chi}(\tau) = \varphi(\tau), \qquad \gamma \in \Gamma$$

A **cusp form** is a modular form with Fourier expansion $\varphi(\tau) = \sum_{n>0} c(n)q^n$, $q = e^{2\pi i\tau}$.

The weight-k **slash operator** is defined as

$$\varphi|_{k,\chi}(\tau) = (c\tau + d)^{-k}\chi(\gamma)^{-1}\varphi(\gamma\tau), \qquad \gamma \in \Gamma$$

where the action of Γ on $\mathcal{H}$ is given by fractional linear transformations and the multiplier system is a map $\chi : \Gamma \to S^1$. From now on, $\Gamma = SL(2,\mathbb{Z})$.

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Quantum Modular Forms

[Zagier (2010)]

- QMFs are defined at the boundary of $\mathcal{H}$ $(\mathbb{Q} \cup \{i\infty\})$
- QMFs are neither analytic nor Γ-covariant functions

A **quantum modular form** of weight k and multiplier χ with respect to Γ is a function $Q : \mathbb{Q} \to \mathbb{C}$ such that for every $\gamma \in \Gamma$ the function $p_\gamma(x) : \mathbb{Q} \backslash \{\gamma^{-1}(\infty)\} \to \mathbb{C}$,

$$p_\gamma(x) := Q(x) - Q|_{k,\chi}\gamma(x)\,,$$

has a better analytic behavior than $Q(x)$.
The function $\gamma \mapsto p_\gamma$ is a cocycle on Γ (i.e. $p_{\gamma_1\gamma_2} = p_{\gamma_1}|_{k,\chi}\gamma_2 + p_{\gamma_2}$).

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Strong QMFs

[Zagier (2010)]

A **strong quantum modular form** is a function Q which associates to each element $x \in \mathbb{Q}$ a formal power series over $\mathbb{C}$, so that the identity

$$p_\gamma(x+it) = Q(x+it) - Q|_{k,\chi}\gamma(x+it)\,, \qquad t \to 0^+\,,\, \gamma \in \Gamma$$

holds as an identity between countable collections of formal power series.

The formal function Q might extend to a globally defined function $Q : (\mathbb{C} \setminus \mathbb{R}) \cup \mathbb{Q} \to \mathbb{C}$

Homological blocks - False thetas

[Cheng, Chun, Ferrari, Gukov, Harrison (2018)], [Bringmann, Mahlburg, Milas (2018)]

The quantum invariants $\widehat{Z}_a(M_3;\tau)$ for a weakly negative definite plumbing graph with are given by

$$q^{-c}\widehat{Z}_a(M_3;\tau)=\sum_{r\in S}\Psi_{m,r}(\tau)+p(\tau)$$

where $p(\tau)$ is a polynomial, $c\in\mathbb{Q}$, S is a subset of $\mathbb{Z}/2m\mathbb{Z}$.

The function $\Psi_{m,r}(\tau)$ is a **false theta function**

$$\Psi_{m,r}(\tau):=\widetilde{\vartheta^1_{m,r}}(\tau)=\sum_{\substack{\ell\in\mathbb{Z}\\ \ell=r \bmod 2m}}\operatorname{sgn}(\ell)\,q^{\ell^2/4m}$$

$\vartheta^1_{m,r}$ is the weight $3/2$ **unary theta function**

$$\vartheta^1_{m,r}(\tau,z)=\frac{1}{2\pi i}\partial_z\vartheta_{m,r}(\tau,z)|_{z=0}=\sum_{\substack{\ell\in\mathbb{Z}\\ \ell\equiv r\,(2m)}}\ell q^{\ell^2/4m}$$

Holomorphic Eichler integral

[Lawrence, Zagier (1999)]

The **holomorphic Eichler integral** of a weight w cusp form $g(\tau)$ is defined as

$$\widetilde{g}(\tau) := \sum_{n\geq 1} c(n) n^{1-w} q^n \,,$$

where the coefficients $c(n)$ are the Fourier coefficients of $g(\tau)$ and $w \in \frac{1}{2}\mathbb{Z}$.

