Scaling Black Holes and Modularity

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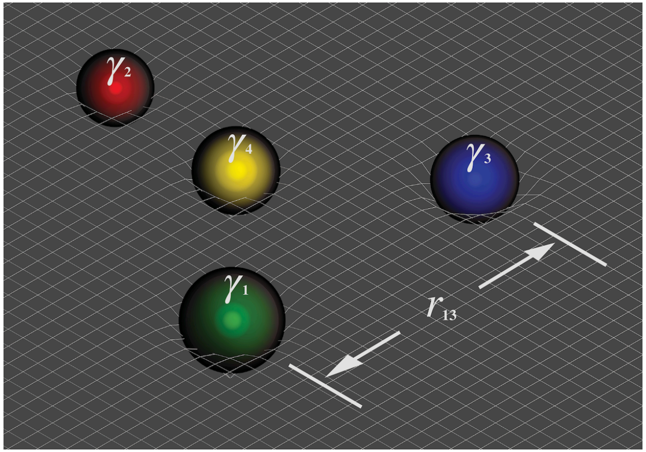
This talk is based on work in progress with Swapnamay Mondal and Aradhita Chattopadhyaya.

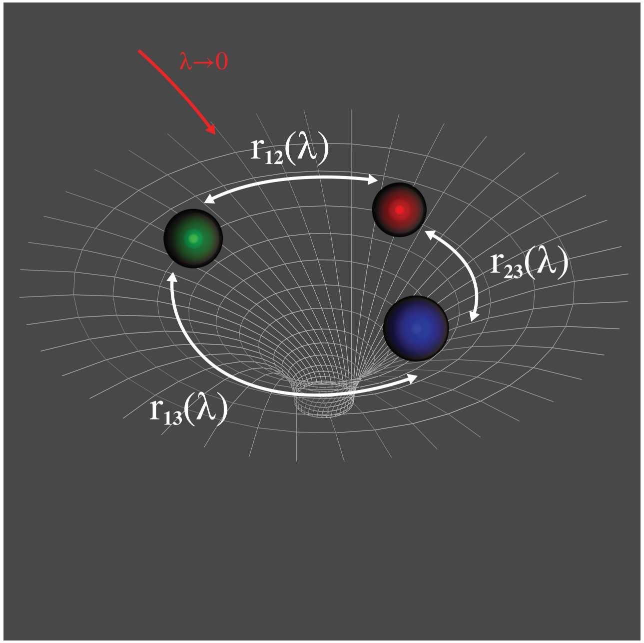
BPS Black Holes

BPS black holes provide a rich context for the study of quantum gravity. While the Cardy formula gives an accurate microscopic derivation of the entropy of large black holes, the existence of bound states of black holes diffuses the spectrum of single center $\mathcal{N} = 2$ black holes.



BPS Black Holes

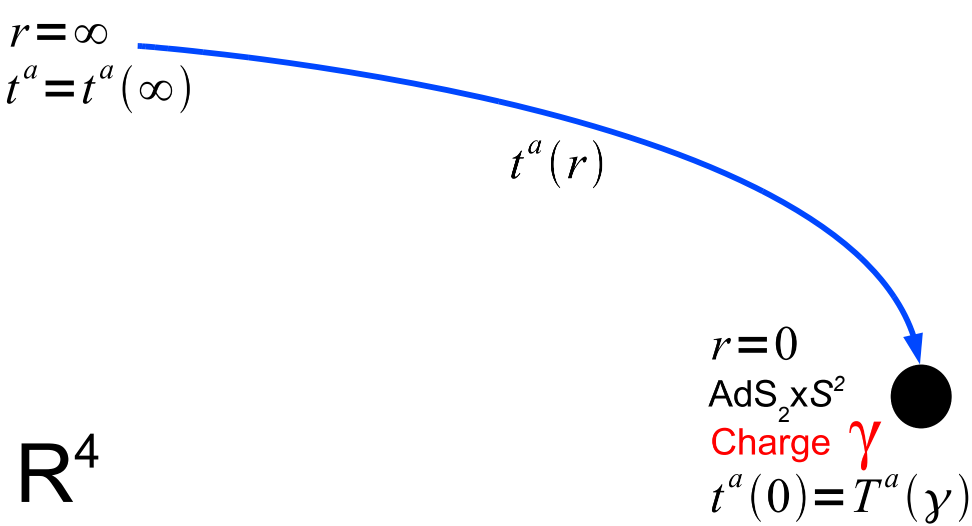
This is in particular the case for “scaling black holes”, which are solutions of supergravity which can be adiabatically connected to the solution with a single black hole singularity by scaling the distances between the centers. We will consider in this talk the spectrum of scaling black holes with three centers.



