Rademacher expansion from symplectic symmetries of the Siegel modular form: An $\mathcal{N} = 4$ counting story

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based on work with Gabriel Cardoso and Suresh Nampuri

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Motivation

Modular forms: uncannily effective in BPS black hole counting

$$f(\sigma) = \sum_{n \geq n_0} c(n) q^n \qquad q = e^{2\pi i \sigma}$$

Hardy-Ramanujan-Rademacher expansion: Modular symmetry so powerful that microscopic degeneracies have exact expression from finite set of data: the polar part.

$$c(n>0) = \sum_{\gamma \geq 1} c(\tilde{n}<0) Kl(n,\tilde{n},\gamma) I(n,\tilde{n},\gamma)$$

Phases **Bessel functions**

$c(\tilde{n}<0)$ is the minimum amount of information to reconstruct $f(\sigma)$

Examples

Modular objects are generating functions for BPS black hole degeneracies

$$Z^{\mathcal{N}=4}_{1/2\,BPS} = \frac{1}{\eta^{24}(\sigma)} = \sum_{n\geq -1} d(n) q^n$$

[Dabholkar '05]

$$Z^{\mathcal{N}=8}_{1/8\,BPS} = \frac{\theta_1(\sigma,v)^2}{\eta^6(\sigma)} = \sum_{4n-\ell^2\geq -1} C(\Delta = 4n-\ell^2) q^n y^\ell$$

[Maldacena, Moore, Strominger '99]

Only one polar term

$d(-1), C(-1)$

Hardy-Ramanujan-Rademacher expansion

$$d(n) = \sum_{\gamma=1}^{\infty} \left(\frac{2\pi}{\gamma}\right)^{14} Kl(n,-1,\gamma) I_{13}\left(\frac{4\pi\sqrt{n}}{\gamma}\right) \qquad \text{for } n > 0$$

$$C(\Delta) = 2\pi \left(\frac{\pi}{2}\right)^{7/2} i^{5/2} \sum_{\gamma=1}^{\infty} c^{-9/2} Kl_\gamma(\Delta) I_{7/2}\left(\frac{\pi\sqrt{\Delta}}{\gamma}\right) \qquad \text{for } \Delta > 0$$

1/4-BPS generating function

The generating function for $1/4$-BPS dyonic degeneracies in $\mathcal{N} = 4, D = 4$ heterotic string theory is a modular form of the genus-2 modular group $Sp(2,\mathbb{Z})$. Φ_{10} Igusa cusp form.

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{\substack{m,n \geq -1 \\ m,n,\ell \in \mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho+n\sigma+\ell v)}$$

[Dijgkraaf, Verlinde, Verlinde '96]

Single-centered black holes have $\Delta = 4mn - \ell^2 > 0$. What are the polar terms?
$\Delta = 4mn - \ell^2 < 0$, 2-centered states, bound states of $1/2$-BPS states.

$SL(2,\mathbb{Z}) \subset Sp(2,\mathbb{Z})$, can we use the $Sp(2,\mathbb{Z})$ symmetries to specify the minimal amount of information needed to reconstruct $d(m,n,\ell)$?

(See Valentin Reys' talk)

Can this give us insight into the saddle points for the gravity path integral?

Results

Symplectic symmetries do yield a generalized Rademacher expansion, with the the polar terms specified by the coefficients of $\eta^{-24}(\sigma)$.

[Murthy, Pioline '09]

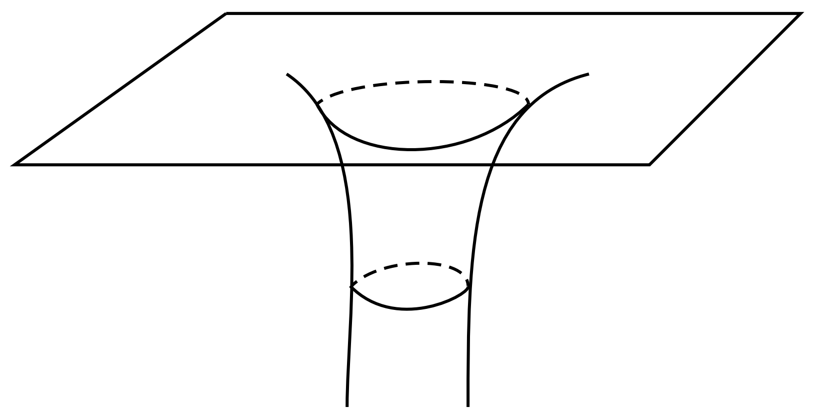
Answer parametrised by two sets of $SL(2,\mathbb{Z})$ matrices inside $Sp(2,\mathbb{Z})$. One set of matrices parametrizes the Rademacher expansion and the other the polar coefficients.

Structure of polar coefficients reproduces exactly the continued fraction structure found in previous work.