Holomorphic Eichler integrals were first constructed to describe the $(w-1)$-fold primitive of a weight $w \in \mathbb{Z}$ cusp form $g(\tau)$. For integral weights $\widetilde{g}(\tau)$ can be expressed as

$$\widetilde{g}(\tau) = \int_{\tau}^{i\infty} d\tau' \, \frac{g(\tau')}{(\tau' - \tau)^{2-w}} \,, \qquad \tau \in \mathcal{H}$$

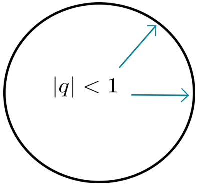
Radial Limit

[Lawrence, Zagier (1999)], [Cheng, Chun, Ferrari, Gukov, Harrison (2018)]

A false theta function does not transform nicely under the modular group, however its radial limit define a strong quantum modular form $Q(x)$

$$Q(x+it) = \lim_{t\to 0^+} \Psi_{m,r}(x+it)\,,$$

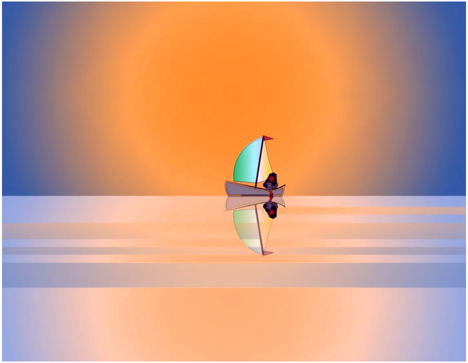
where $\tau = x+it$ with $x \in \mathbb{Q}$ and $t \in \mathbb{R}_+$.



The radial limit of false theta functions reproduces the WRT invariants $\longrightarrow$ **topological information** of M_3 can be easily extrapolated.

$M_3 \longrightarrow -M_3$

- **Q**: What happens in the lower half-plane?
- **Q**: What is $\widehat{Z}_a(-M_3;\tau)$ for a weakly positive definite 3-manifolds?



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∘ The Ohtsuki series obeys

$$Z_{\mathrm{CS}}(-M_3;k) = Z_{\mathrm{CS}}(M_3)(-k), \qquad k \to \infty$$

∘ $\widehat{Z}_a(-M_3;\tau)$ is expected to be a holomorphic function on $\mathcal{H}$ with well-defined q-expansions and integral coefficients.

Non-holomorphic Eichler integral

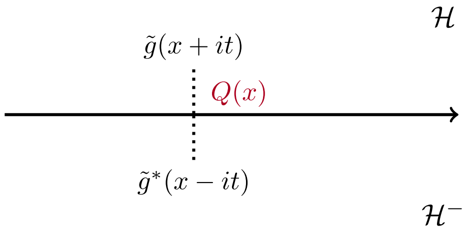
[Lawrence, Zagier (1999)], [Bringmann, Rolen (2015)]

Consider the non-holomorphic Eichler integral $\tilde{g}^* : \mathcal{H}^- \to \mathbb{C}$

$$\tilde{g}^*(\tau) := \int_{\bar{\tau}}^{i\infty} (\tau' - \tau)^{-k} g(\tau')\, d\tau'$$

The two Echler integrals $\tilde{g}^*$ and $\tilde{g}$ agree to infinite order at any $x \in \mathbb{Q}$, so that for $t > 0$

$$\tilde{g}(x+it) \sim \sum_{n \geq 0} \alpha_n t^n, \qquad \tilde{g}^*(x-it) \sim \sum_{n\geq 0} \alpha_n (-t)^n$$



Mock modular forms

[Zwegers (2008)]

A **mock modular form** $f(\tau)$ of weight k is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, whose completion

$$\hat{f}(\tau) = f(\tau) + g^*(\tau)$$

transforms like a modular form of weight k, $\hat{f}|_{k,\chi}\gamma(\tau) = \hat{f}(\tau)$.
The shadow, $g(\tau)$, is a holomorphic modular form of weight $2-k$ and

$$g^*(\tau) := \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-k} \overline{g(-\bar{\tau}')}\, d\tau'$$

The non-holomorphic Eichler integral g^* transforms as

$$g^*(\tau) - g^*|_{k,\chi}\gamma(\tau) = \int_{-\gamma^{-1}(i\infty)}^{i\infty} g(\tau')(\tau' + \tau)^{-k} dw.$$

Mock modular forms

[Griffin, Ono, Rolen (2013)], [Choi, Lim, Rhoades (2016)]

One of the distinctive features of mock theta functions is the infinite number of exponential singularities at roots of unity.

