# Generalised Structures for Supersymmetric Backgrounds

André Janeiro Coimbra

Institut des Hautes Études Scientifiques

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Based on work with Charles Strickland-Constable and Daniel Waldram
1411.5721, 1504.02465 and 1606.09304

# Geometry of Supersymmetric Backgrounds

Consider a compactification of string theory to four-dimensional Minkowski spacetime. If the background is supersymmetric, then in the absence of fluxes the 6d internal manifold must satisfy

*(Candelas, Horowitz, Strominger, Witten '85)*

$$\nabla\epsilon = 0$$
$$\downarrow$$
$$\text{Calabi-Yau 3-fold}$$

This is an example of a compact manifold with special holonomy, i.e. a manifold in which there exist spinor fields parallel with respect to the Levi–Civita connection. For a CY3-fold we have a torsion-free $SU(3)$ structure.

# Geometry of Supersymmetric Backgrounds

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$$[\nabla + (\mathrm{Flux})\,] \cdot \epsilon = 0$$

If fluxes are turned on, the compatible connection is not torsion-free, so it is not a special holonomy manifold. So what is the geometry of the internal manifold?

Some of the strategies that have been used:

- $G$-structures – classify solutions based on torsion classes of the structure defined by Killing spinor
- Generalised Complex Geometry – "geometrise" NSNS sector
  *(review: Graña '05 [hep-th/0509003])*
- Exceptional Generalised Geometry – "geometrise" NSNS and RR
  *(Hull '07; Pacheco, Waldram '08; Graña, Orsi '11)*

# Field Ansatz for Eleven-Dimensional Supergravity

Focus on eleven-dimensional supergravity reduced to four dimensions – results also hold for IIA and IIB and for internal spaces with $d \leq 7$.

We keep only the components of the eleven-dimensional fields which are scalars in the external space.

Therefore we take the metric to be

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{2\Delta}\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu + g_{mn}\mathrm{d}x^m \mathrm{d}x^n,$$

and keep the components of the 4-flux $\mathcal{F}$

$$F_{m_1 \dots m_4} = \mathcal{F}_{m_1 \dots m_4}, \qquad \tilde{F}_{m_1 \dots m_7} = (*_{11}\mathcal{F})_{m_1 \dots m_7}\,.$$

These field strengths are globally defined closed forms, which means that we have "gerbe"-like gauge fields, the 3-form $A_{mnp}$ and the 6-form $\tilde{A}_{mnpqrs}$.

The fermionic content is given by two components of the gravitino $\Psi_M$, the internal gravitino $\psi_m$ and the trace of the external component $\rho$.

## The Killing Spinor Equations

For supersymmetric vacua we set the fermions to zero and require the existence of at least one spinor $\varepsilon$ globally defined on $M$ such that the supersymmetric variations of all the fields with respect to $\varepsilon$ vanish.

This implies that

$$\delta\rho = \left[\nabla\!\!\!/ - \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/ + (\partial\!\!\!/\Delta)\right]\varepsilon = 0$$

$$\delta\psi_m = \big[\nabla_m + \tfrac{1}{288}F_{n_1\ldots n_4}\left(\Gamma_m{}^{n_1\ldots n_4} - 8\delta_m{}^{n_1}\Gamma^{n_2 n_3 n_4}\right) - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{mn_1\ldots n_6}\Gamma^{n_1\ldots n_6}\big]\varepsilon = 0$$

These are the Killing Spinor Equations and we call $\varepsilon$ the Killing spinor.

More independent Killing spinors imply that more supersymmetry is preserved.

## $E_{7(7)} \times \mathbb{R}^+$ Generalised Geometry

Generalised Geometry was first introduced by Hitchin in 2002 as a form of unifying symplectic and complex geometry. Physically, in analogy to the relation between Riemannian geometry and general relativity, we can think of generalised geometries as a new attempt at "geometrising" the bosonic symmetries of supergravity.

We introduce an extended notion of tangent space, where generalised vectors are patched together precisely according to the supergravity symmetries. By studying structures on these generalised tangent spaces, we can gain new insights into supergravity.

In previous work we showed that $E_{d(d)} \times \mathbb{R}^+$ generalised geometry can be used to fully reformulate eleven-dimensional supergravity restricted on a $d \leq 7$-dimensional compact manifold, making its larger local symmetries manifest.

Since we are looking at reductions down to four dimensions, we will use $E_{7(7)} \times \mathbb{R}^+$ generalised geometry

# The Generalised Tangent Space

Let $M$ be a 7-dimensional spin manifold.
The *generalised tangent space* $E$ of $M$ is given by

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)$$

Globally, $E$ is actually defined as a series of extensions, twisted by gerbes which encode the topology of the gauge fields.