$Sp(2,\mathbb{Z})$ symmetry powerful enough to systematically encode all this data

Outline

Dyonic degeneracies

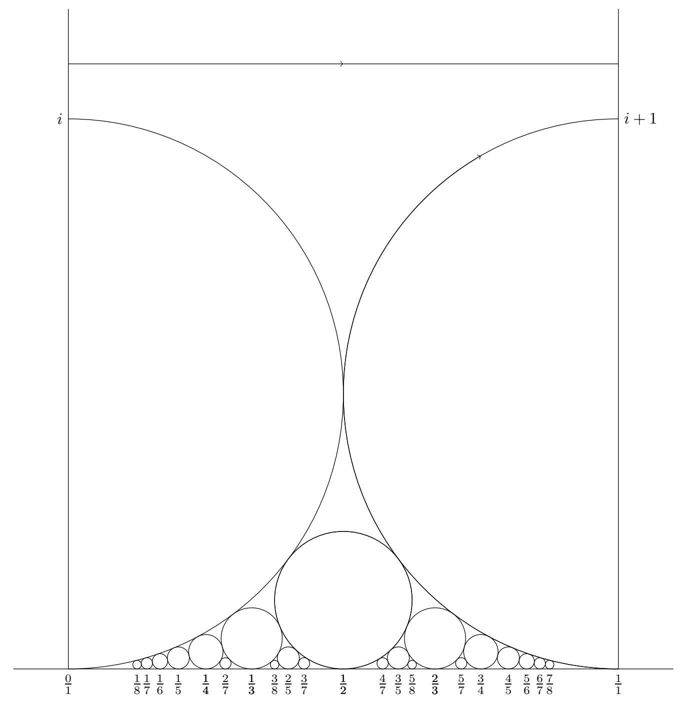


Siegel modular forms

Mock Jacobi forms

$$\frac{1}{\Phi_{10}} \qquad \psi_m^F$$

Rademacher expansion



Continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Dyonic degeneracies

Heterotic string theory on T^6, $\mathcal{N} = 4$ supersymmetry, $S-$duality group is $SL(2,\mathbb{Z})$

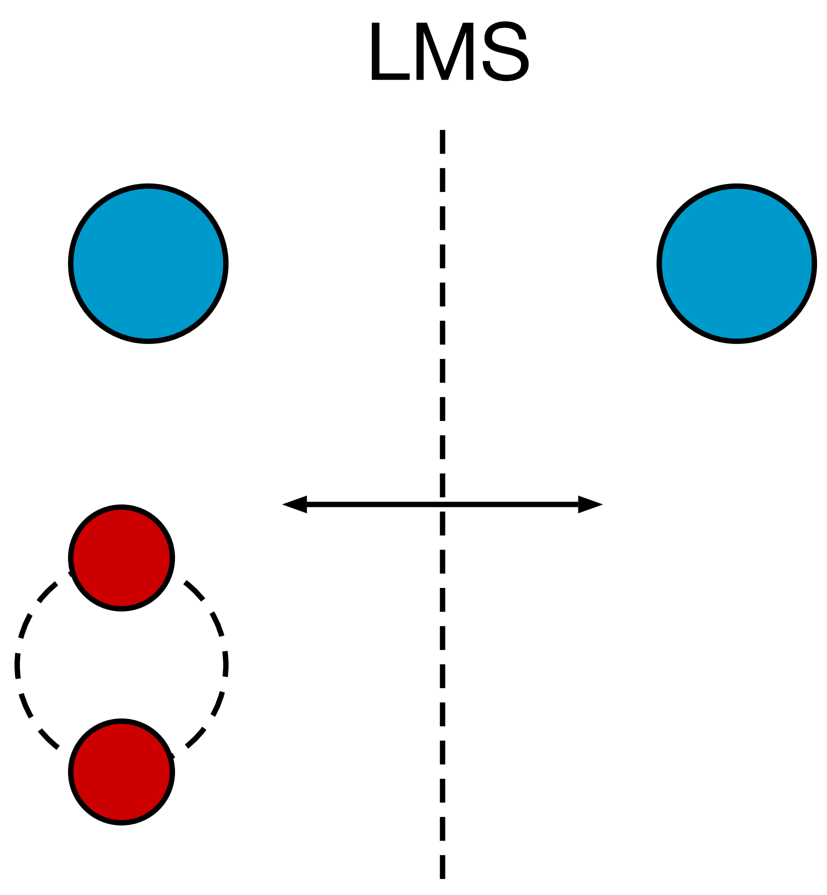
$1/4$-BPS states carry electric $\vec{Q}$ and magnetic $\vec{P}$ charges: **Dyons**

Degeneracies characterized by $m = P^2/2 \in \mathbb{Z}$, $n = Q^2/2 \in \mathbb{Z}$, $\ell = P \cdot Q \in \mathbb{Z}$

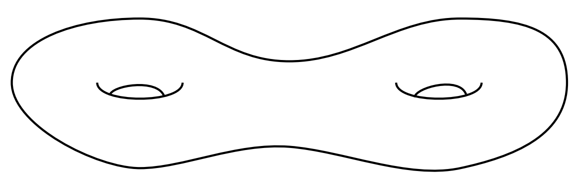
$$d(\vec{P}, \vec{Q}) = d(m, n, \ell)$$

$S-$duality invariant: $\quad \Delta = Q^2P^2 - (Q \cdot P)^2 = 4mn - \ell^2$

Two types of state in the spectrum, single-centered: immortal, and two-centered: can (dis)appear across Lines of Marginal Stability.