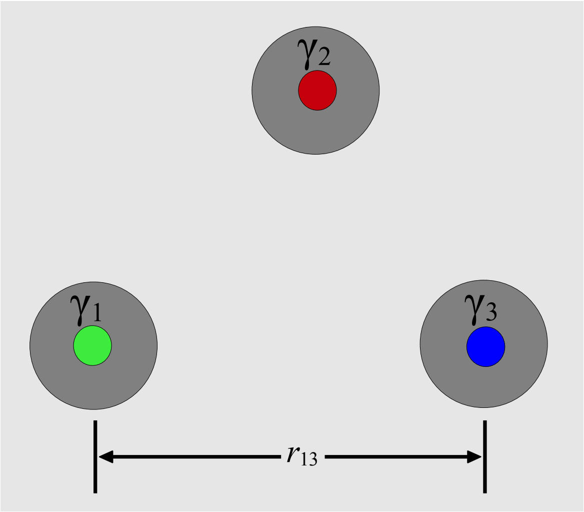
Attractor mechanism

The vectormultiplet scalar fields of are position dependent and required to satisfy the attractor equations in the near-horizon $AdS_2 \times S^2$.

For single center black hole:



BPS bound states of $\mathcal{N} = 2$ supergravity



N BPS black holes with charges γ_i located at $\vec{r}_i$ in $\mathbb{R}^3$

Static BPS bound states exist due to interplay between gravitational attraction and electro-magnetic repulsion

$\Longrightarrow$ Bound states are static and therefore part of the 1-particle Hilbert space $\mathcal{H}_{\rm BPS}(\gamma; t)$

Denef equations

$\mathcal{N} = 2$ BPS equations of motion require the distances $r_{ij} = |\vec{r}_i - \vec{r}_j| \in \mathbb{R}_+$ to satisfy:

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$$

- $\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle \in \mathbb{Z}$: Dirac-Schwinger-Zwanziger innerproduct
- $c_i(\{\gamma_j\}; t) \in \mathbb{R}$: stability parameters depending on $Z(\gamma_i, t)$

Denef (2000)

Phase space $M_N(\{\gamma_i\}, \{c_i\})$:

- parametrizes $\vec{r}_i \in \mathbb{R}^3, i = 1, \ldots, N$
- has dimension $2N - 2$

De Boer, El-Showk, Messamah, Van den Bleeken (2008)

Denef equations: Two aspects

Wall-crossing:

Solutions might decay or recombine upon varying $c_i \in \mathbb{R}$:

Denef (2000); Denef, Moore (2007),...

For example $N = 2$: $\lim_{c_1 \to 0} r_{12} = \lim_{c_1 \to 0} \frac{\gamma_{12}}{c_1} = \pm\infty$

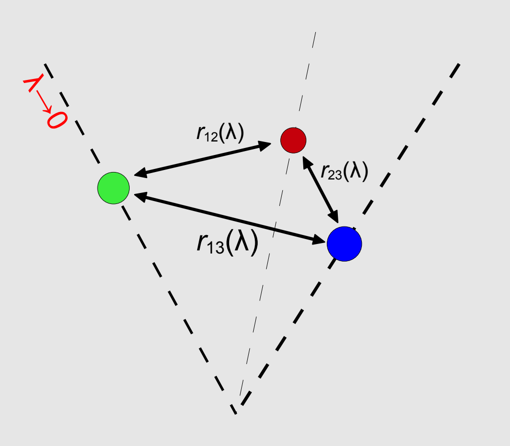
Scaling solutions:

Centers could get arbitrarily close, depending on $\{\gamma_i\}$

Bena, Wang, Warner (2006); Denef, Moore (2007),...