On an open subset $U_{(i)} \subset M$ we can write

$$V_{(i)} \in \Gamma(TU_i \oplus \Lambda^2 T^*U_i \oplus \Lambda^5 T^*U_i \oplus (T^*U_i \otimes \Lambda^7 T^*U_i))$$

Then the patching on the overlap $U_{(i)} \cap U_{(j)}$ is given by

$$\begin{aligned} V_{(i)} &= v_{(i)} + \omega_{(i)} + \sigma_{(i)} + \tau_{(i)} \\ &= v_{(j)} + \omega_{(j)} + i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)} + \sigma_{(j)} + i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)} + \omega_{(j)} \wedge \mathrm{d}\Lambda_{(ij)} + \ldots \end{aligned}$$

where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally 2- and 5-forms which satisfy certain consistency conditions on higher order overlaps. This matches precisely the gauge transformations of supergravity.

Crucially, the symmetry transformations $GL(7,\mathbb{R}) \ltimes$ “Gauge” $\subset E_{7(7)} \times \mathbb{R}^+$.

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P. Pacheco, D. Waldram ’08
A.C., C. Strickland-Constable, D. Waldram ’11

# The Generalised Tangent Space

In fact, the fiber $E_x$ at $x \in M$ forms the $\mathbf{56_1}$ representation space of $E_{7(7)} \times \mathbb{R}^+$.

Frames for $E$ form an $E_{7(7)} \times \mathbb{R}^+$ principal bundle, the generalised frame bundle $\tilde{F}$. *Generalised tensors* will then be associated to different representations of $E_{7(7)} \times \mathbb{R}^+$.

Several familiar notions from Riemannian geometry can be defined for the $E_{7(7)} \times \mathbb{R}^+$ generalised tangent bundle.

# Dorfman Bracket

The differential structure of $E$ is given by the *Dorfman bracket*, a generalisation of the Lie derivative which combines the action of infinitesimal diffeomorphisms and gauge transformations

$$\begin{aligned} L_V V' = \mathcal{L}_v v' + \left(\mathcal{L}_v \omega' - i_{v'} \mathrm{d}\omega\right) + \left(\mathcal{L}_v \sigma' - i_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega\right) \\ + \left(\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma\right) . \end{aligned} \tag{1}$$

The Dorfman bracket is not antisymmetric, but it does satisfy the Leibniz property, i.e. $E$ is a Leibniz algebroid.

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I. Dorfman '87; T. Courant '90

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In $E_{7(7)} \times \mathbb{R}^+$ language:

$$L_V W^M = V^N \partial_N W^M - W^N (\partial \times_{\mathrm{ad}} V)^M{}_N$$

where

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m \\ 0 & \text{else} \end{cases} \in E^*$$

I. Dorfman '87; T. Courant '90

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# Generalised Connections

A *generalised connection* is a first-order linear differential operator which acts on generalised vectors as

$$D_M W^N = \partial_M W^N + \Omega_M{}^N{}_P W^P$$

where $\Omega_V = V^M \Omega_M{}^N{}_P \in \mathrm{ad}\, \tilde{F}$.

For $d = 7$ the space of generalised connections is therefore $\mathbf{56} \times (\mathbf{133} + \mathbf{1})$ dimensional.

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The *generalised torsion* of a generalised connection is defined as usual by

$$T(V, W) = L_V^D W - L_V W$$

now with the Dorfman derivative instead of the Lie derivative.

We find that the generalised torsion constraints some components of the connection

$$T \in W \subset E^* \otimes \mathrm{ad}\,\tilde{F}$$

with $W$ in the $\mathbf{912}_{-1} + \mathbf{56}_{-1}$ representation of $E_{7(7)} \times \mathbb{R}^+$.

M. Gualtieri '04

# Generalised Metric and Spinors

We now introduce extra structure, in analogy with Riemannian geometry. Consider the maximal compact subgroups $SU(8)/\mathbb{Z}_2 \subset E_{7(7)}$.

An $SU(8)/\mathbb{Z}_2$ structure on $E$ is defined by a *generalised metric* $\mathcal{G}$ which at each point parametrises the coset

$$\mathcal{G} \in \frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)/\mathbb{Z}_2}$$

This precisely corresponds to the degrees of freedom of the bosonic supergravity fields, which are thus unified in a single object

$$\{g, A, \tilde{A}, \Delta\} \in \mathcal{G}$$

$Spin(7)$ spinors can be identified as transforming under the double cover $SU(8)$. The fermion fields $\psi_m$ and $\rho$ are thus thought of as $SU(8)$ objects.

M. Gualtieri '04
C. Hull '07

# Generalised Levi–Civita

With a generalised metric, we can restrict to covariant derivatives that preserve the metric

$$D\mathcal{G} = 0$$

These are then $SU(8)$ connections, the generalised analogue of spin connections.