For immortal dyons, $\Delta > 0$, Area $\sim \sqrt{\Delta}$



Siegel modular forms

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

$\Phi(\Omega)$ a Siegel modular form of degree 2 and weight ω if

$$\det \mathrm{Im}\Omega > 0$$

$$\Phi((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^{\omega}\Phi(\Omega) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z})$$

Fourier-Jacobi expansion into Jacobi forms $\psi_m(\sigma, v)$ of weight ω and index m,

$$\Phi(\Omega) = \sum_{m \geq m_0} \psi_m(\sigma, v) p^m \qquad p = e^{2\pi i \rho}$$

$$\psi_m\left(\frac{a\sigma + b}{c\sigma + d}, \frac{v}{c\sigma + d}\right) = (c\sigma + d)^{\omega}\, e^{\frac{2\pi i m c v^2}{c\sigma + d}}\, \psi_m(\sigma, v)\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\psi_m(\sigma, v + \lambda\sigma + \mu) = e^{-2\pi i m(\lambda^2\sigma + 2\lambda v)}\, \psi_m(\sigma, v) \qquad \lambda, \mu \in \mathbb{Z}$$

Siegel modular form

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{\substack{m,n\geq -1 \\ m,n,\ell\in\mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho+n\sigma+\ell v)}$$

$$\Phi_{10}^{-1}((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^{-10}\Phi_{10}^{-1}(\Omega)$$

$\Phi_{10}^{-1}(\Omega)$ is meromorphic. To extract the coefficients specify contour,

[Cheng,Verlinde '07]

[Sen '07]

$$d(m,n,\ell)_{imm} = \int_{\mathcal{C}_{imm}} d\sigma\, dv\, d\rho\, (-1)^{\ell+1} \frac{e^{-2\pi i(m\rho+n\sigma+\ell v)}}{\Phi_{10}(\rho,\sigma,v)}$$

$$\mathcal{C}_{imm} : \rho_2 = 2nK,\ \sigma_2 = 2mK,\ v_2 = -\ell K,\ \ K \gg 1$$

Poles

Poles labelled by 5 integers (n_2, n_1, j, m_2, m_1)

$$n_2(\rho\sigma - v^2) + jv + n_1\sigma - m_1\rho + m_2 = 0$$

Satisfying

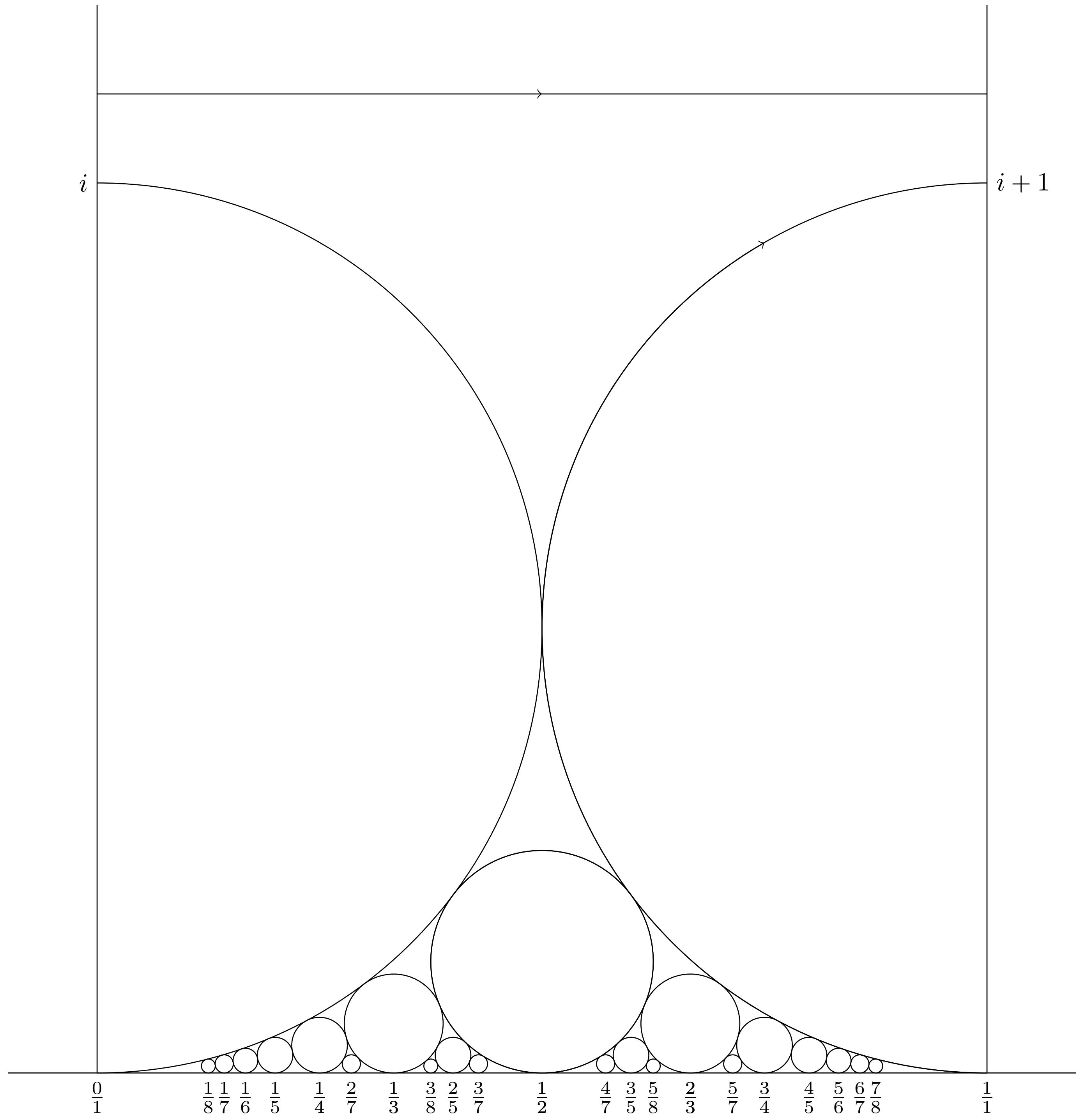
$$m_1 n_1 + m_2 n_2 = \frac{1}{4}\left(1 - j^2\right)$$

Two types, $n_2 = 0$ linear poles and $n_2 \neq 0$ quadratic poles. Associated to two-centered and single-centered states, resp. Linear poles parametrized by $SL(2,\mathbb{Z})$ matrices. Represent lines of marginal stability.