There is a collection of weakly holomorphic modular forms $\{G_j\}_{j=1}^n$ such that $(f - G_j)$ is bounded towards all cusps equivalent to x_j.
Given a choice of $\{G_j\}_{j=1}^n$, $f(\tau)$ defines a quantum modular form

$$Q(x) := \lim_{t \to 0^+} (f - G_x)(x + it),$$

Given a mock modular form $f(\tau)$ whose shadow is a cusp form $g(\tau)$ and $\tilde{g}(\tau)$ is its Eichler integral, then $f(\tau)$ and $\tilde{g}(\tau)$ have the "same" asymptotic series at $x \in \mathbb{Q}$

$$(f - G_x)(-x + it) \sim \sum_{n \geq 0} \alpha_x(n)(-t)^n, \quad \tilde{g}(x + it) \sim \sum_{n \geq 0} \alpha_x(n) t^n.$$

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Mock-False Conjecture

[Cheng, Chun, Ferrari, Gukov, Harrison (2018)]

Mock-False Conjecture

If the homological block $q^{-c}\widehat{Z}_a(M_3;\tau) = \sum_{r\in S}\Psi_{m,r}$ is a false theta function for some $c \in \mathbb{Q}$, then

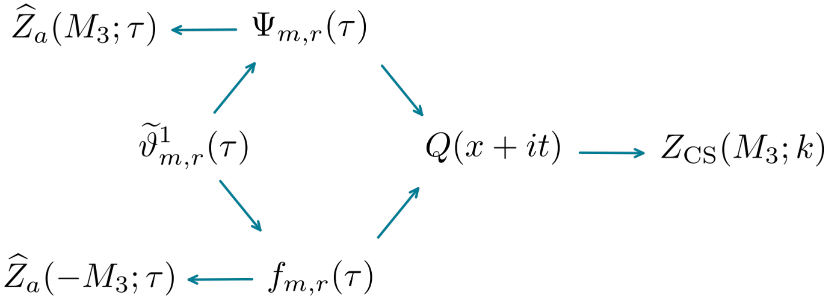
$$q^{c}\widehat{Z}_a(-M_3;\tau) = \sum_{r\in S} f_{m,r}(\tau)$$

is a mock theta function, whose shadow is a linear combination of the weight 3/2 unary theta series $\vartheta^1_{m,r}(\tau)$.

See also the more recent works

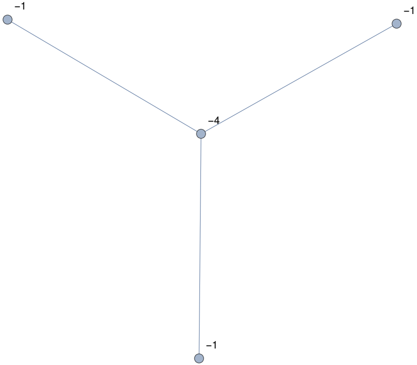
[Gukov, Manolescu (2019)], [Cheng, Ferrari, Sgroi (2019)], [Cheng, Sgroi (20xx)]

A little summary



Sphere plumbings

[Ferrari, Putrov (2020)]



The homological block associated to the 3-sphere turns out to be

$$\widehat{Z}^{\mathfrak{sl}(2|1)}(S^3;\tau) = -\frac{1}{6} + 2\sum_{n\geq 1} d(n)q^n = -\frac{1}{6} + 2q + 4q^2 + 4q^3 + 6q^4 + \ldots$$

where $d(n)$ is the number of positive divisors of n.

Eisenstein series and strong QMFs

[Ferrari, Putrov (2020)]

The homological block associated to the 3-sphere

$$\widehat{Z}^{\mathfrak{sl}(2|1)}(S^3;\tau) = 1 + 2\zeta(0) + 2\zeta(-1) + 2\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = -\frac{1}{6} + 2\sum_{n=1}^{\infty} d(n)q^n$$

can be expressed in terms of the **weight 1 holomorphic Eisenstein series**

$$\widehat{Z}(S^3;\tau) = \frac{1}{3} + 2G_1(\tau)\,, \quad G_1(\tau) := \frac{1}{2}\zeta(0) + \sum_{m\geq 1} d(m)q^m$$

The Eisenstein series G_k is defined as

$$G_k(\tau) := \frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

For k even and greater than two, $G_k(\tau)$ is a modular from of weight k.

Eisenstein series and strong QMFs

[Lewis, Zagier (2001)], [Bettin, Conrey (2011)]

The Eisenstein series $G_1(\tau)$ satisfies the following equation

$$G_1(\tau) - \frac{1}{\tau} G_1(-1/\tau) = p_1(\tau)$$

$p_1(\tau)$ is a real analytic function with growth $p_1(\tau) = O(|\log(\tau)|/\tau)$ as $\tau \to 0^+$ and $p_1(\tau) = O(\log(\tau))$ as $\tau \to \infty$ and it satisfies the three term relation

$$p_1(\tau) - p_1(\tau+1) = (\tau+1)^{-1} p_1\left(\frac{\tau}{\tau+1}\right)$$

and extends to an analytic function in the slit plane ($\mathbb{C}' := \mathbb{C} \backslash \mathbb{R}_{\leq 0}$).