Like the Levi–Civita connection of Riemannian geometry, it is always possible to find a generalised $D$ which is both *metric-compatible and torsion-free*.

In fact, there exists entire families of "generalised Levi–Civita" connections – they are not uniquely determined, unlike ordinary geometry.

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A.C., C. Strickland-Constable, D. Waldram '11

# Generalised Curvatures

However, the generalised analogues of *Ricci curvatures are uniquely determined*. They can be computed from any generalised Livi–Civita and give the same result.

A simple way of obtaining the generalised Ricci scalar in $SU(8)$ is via an analogue of the Lichnerowicz relation

$$D^{\alpha\beta}\bar{D}_{\beta\gamma}\varepsilon^{\gamma} - \tfrac{1}{2}\bar{D}_{\beta\gamma}D^{[\alpha\beta}\varepsilon^{\gamma]} = \mathcal{R}\,\varepsilon^{\alpha}.$$

Remarkably, this allows us to rewrite the entire bosonic sector of the supergravity as just generalised Einstein gravity

$$S_B = \int |\det \mathcal{G}|^{\frac{\dim E}{9-d}}\,\mathcal{R}$$

Even though the connection used to compute $\mathcal{R}$ is ambiguous, the scalar itself is not – good thing, otherwise could not do physics!

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A.C., C. Strickland-Constable, D. Waldram '11, '12

# Killing Spinor Equations Revisited

To complete the rewrite of the bosonic sector of supergravity, we need the SUSY variations. Like the Riccis, these are independent of the choice of generalised Levi–Civita. In particular, given any metric-compatible $D$ such that $T(D) = 0$, the supersymmetry transformations of the fermions are just

$$\delta\psi^{\alpha\beta\gamma} = D^{[\alpha\beta}\varepsilon^{\gamma]}$$

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So the Killing Spinor Equations

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So the Killing Spinor Equations are now simply

$$\bar{D}_{\alpha\beta}\varepsilon^{\beta} = 0, \qquad D^{[\alpha\beta}\varepsilon^{\gamma]} = 0$$

Already looks very close to the special holonomy equations.
Clearly if we have a torsion-free connection with

$$D\varepsilon = 0 \Rightarrow \bar{D}_{\alpha\beta}\varepsilon^{\beta} = 0, \qquad D^{[\alpha\beta}\varepsilon^{\gamma]} = 0$$

so the background is supersymmetric.
Does the converse hold? Given a supersymmetric background can we find a generalised connection $D$ such that $D\varepsilon = 0$ *and* is torsion-free $T(D) = 0$?

## The Generalised Intrinsic Torsion of $SU(7)$-structures

The obstruction to finding a torsion-free connection which is compatible with a given $G$-structure is measured by the *intrinsic torsion* of the structure. The vanishing of the intrinsic torsion is generically a first-order differential condition on the objects defining the structure.

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If $M$ admits a reduced generalised $G$-structure which is torsion-free we say that it is a generalised special holonomy space.

In Euclidean signature, it is easy to show the generalised Ricci tensor vanishes as well – physically, the background solves the equations of motion.

A nowhere vanishing spinor $\varepsilon \in \mathbf{8}$ defines an $SU(7) \subset SU(8)$ structure in the generalised tangent bundle.

So we must look in the space of $SU(7)$-compatible connections and see if it is possible to find one which is torsion-free in backgrounds which are supersymmetric.

# The Generalised Intrinsic Torsion of $SU(7)$-structures

We can explicitly compute the generalised intrinsic torsion space of this $SU(7)$ structure. We have the $SU(8)$ representations

$$\begin{aligned} E &= \mathbf{56} \to \mathbf{28} + \bar{\mathbf{28}} \text{ (generalised vector space)} \\ W &= \mathbf{56} + \mathbf{912} \to \mathbf{28} + \bar{\mathbf{28}} + \mathbf{36} + \bar{\mathbf{36}} + \mathbf{420} + \bar{\mathbf{420}} \text{ (torsion space)} \\ \text{KSEs} &= \mathbf{8} + \bar{\mathbf{8}} + \mathbf{56} + \bar{\mathbf{56}} \end{aligned}$$

(to avoid clutter we will omit the complex conjugates from now on)

# The Generalised Intrinsic Torsion of $SU(7)$-structures

The next step is to calculate their $SU(7)$ decompositions:

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$$K_{SU(7)} = E^* \otimes \mathfrak{su}_7 = (\mathbf{7} + \mathbf{21}) \times \mathbf{48}$$

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and finally we can compute the space *of intrinsic torsion of $SU(7)$-structures*

$$\begin{aligned} W_{\text{int}} = \frac{W}{W_{SU(7)}} &= \mathbf{1} + \mathbf{7} + \mathbf{21} + \mathbf{35} \\ &= \text{KSEs} \end{aligned}$$

# $SU(7)$ Generalised Special Holonomy Manifolds

Thus setting the Killing Spinor Equations to zero is equivalent to demanding the vanishing of the generalised intrinsic torsion.

$$\delta\rho = 0, \quad \delta\psi = 0$$
$$\Updownarrow$$
$$\exists\, D:\ D\varepsilon = 0, \quad T(D) = 0$$

in which case we have the generalised analogue of special holonomy.