Using $Sp(2,\mathbb{Z})$, can map any pole to 'simplest' pole given by $v = 0$, where

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = -\frac{1}{4\pi^2}\frac{1}{v^2}\frac{1}{\eta^{24}(\rho)}\frac{1}{\eta^{24}(\sigma)} + \mathcal{O}(v^0)$$

Rademacher expansion



$$\frac{1}{\eta^{24}(\rho)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n \rho} \qquad d(n) = \int_z^{z+1} d\rho\, e^{2\pi i n \rho} \frac{1}{\eta^{24}(\rho)}$$

Deform contour along Ford circles, use modular symmetry to find behaviour near rational points $-\delta/\gamma$

$$\eta^{-24}(\sigma) = (\gamma\sigma + \delta)^{10} \eta^{-24}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right) \qquad d(n) = \frac{2\pi}{n^{\frac{13}{2}}} \sum_{\gamma>0} \frac{K(-1,n,\gamma)}{\gamma} I_{13}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)$$

where $Kl(n, m, \gamma)$ is the classical Kloosterman sum

$$Kl(n,m,\gamma) = \sum_{\substack{\delta \in \mathbb{Z}/\gamma\mathbb{Z} \\ \alpha\delta \equiv 1 \mod \gamma}} e^{2\pi i\left(n\frac{\delta}{\gamma} + m\frac{\alpha}{\gamma}\right)}$$

and $I_\rho(z)$ the Bessel function of index ρ

$$I_\rho(z) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{t^{\rho+1}} e^{t+\frac{z^2}{4t}}$$

Rademacher expansion

Jacobi form of weight $\omega < 0$ and index m for $\Delta > 0$

$$C_\nu(\Delta) = i^{-\omega+\frac{1}{2}} \sum_{\gamma=1}^{\infty} \left(\frac{c}{2\pi}\right)^{\omega-\frac{5}{2}} \sum_{\tilde{\Delta}<0} C_\mu(\tilde{\Delta}) Kl\left(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}, \gamma\right)_{\mu\nu} \left|\frac{\tilde{\Delta}}{4m}\right|^{\frac{3}{2}-\omega} I_{\frac{3}{2}-\omega}\left(\frac{\pi}{\gamma}\sqrt{|\tilde{\Delta}|\Delta}\right)$$

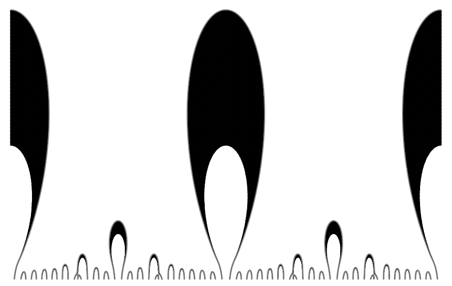
$C_\mu(\tilde{\Delta})$ with $\tilde{\Delta} < 0$, the polar coefficients, are the only input of the formula

$$Kl\left(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}; \gamma, \psi\right)_{\ell\tilde{\ell}} = \sum_{\substack{0 \le -\delta < \gamma \\ (\delta,\gamma)=1, \alpha\delta\equiv 1 \bmod \gamma}} e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{\tilde{\Delta}}{4m} + \frac{\delta}{\gamma}\frac{\Delta}{4m}\right)} \psi(\Gamma)_{\tilde{\ell}\ell}$$

Generalized Kloosterman sum

$$\psi(\Gamma)_{\ell j} = \frac{1}{\sqrt{2m\gamma i}} \sum_{T\in\mathbb{Z}/\gamma\mathbb{Z}} e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{(\ell-2mT)^2}{4m} - \frac{j(\ell-2mT)}{2m\gamma} + \frac{\delta}{\gamma}\frac{j^2}{4m}\right)}$$

Multiplier system



Mock Jacobi forms

[Ramanujan ’1920]

[Zwegers ’2001]

Fourier-Jacobi expansion $$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m\geq -1} \psi_m(\sigma,v)e^{2\pi i m\rho}$$

$$\psi_m(\sigma,v) = \psi_m^F(\sigma,v) + \psi_m^P(\sigma,v)$$

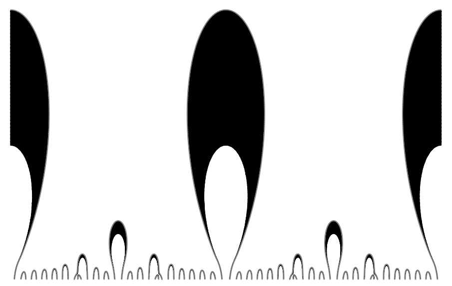
[Dabholkar, Murthy, Zagier ’12]

Split into **mock** Jacobi forms: a finite and a polar part.

$$\psi_m^F(\sigma,v) = \sum_{n,\ell} c_m^F(n,\ell)q^n y^\ell$$ has no poles in (σ,v) **Immortal**

Modularity can be restored at the expense of holomorphicity.

$$d_{imm}(m,n,\ell) = (-1)^{\ell+1}c_m^F(n,\ell)$$ (for $n > m$)



Mock Jacobi forms

$$\psi^F_m(\sigma, v) = \sum_{n,\ell} c^F_m(n,\ell) q^n y^\ell$$ transforms anomalously