For example $N = 3$: If $\gamma_{12} + \gamma_{23} \geq \gamma_{31}$, and cyclic perm. $\Rightarrow$

$$\lim_{\lambda \to 0} r_{ij}(\lambda) = \lambda \gamma_{ij} + \mathcal{O}(\lambda^2) \in M(\{\gamma_i\}, \{c_i\})$$



Invariants

We have various types of BPS indices occurring:

- refined BPS index $\Omega(\gamma, y; t)$
- BPS index $\Omega(\gamma; t) = \Omega(\gamma; 1, t)$
- refined single-centered invariant $\Omega_S(\gamma, y)$
- single-centered invariant $\Omega_S(\gamma) = \Omega_S(\gamma, 1)$
- total invariant

$$\Omega_T(\gamma, y) = \Omega_S(\gamma, y) + \sum_{\sum_{j=1}^n m_j\gamma_j = \gamma} H(\{\gamma_j, m_j\}, y) \prod_{i=1}^n \Omega_S(\gamma_i, y^{m_i})$$

with $H(\{\gamma_j, m_j\}, y)$ determined by the "minimal modification hypothesis".

Invariants

We also need a numerical counterpart to $\Omega_T(\gamma, y)$, however $\lim_{y\to 1} \Omega_T(\gamma, y)$ diverges generically

We use instead the prescription

$$\frac{f(y)}{(y - y^{-1})^\ell} \longrightarrow \frac{1}{2^\ell \, \ell!} \left(y \frac{d}{dy} \right)^\ell f(y) \Big|_{y=1} ,$$

for $f(1) \neq 0$.

Decomposition formula

In terms of these invariants, the $\Omega(\gamma, y, t)$ can be expressed as

$$\bar{\Omega}(\gamma, y; t) = \sum_{\substack{\sum_i N_i\gamma_i = \gamma, \\ \gamma_i \neq \gamma_j, i \neq j}} g_C(\{N_i\gamma_i\}; \{c_i(t)\}, y) \prod_j \frac{\bar{\Omega}_T(\gamma_j, y)^{N_j}}{N_j!}$$

$|\mathrm{Aut}(\{\gamma_j\})|$

JM, Pioline, Sen (2010)

$g_C(\{\gamma_i\}, \{c_i\}, y)$ is the (twisted) Dirac index of the space $M_N(\{\gamma_i\}, \{c_i\})$

Coulomb branch: Localization

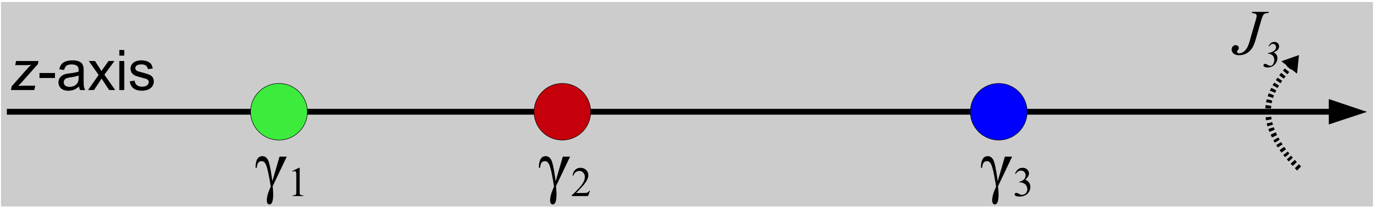
Evaluate integral by localization with respect to J_3

Duistermaat, Heckman (1982); Berline, Vergne (1985);...

$\Downarrow$

Sum over isolated fixed points $\in M_N(\{\gamma_i\}, \{c_i\})$ of J_3

The solutions which contribute are of the form:



JM, Pioline, Sen (2011)

Coulomb branch formula

Fixed point formula:

$$g_C(\{\gamma_i\}, y; \{c_i\}, y) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+N-1}}{(y-y^{-1})^{N-1}} \sum_{p\in\{\text{f.p. of } J_3\}} s(p)\, y^{2J_3(p)}$$

- angular momentum:

$$J_3(p) = \frac{1}{2}\sum_{i<j}\gamma_{ij}\,\mathrm{sign}(z_j - z_i)$$

- sign:

$$s(p) = \mathrm{sign}\left(\det\left(\frac{\partial^2 W}{\partial z_i \partial z_j}\right)\right)$$

with $W(\{z_i\}) = -\sum_{i<j}\gamma_{ij}\,\mathrm{sign}(z_j - z_i)\,\log|z_i - z_j| - \sum_{i=1}^{N} c_i z_i$