A.C., C. Strickland-Constable, D. Waldram '14
cf. M. Graña, F. Orsi '11

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> *Manifolds with a generalised torsion-free $SU(7)$-structure are $\mathcal{N} = 1$ supersymmetric backgrounds of M theory and vice-versa.*

(can think of these manifolds as "exceptional generalised Calabi-Yau")

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A.C., C. Strickland-Constable, D. Waldram '14
cf. M. Graña, F. Orsi '11

## Other dimensions, more SUSY

| $d$ | $\tilde{H}_d$ | $G_{\mathcal{N}}$ |
|---|---|---|
| 7 | $SU(8)$ | $SU(8-\mathcal{N})$ |
| 6 | $USp(8)$ | $USp(8-2\mathcal{N})$ |
| 5 | $USp(4)\times USp(4)$ | $USp(4-2\mathcal{N}_+)\times USp(4-2\mathcal{N}_-)$ |
| 4 | $USp(4)$ | $USp(4-2\mathcal{N})$ |

Table: Generalised structure subgroups $G_{\mathcal{N}} \subset \tilde{H}_d$ preserving $\mathcal{N}$ supersymmetry in $(11-d)$-dimensional Minkowski backgrounds. Note that for $d=5$ we have six-dimensional supergravity with $(\mathcal{N}_+, \mathcal{N}_-)$ supersymmetry.

The result is more general

> *The internal spaces of $\mathcal{N}$ supersymmetric Minkowski backgrounds are precisely the spaces of generalised $G_{\mathcal{N}}$ special holonomy.*

The proof has some subtleties for $\mathcal{N}>2$, requires the use of the fact that Killing spinors form a superalgebra – neat formulation in terms of generalised geometry, uses the Kosman-Dorfman bracket of spinors.

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A.C., C. Strickland-Constable '16

cf. M. Graña, F. Orsi '12

# AdS backgrounds

| $d$ | $G$ | $G_{\text{com}}$ | R-symmetry | $W_{\text{int}}$ |
|---|---|---|---|---|
| 7 | $SU(7)$ | $U(1)$ | $\mathbb{Z}_2$ | $\mathbf{1}_2$ |
| 6 | $USp(6)$ | $USp(2)$ | $U(1)$ | $(\mathbf{3},\mathbf{1})$ |
| 5 | $USp(2)\times USp(4)$ | $USp(2)$ | — | no singlets |
| 4 | $USp(2)$ | $USp(2)$ | $USp(2)$ | $(\mathbf{1},\mathbf{1})$ |

Table: Generalised structure subgroups $G \subset \tilde{H}_d$, commutant groups $G_{\text{com}}$ of $G$ in $\tilde{H}_d$, AdS R-symmetry groups and non-vanishing generalised intrinsic torsion as representations of $G_{\text{com}} \times G$ for minimal supersymmetry in AdS backgrounds.

Can similarly show that generic AdS backgrounds correspond to spaces which have generalised "weak special holonomy" – they are not torsion-free, they have a *constant singlet torsion* component, which gives the cosmological constant.

For $\mathrm{AdS}_4 \times M_7$ we have

$$\begin{aligned} \delta\psi &= D^{[\alpha\beta}\epsilon^{\gamma]} = 0, \\ \delta\rho &= D_{\alpha\beta}\epsilon^{\beta} = \Lambda\bar{\epsilon}_\alpha \end{aligned} \qquad (2)$$

A.C., C. Strickland-Constable '15

M. Graña, P. Ntokos '16

A. Ashmore, M. Petrini, D. Waldrama '16

# Conclusion

$E_{7(7)} \times \mathbb{R}^+$ generalised geometry allows us to "geometrise" the full bosonic sector of four-dimensional backgrounds of eleven-dimensional supergravity.

This enabled us to re-interpret $\mathcal{N} = 1$ flux backgrounds as manifolds with $SU(7)$ generalised special holonomy for Minkowski/singlet $SU(7)$ torsion for AdS $\rightarrow$ "integrability" condition that works for all possible fluxes.

Also works in other dimensions, and with higher $\mathcal{N}$.

Would be interesting to know if similar results also hold for other types of generalised geometry, which are used to describe other supergravities. Easy to show it holds for $O(d, d)$ generalised geometry. Is it a general feature?

Can this result be used to find new solutions?

Describe the moduli space of generic backgrounds?

Applications to higher-derivative corrections?

# Conclusion

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Thank you very much.