[Dabholkar, Murthy, Zagier '12]

$$\psi^F_m(\sigma, v) = (c\sigma + d)^{10} e^{-2\pi i m \frac{cv^2}{c\sigma + d}} \psi^F_m\left(\frac{a\sigma + b}{c\sigma + d}, \frac{v}{c\sigma + d}\right)$$

$$-\frac{d(m)}{\eta^{24}\left(\frac{a\sigma+b}{c\sigma+d}\right)} \sqrt{\frac{m}{8\pi^2}} i^{1/2} (c\sigma + d)^{21/2} \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \vartheta_{m,\ell}(\sigma, v) \int_{-a/c}^{i\infty} \left(v + \frac{a\sigma + b}{c\sigma + d}\right)^{-3/2} \sum_{j\mathbb{Z}/2m\mathbb{Z}} \psi(\gamma)_{\ell j} \overline{\vartheta^0_{m,j}(-\overline{v})} dv$$

will have a modified Rademacher expansion

Generalized Rademacher expansion

$$\boxed{c_m^{\mathrm{F}}(n,\ell)} = 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ \boxed{4m\widetilde{n}-\widetilde{\ell}^2<0}}} \boxed{c_m^{\mathrm{F}}(\widetilde{n},\widetilde{\ell})} \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right)$$

$\boxed{4mn-\ell^2>0}$

$$+\sqrt{2m}\sum_{k=1}^{\infty}\frac{Kl\big(\frac{\Delta}{4m},-1;k,\psi\big)_{\ell 0}}{\sqrt{k}}\left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \qquad \text{(A.12)}$$

$$-\frac{1}{2\pi}\sum_{k=1}^{\infty}\sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z}\\ g\in\mathbb{Z}/2mk\mathbb{Z}\\ g\equiv j\,(\mathrm{mod}\ 2m)}}\frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\big)_{\ell j}}{k^2}\left(\frac{4m}{\Delta}\right)^{25/4}\times$$

$$\times\int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,\mathrm{d}u\,,$$

[Ferrari, Reys, ’17]

[Bringmann, Manschot ’10]

computes the coefficients $c_m^F(n,\ell)$ with $\Delta > 0$ in terms of $c_m^F(\tilde{n},\tilde{\ell})$ with $\Delta < 0$.

Polar coefficients

Study states in $\mathcal{N} = 4$ string theory with discriminant $\tilde{\Delta} = 4m\tilde{n} - \tilde{\ell}^2 < 0$.
They are bound states of two $1/2$-BPS states: Study their decays. [Sen '11]

Show there is a finite, computable set $W(m, \tilde{n}, \tilde{\ell}) \subset SL(2,\mathbb{Z})$ such that

$$d(m,\tilde{n},\tilde{\ell})_{\tilde{\Delta}<0} = \sum_{\substack{\gamma\in W \\ W\subset SL(2,\mathbb{Z})}} (-1)^{\ell_\gamma+1}\ell_\gamma d(m_\gamma)d(n_\gamma)$$

$$(m,\tilde{n},\tilde{\ell}) \underset{\gamma}{\leftrightarrow} (m_\gamma, n_\gamma, \ell_\gamma)$$

[Chowdhury, Kidambi, Murthy, Reys, Wrase '19]

The set W is generated by the continued fraction of $\tilde{\ell}/2m$ $\quad \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$

$$\frac{\tilde{\ell}}{2m} = [a_0; a_1, \ldots, a_r] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_r}}}$$

$$0 \le \frac{\tilde{\ell}}{2m} - \frac{q}{s} \le \frac{1}{rs}$$

[Cardoso, Nampuri, MR '20]

Rademacher from Siegel

$$d(m,n,\ell)_{imm} = \int_{\mathcal{C}_{imm}} d\sigma\, dv\, d\rho\, (-1)^{\ell+1} \frac{e^{-2\pi i(m\rho+n\sigma+\ell v)}}{\Phi_{10}(\rho,\sigma,v)}$$

$$\mathcal{C}_{imm} : \rho_2 = 2nK,\ \sigma_2 = 2mK,\ v_2 = -\ell K,\ \ K \gg 1 \qquad 0 \le \rho_1, \sigma_1, v_1 \le 1$$

Working assumptions: Poles encode the full degeneracy and we can interchange sums and integrations

Perform first integral as a sum over residues

Use $Sp(2,\mathbb{Z})$ symmetries to bring any pole (n_2, n_1, j, m_2, m_1) to a $v' = 0,\ (m_2', m_1', j', n_2', n_1') = (0,0,1,0,0)$ pole

Pole transformations

Two $SL(2,\mathbb{Z}) \in Sp(2,\mathbb{Z})$ transformations needed,

[Murthy, Pioline '09]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma & 0 & \alpha \end{pmatrix}, \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix} \in Sp(2,\mathbb{Z})$$

First one takes $n_2 \neq 0$ to $n_2' = 0$. Second takes general $n_2' = 0$ to $\tilde{v} = \Sigma \in \mathbb{Z}$

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \Sigma \in \mathbb{Z}$$

Constrained to $a, -c, \gamma > 0, \quad \alpha, \delta \in \mathbb{Z}/\gamma\mathbb{Z}, \quad \Sigma \in \mathbb{Z}/(-ac)\mathbb{Z},$
these transformations take (n_2, n_1, j, m_2, m_1) to $(0,0,1,0,0)$ pole