Coulomb branch formula: Example

Example: $\gamma_i, i = 1, \ldots, 3$, such that $\gamma_{12}, \gamma_{13}, \gamma_{23} > 0$,
$c_3 < c_2 < 0 < c_1$

- Fixed points have orderings:

$$\{1,2,3;+\}, \{2,1,3;-\}, \{3,1,2;-\}, \{3,2,1;+\},$$

with $\pm = s(p)$

- Enumerate:

$$g_C(\{\gamma_i\}, y; \{c_i\}) = (-1)^{\gamma_{12}+\gamma_{23}+\gamma_{13}} (y - y^{-1})^{-2}$$
$$\left(y^{\gamma_{12}+\gamma_{13}+\gamma_{23}} - y^{\gamma_{12}-\gamma_{23}-\gamma_{13}} - y^{\gamma_{13}+\gamma_{23}-\gamma_{12}} + y^{-\gamma_{12}-\gamma_{13}-\gamma_{23}} \right)$$

Minimal modification hypothesis

With loops/generic superpotential:

- scaling solutions are possible
- explicit algorithm, recursive in the number of centers
- sum over regular fixed points $\neq$ SU(2) character

Problem: What is the contribution of the scaling fixed point?

For 3-center, determine H such that

$$g_C(\{\gamma_i\}, y; \{c_i\}, y) + H(\{\gamma_i\}, y)$$

is an $SU(2)$ character, with "minimal amount" angular momentum.

Minimal modification hypothesis

Consider a 3-center scaling black hole. Let $a = \gamma_{12}$, $b = \gamma_{23}$ and $c = \gamma_{31}$.

$$H(\{\gamma_j\}, y) = \begin{cases} -\frac{2}{(y-y^{-1})^2}, & \text{if } a+b+c \in 2\mathbb{Z}, \\ \frac{y+y^{-1}}{(y-y^{-1})^2}, & \text{if } a+b+c \in 2\mathbb{Z}+1, \end{cases}$$

Numerical version:

$$H(\{\gamma_j\}) = \begin{cases} 0, & \text{if } a+b+c \in 2\mathbb{Z}, \\ \frac{1}{4}, & \text{if } a+b+c \in 2\mathbb{Z}+1, \end{cases}$$

$$\Omega_T(\gamma) = \Omega_S(\gamma) + \begin{cases} 0, & \text{if } a+b+c \in 2\mathbb{Z}, \\ \frac{1}{4}\prod_{j=1}^{3} \Omega_S(\gamma_j), & \text{if } a+b+c \in 2\mathbb{Z}+1. \end{cases}$$

Black Hole Bound States

We have

$$g_C\left(\{\gamma_j\};\{c_j^*\}\right) = \frac{(-1)^{a+b+c}}{4}\left[F^*(123)\,(a+b-c)^2 + \text{perm} \right.$$
$$\left. + \frac{1}{4} A\,\delta_{a,c}\,\delta_{b,c}\,a^2 \right]$$

cycl.

with

$$F^*(123) = \frac{1}{4}(1+\text{sgn}(a-c)\,\text{sgn}(b-c)+\text{sgn}(b-c)\,\text{sgn}(c-a-b)$$
$$+\,\text{sgn}(c-a-b)\,\text{sgn}(a-c)\,).$$

A above is introduced to deal with exceptions where arguments of sgn's vanish. We will see that modularity provides a definite answer. $\Rightarrow$ expect $A=1$

Black Hole Bound States

To enumerate scaling configurations, we also introduce the quantity f_C which determines whether a scaling configuration exists for these charges or not,

$$
\begin{aligned}
f_C\left(\{\gamma_j\};\{c_j^*\}\right) = \frac{(-1)^{a+b+c}}{4}\,&[1+\mathrm{sgn}(a+b-c)\,\mathrm{sgn}(a-b+c)+\mathrm{perm} \\
&+A_1\,\delta_{a,0}\,\delta_{b,c}+\mathrm{perm}]
\end{aligned}
$$

The A_ℓ, $\ell = 1,2,3$ is introduced to deal with exceptions where arguments of sgn's vanish. We will see that modularity provides a again a definite answer.