$\longrightarrow$ Up to a non-smooth term, the homological block $\widehat{Z}^{\mathfrak{sl}(2|1)}(S^3;\tau)$ provides an example of a quantum modular form.

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$SU(2|1)$ homological blocks for Lens spaces

[Ferrari, Putrov (2020)]

For $p > 0$, consider $M^3 = L(p,1)$. In this case $H_1(L(p,1),\mathbb{Z}) \cong \mathbb{Z}_p$, and so there are p^2 homological blocks labelled by pairs $(b,c) \in \mathbb{Z}_p$.
The homological blocks turn out to be

$$\widehat{Z}^{\mathfrak{sl}(2|1)}_{b,c}[L(p,1)] = \mathrm{const}_{b,c} + 2q^{\frac{(p-b)(p-c)}{p} - (p-b)} \sum_{k\geq 1} \frac{q^{ck}}{1-q^{pk-(p-b)}},$$

taking $1 \leq b, c \leq p$ and

$$\mathrm{const}_{b,c} = \begin{cases} 1 + 2p\zeta(-1) + 2\zeta(0), & b = c = 0 \mod p, \\ p\zeta(-1, b/p) + \zeta(0, b/p), & c = 0 \mod p,\ b \neq 0 \mod p, \\ p\zeta(-1, c/p) + \zeta(0, c/p), & b = 0 \mod p,\ c \neq 0 \mod p, \\ 0, & b, c \neq 0 \mod p. \end{cases}$$

where $\zeta(s,x)$ denotes the Hurwitz zeta function.

$SU(2|1)$ homological blocks & QMFs

[Ferrari, Putrov (2020)]

The homological blocks of the Lens space $L(2,1)$ can be proven to satisfy the following transformation properties under the action of the generators of $SL_2(\mathbb{Z})$,

$$\widehat{Z}^{\mathfrak{sl}(2|1)}_{a,b}(\tau) - \frac{1}{2\tau}\sum_{a',b'}(-1)^{aa'+bb'}\,\widehat{Z}^{\mathfrak{sl}(2|1)}_{a',b'}(-1/\tau) = \psi_{2,(a,b)}(\tau)$$

$$\widehat{Z}^{\mathfrak{sl}(2|1)}_{a,b}(\tau) = (-1)^{ab}\,\widehat{Z}^{\mathfrak{sl}(2|1)}_{a,b}(\tau+1)$$

The function $\psi_{2,(a,b)}(\tau)$ extends to an analytic function in the slit plane $\mathbb{C}'$.

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with

$$\psi_{2,(a,b)}(\tau) = \widetilde{r}_{2,(a,b)} + \frac{1}{\pi}\int_{\mathrm{Re}(s)=-1/2} ds\,(2\pi\tau)^{-s}\frac{\Gamma(s)}{\sin\left(\frac{s\pi}{2}\right)}2^{-s}\zeta(s,a/2)\zeta(s,b/2),$$

$$\widetilde{r}_{2,(a,b)}(\tau) = \mathrm{const}_{a,b} + \frac{1}{2}(a-1)(b-1) + \frac{1}{2\pi i\tau}\times$$
$$\times\left(\log(-4\pi i\tau) + \gamma - \gamma_0(a/2) - \gamma_0(b/2) - \pi i\sum_{a',b'}(-1)^{aa'+bb'}\mathrm{const}_{a',b'}\right)$$

The limit of $\widehat{Z}^{\mathfrak{sl}(2|1)}(L(2,1),\tau)$ when τ tends to the real line, provides, up to a non-smooth correction term, new examples of vector-valued quantum modular forms.

Some open questions

- Physical interpretation of the choice of contours
- How to regularize the integral for oppositely oriented 3-manifolds
- Positivity of the coefficients
- The homological blocks for Seifert 3-manifolds with 3 exceptional fibers can be given in terms of

$$\sum_{m\geq 0} \frac{q^{\alpha m^2+\beta m}}{1-q^{Am+B}}$$

- Provide a categorification of the homological blocks

• more general supergroups • more complicated 3-manifolds
• connection to logarithmic VOAs • homological blocks for supergroups

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