Residues

Use $Sp(2,\mathbb{Z})$ covariance, $\dfrac{1}{\Phi_{10}(\rho,\sigma,v)} = (\gamma\sigma+\delta)^{10}\dfrac{1}{\Phi_{10}(\rho',\sigma',v')}$

$$\rho' = a^2\left(\rho - \frac{\gamma v^2}{\gamma\sigma+\delta}\right) + b^2\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right) - 2ab\frac{v}{\gamma\sigma+\delta}, \qquad \sigma' = c^2\left(\rho - \frac{\gamma v^2}{\gamma\sigma+\delta}\right) + d^2\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right) - 2cd\frac{v}{\gamma\sigma+\delta}$$

$$v' = -ac\left(\rho - \frac{\gamma v^2}{\gamma\sigma+\delta}\right) - bd\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right) + (ad+bc)\frac{v}{\gamma\sigma+\delta} - \Sigma$$

Near $v'=0$, $\dfrac{1}{\Phi_{10}(\rho',\sigma',v')} \xrightarrow[v'\to 0]{} -\dfrac{1}{4\pi^2}\dfrac{1}{v'^2}\dfrac{1}{\eta^{24}(\rho')}\dfrac{1}{\eta^{24}(\sigma')}$ compute residue w.r.t. ρ,

$$(-1)^{\ell+1}\frac{(\gamma\sigma+\delta)^{10}}{ac}\left(\frac{m}{ac} + \frac{a}{c}E_2(\rho'_*) + \frac{c}{a}E_2(\sigma'_*)\right)\frac{1}{\eta^{24}(\rho'_*)}\frac{1}{\eta^{24}(\sigma'_*)}e^{-2\pi i(m\Lambda+n\sigma+\ell v)}$$

Fourier expansion

$$(-1)^{\ell+1}\frac{(\gamma\sigma+\delta)^{10}}{ac}\left(\frac{m}{ac}+\frac{a}{c}E_2(\rho''_*)+\frac{c}{a}E_2(\sigma''_*)\right)\frac{1}{\eta^{24}(\rho''_*)}\frac{1}{\eta^{24}(\sigma''_*)}e^{-2\pi i(m\Lambda+n\sigma+\ell v)}$$

Fourier expanding, full expression becomes

$$(-1)^{\ell}\sum_{P'}\sum_{M,N\geq -1}(\gamma\sigma+\delta)^{10}\boxed{\ell_\gamma d(M)d(N)}\exp\left(-2\pi i\left[-\tilde{n}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)-\tilde{\ell}\frac{v}{\gamma\sigma+\delta}+m\frac{\gamma v^2}{\gamma\sigma+\delta}+\ell v+n\sigma\right]\right)$$

where $\ell_\gamma=-\frac{m}{ac}+\frac{a}{c}M+\frac{c}{a}N$, $\tilde{\ell}=-\frac{ad+bc}{ac}m+\frac{a}{c}M-\frac{c}{a}N$, $\tilde{n}=\frac{bd}{ac}m-\frac{b}{c}M+\frac{d}{a}N$

Sum over $\Sigma\in\mathbb{Z}/(-ac)\mathbb{Z}$ forces $\ell_\gamma,\tilde{\ell},\tilde{n}\in\mathbb{Z}$. Extra pole restriction $\ell_\gamma>0$.

$$(m,\tilde{n},\tilde{\ell})\leftarrow\begin{pmatrix}a&b\\c&d\end{pmatrix}\rightarrow(M,N,\ell_\gamma),\quad \tilde{\Delta}=4MN-\ell_\gamma^2=4m\tilde{n}-\tilde{\ell}^2$$

v integral

Condition det Im$(\Omega) > 0$ for poles imposes restricted v contour

$$\frac{v_2}{\sigma_2}\sigma_1 + \frac{\delta}{\gamma}\frac{v_2}{\sigma_2} - \frac{b}{\gamma a} < v_1 < \frac{v_2}{\sigma_2}\sigma_1 + \frac{\delta}{\gamma}\frac{v_2}{\sigma_2} - \frac{d}{\gamma c}$$

for immortal contour values $\frac{v_2}{\sigma_2} = -\frac{\ell}{2m}$ this becomes

$$v : -\frac{\ell}{2m\gamma}(\gamma\sigma + \delta) - \frac{b}{\gamma a} \quad \rightarrow \quad -\frac{\ell}{2m\gamma}(\gamma\sigma + \delta) - \frac{b}{\gamma a} - \frac{1}{ac\gamma}$$

Integrate with these extrema. v_1 in interval of length 1 implies

$$b \in \mathbb{Z}/a\gamma\mathbb{Z}$$

v integral

v dependence only in exponential, complete square

$$\int dv\, \exp\left(-2\pi i \frac{m\gamma}{\gamma\sigma+\delta}\left(v+\frac{1}{2m\gamma}\left((\gamma\sigma+\delta)\ell-\tilde{\ell}\right)\right)^2\right)$$

use

$$\int dx\, e^{-a(x+b)^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{Erf}\left[\sqrt{a}(x+b)\right]$$

get

$$(-1)^\ell \sum_{P'} \sum_{M,N\geq -1} (\gamma\sigma+\delta)^{10} \ell_\gamma d(M) d(N) e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{\tilde{\Delta}}{4m}+\frac{\delta}{\gamma}\frac{\Delta}{4m}\right)}$$

$$e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{\tilde{\ell}^2}{4m}-\frac{\ell\tilde{\ell}}{2m\gamma}+\frac{\delta}{\gamma}\frac{\ell^2}{4m}\right)} e^{\left(-2\pi i\left[\frac{\tilde{\Delta}}{4m}\frac{1}{\gamma}\frac{1}{\gamma\sigma+\delta}+\frac{\Delta}{4m}\frac{\gamma\sigma+\delta}{\gamma}\right]\right)}$$

$$\frac{1}{2}\frac{\sqrt{\gamma\sigma+\delta}}{\sqrt{2m\gamma i}}\left(\mathrm{Erf}\left[\sqrt{\frac{2\pi i m\gamma}{\gamma\sigma+\delta}}\left(-\frac{b}{a\gamma}-\frac{1}{ac\gamma}-\frac{\tilde{\ell}}{2m\gamma}\right)\right]-\mathrm{Erf}\left[\sqrt{\frac{2\pi i m\gamma}{\gamma\sigma+\delta}}\left(-\frac{b}{a\gamma}-\frac{\tilde{\ell}}{2m\gamma}\right)\right]\right).$$

T trick and Erf trick

Use right action $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & aT+b \\ c & cT+d \end{pmatrix}$

$\tilde{\ell} = -\frac{ad+bc}{ac}m + \frac{a}{c}M - \frac{c}{a}N$ transforms, $\tilde{\ell} \to \tilde{\ell} - 2mT$

constraints now $\tilde{\ell} \in \mathbb{Z}/2m\mathbb{Z}$ $T \in \mathbb{Z}/\gamma\mathbb{Z}$

Want to use

$$\mathrm{Erf}(x) = 1 - \mathrm{Erfc}(x) = 1 - \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x} + \frac{1}{2\sqrt{\pi}}\int_{x^2}^{\infty} t^{-3/2}e^{-t}dt, \quad \text{for } \mathrm{Re}(x) > 0.$$

Take error functions to have positive real part using $\mathrm{Erf}(-x) = -\mathrm{Erf}(x)$

Continued fraction condition

$$\mathrm{Erf}(x) = 1 - \mathrm{Erfc}(x) = 1 - \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x} + \frac{1}{2\sqrt{\pi}}\int_{x^2}^{\infty} t^{-3/2}e^{-t}dt, \quad \text{for } \mathrm{Re}(x) > 0.$$

$$\mathrm{Erf}\left[\sqrt{\frac{2\pi i m}{\tilde{\sigma}}}\left(\frac{b}{a}+\frac{\tilde{\ell}}{2m}\right)\right] + \mathrm{Erf}\left[\sqrt{\frac{2\pi i m}{\tilde{\sigma}}}\left(-\frac{b}{a}-\frac{1}{ac}-\frac{\tilde{\ell}}{2m}\right)\right].$$

$$\gamma(\gamma\sigma+\delta) = \tilde{\sigma}, \qquad d\sigma = \frac{1}{\gamma^2}d\tilde{\sigma}$$

Continued fraction condition!

$$0 \le \frac{\tilde{\ell}}{2m} - \frac{q}{s} \le \frac{1}{rs}$$

Both Erf functions will have positive real part iff

$$0 < \frac{b}{a} + \frac{\tilde{\ell}}{2m} < -\frac{1}{ac}$$

Regrouping

Group terms from the '1' contribution with positive real part

$$(-1)^\ell \sum_{P'} \sum_{M,N\geq -1} \int d\tilde\sigma \, \frac{\tilde\sigma^{21/2}}{\gamma^{25/2}} \, \ell_\gamma d(M) d(N) e^{2\pi i \left(\frac{\alpha}{\gamma}\frac{\tilde\Delta}{4m} + \frac{\delta}{\gamma}\frac{\Delta}{4m}\right)} \frac{1}{\sqrt{2m\gamma i}} e^{2\pi i \left(\frac{\alpha}{\gamma}\frac{(\tilde\ell - 2mT)^2}{4m} - \frac{\ell(\tilde\ell-2mT)}{2m\gamma} + \frac{\delta}{\gamma}\frac{\ell^2}{4m}\right)} e^{\left(-2\pi i\left[\frac{\tilde\Delta}{4m}\frac{1}{\tilde\sigma} + \frac{\Delta}{4m}\frac{\tilde\sigma}{\gamma^2}\right]\right)}$$

Recognize

$$Kl\left(\frac{\Delta}{4m}, \frac{\tilde\Delta}{4m}; \gamma, \psi\right)_{\ell\tilde\ell} = \sum_{\substack{0\leq -\delta < \gamma \\ (\delta,\gamma)=1, \alpha\delta\equiv 1 \bmod \gamma}} e^{2\pi i \left(\frac{\alpha}{\gamma}\frac{\tilde\Delta}{4m} + \frac{\delta}{\gamma}\frac{\Delta}{4m}\right)} \psi(\Gamma)_{\tilde\ell \ell}, \quad \psi(\Gamma)_{\ell j} = \frac{1}{\sqrt{2m\gamma i}} \sum_{T\in\mathbb{Z}/\gamma\mathbb{Z}} e^{2\pi i \left(\frac{\alpha}{\gamma}\frac{(\ell-2mT)^2}{4m} - \frac{j(\ell-2mT)}{2m\gamma} + \frac{\delta}{\gamma}\frac{j^2}{4m}\right)}$$

The integral over $\tilde\sigma$ along Ford circles imposes $\tilde\Delta < 0$. Sum over M, N, a, c with fixed $\tilde\Delta < 0$ satisfying $0 \leq \frac{\tilde\ell}{2m} + \frac{b}{a} < -\frac{1}{ac}$ and $\tilde\ell \in \mathbb{Z}/2m\mathbb{Z}$ becomes

$$c_m^F(\tilde n, \tilde\ell)$$

Bessel 23/2

Final answer is then

$$(-1)^{\ell+1} 2\pi \sum_{\substack{\gamma>0,\tilde{\Delta}<0\\ \tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}}} c_m^F(\tilde{n},\tilde{\ell}) \frac{Kl(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m},\gamma,\psi)_{\ell\tilde{\ell}}}{\gamma} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde{\Delta}|}\right)$$

compare with

[Ferrari, Reys, '17]

$$\begin{aligned} c_m^{\mathrm{F}}(n,\ell) &= 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ 4m\tilde{n}-\tilde{\ell}^2<0}} c_m^{\mathrm{F}}(\tilde{n},\tilde{\ell}) \frac{Kl(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m};k,\psi)_{\ell\tilde{\ell}}}{k} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\tilde{\Delta}|\Delta}\right) \\ &+ \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl(\frac{\Delta}{4m},-1;k,\psi)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \\ &- \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z}\\ g\in\mathbb{Z}/2mk\mathbb{Z}\\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\ &\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right) (1-mu^2)^{25/4}\, \mathrm{d}u\,, \end{aligned} \qquad \text{(A.12)}$$

First term!

Mock part

Other terms will come from

$$\mathrm{Erf}(x) = 1 - \mathrm{Erfc}(x) = 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} + \frac{1}{2\sqrt{\pi}} \int_{x^2}^{\infty} t^{-3/2} e^{-t} dt, \quad \text{for } \mathrm{Re}(x) > 0.$$

work in progress

$$\sim$$

$$\begin{aligned} c_m^{\mathrm{F}}(n,\ell) \; &= \; 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell} \in \mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n} - \widetilde{\ell}^2 < 0}} c_m^{\mathrm{F}}(\widetilde{n},\widetilde{\ell}) \frac{Kl\left(\frac{\Delta}{4m}, \frac{\widetilde{\Delta}}{4m}; k, \psi\right)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right) \\ &+ \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl\left(\frac{\Delta}{4m}, -1; k, \psi\right)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \qquad \text{(A.12)} \\ &- \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\substack{j \in \mathbb{Z}/2m\mathbb{Z} \\ g \in \mathbb{Z}/2mk\mathbb{Z} \\ g \equiv j \,(\mathrm{mod}\ 2m)}} \frac{Kl\left(\frac{\Delta}{4m}, -1 - \frac{g^2}{4m}; k, \psi\right)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\ &\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1 - mu^2)}\right) (1 - mu^2)^{25/4}\, \mathrm{d}u\,, \end{aligned}$$

A special case, $m = 0$

For $m = 0$,

$$\psi_0^F(\sigma, v) = 2\frac{E_2(\sigma)}{\eta^{24}(\sigma)}$$

and so for $\ell = 0$ and $n \geq 0$, the immortal degeneracies $d_{imm}(0,n,0)$ are given by the Fourier coefficients of a quasi-modular form.

Using
$$\frac{E_2(\sigma)}{\eta^{24}(\sigma)} = (\gamma\sigma + \delta)^{10} \frac{E_2\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)}{\eta^{24}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)} - \frac{6\gamma}{\pi i}\frac{(\gamma\sigma+\delta)^{11}}{\eta^{24}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)}$$

one can obtain a Rademacher expansion for the coefficients

$$d_{imm}(0,n,0) = \sum_{\gamma=1}^{+\infty} Kl(n,-1,\gamma)\left(-\frac{24}{n^6} I_{12}\left(\frac{4\pi\sqrt{n}}{\gamma}\right) + \frac{4\pi}{\gamma\, n^{11/2}} I_{11}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)\right)$$

$d_{imm}(0,n,0)$ from Siegel

Performing the same sum over poles of $1/\Phi_{10}$ fixing $m = \ell = 0$, the exponent has a much simpler form. No term quadratic in v.

When there is no v dependence we can integrate it out. Integrate over σ along Ford circles and obtain

$$\sum_{\gamma=1}^{+\infty} Kl(n,-1,\gamma)\frac{4\pi}{\gamma\, n^{11/2}} I_{11}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)$$

For the linear term in v,

$$\sim \quad \sum_{\gamma=1}^{+\infty} Kl(n,-1,\gamma) \sum_{\substack{0\leq M<c^2\\ -c>0}} \frac{2}{n^6}\frac{-M+c^2}{M+c^2} d(M) I_{12}\left(\frac{4\pi}{\gamma}\sqrt{n}\right)$$

Fix $M = 0$, use $d(0) = 24$. Match! Why?

Conclusions

One can use the $Sp(2,\mathbb{Z})$ symmetries of $1/\Phi_{10}$ to perform the sum over residues with $n_2 \neq 0$ and obtain a finer-grained generalized Rademacher expansion (~): includes the expressions for the polar coefficients. Continued fraction arises naturally.

Only input $d(n)$ of $\eta^{-24}(\sigma)$: non-perturbative from perturbative.

In our parametrization $n_2 = -ac\gamma$. Rademacher expansion with respect to γ and not n_2.

Open questions

Can we use the microscopic structure as a guide for a macroscopic path integral computation? The work [Murthy, Reys '15] matches the expression of our first coefficient, $n_2 = 1$. Can we go further? What is the role of γ? And a, c? [Work in progress w/ Abhiram Kidambi & Valentin Reys]

$1/4$-BPS dyons in CHL orbifols are also counted by Siegel modular forms with a quadratic pole at $v = 0$. Polar coefficients known. Generalized Rademacher expansion not known.

Thank you