D4-D2-D0 Black Holes

Let us review a few aspects of D4-D2-D0 black holes, with D4-brane charge P, D2-brane charge Q and D0-brane charge Q_0, abbreviated to $\gamma = (P, Q, Q_0)$.

Let D_{abc}, $a, b, c = 1, \ldots, b_2(X)$ be the triple intersection numbers of the Calabi-Yau X, then

- $D_{ab} = D_{abc}P^c$ gives us quadratic form on the lattice Λ, and Q takes values in Λ^*, with quadratic form $D^{ab} = (D^{-1})^{ab}$
- Λ has signature $(1, b_2 - 1)$.

Maldacena, Strominger, Witten (1997)

The Kähler modulus is $t = B + i\,J$

The large volume attractor point is

$$t_\gamma^\lambda = D^{ab} Q_b + i\lambda P^a,$$

with sufficiently large $\lambda \gg 1$.

Attractor invariants (or MSW invariants)

$$\Omega(\gamma, t_\gamma^\lambda)$$

These invariants are unchanged under the "spectral flow" symmetry, such that these only depend on the class of $Q \in \Lambda^*$ in Λ^*/Λ. This class is denoted by μ.

Partition function

Partition function is for fixed P, and admits a theta series decomposition due to a symmetry of the attractor invariants:

$$\begin{aligned}\mathcal{Z}_P^\lambda(\tau, C, t) &= \sum_{Q, Q_0} \bar{\Omega}(\gamma, t_\gamma^\lambda)\, e^{-\tau_2 M(\gamma, t) + 2\pi i C^0 Q_0 + 2\pi i C.Q} \\ &= \sum_{\mu \in \Lambda^*/\Lambda} h_{P,\mu}(\tau)\, \Theta_\mu(\tau, \bar{\tau}, C, B)\end{aligned}$$

$$q = e^{2\pi i \tau}$$

with $\tau = C^0 + i\tau_2$, and

$$h_{P,\mu}(\tau) = \sum_{Q_0} \bar{\Omega}(\gamma, t_\gamma^\lambda)\, q^{-Q_0 + (\mu + P/2)^2/2}$$

and

$$\Theta_\mu(\tau, \bar{\tau}, C, B) = \sum_{Q \in \Lambda_\mu^*} (-1)^{P.Q} q^{\hat{Q}_+^2/2} \bar{q}^{-\hat{Q}_-^2/2} e^{2\pi i C.(Q - B/2)},$$

Gaiotto, Strominger, Yin (2006); De Boer *et al* (2006); Denef, Moore (2007)

Partition function

S-duality action on the Type IIB hypermultiplet geometry requires that $h_{P,\mu}$ transforms as a mock modular form $\Rightarrow$ $h_{P,\mu}$ can be completed with non-holomorphic terms to $\widehat{h}_{P,\mu}$, such that the latter transforms as a vector-valued modular form. The depth of the mock modular form corresponds the maximal length of a partition of P.

Alexandrov, Banerjee, JM, Pioline (2016/7), Alexandrov, Pioline (2018),. . .

Transformation law

$$S: \widehat{h}_{P,\mu}(-1/\tau, -1/\bar{\tau}) = -\frac{1}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{-b_2/2-1}\varepsilon(S)^* e^{-i\pi P^2/2} \times \sum_{\delta\in\Lambda^*/\Lambda} e^{-2\pi i\delta.\mu}\widehat{h}_{P,\delta}(\tau,\bar{\tau}),$$

$$T: \widehat{h}_{P,\mu}(\tau+1, \bar{\tau}+1) = \varepsilon(T)^* e^{i\pi(\mu+P/2)^2}\widehat{h}_{P,\mu}(\tau,\bar{\tau}),$$

We also introduce

$$\widehat{\mathcal{Z}}_P^{\lambda}(\tau,\bar{\tau},C,t) = \sum_{\mu\in\Lambda^*/\Lambda} \widehat{h}_{P,\mu}(\tau,\bar{\tau})\,\Theta_\mu(\tau,\bar{\tau},C,B),$$

which transforms as a modular form.

Charge Lattices for Bound States

We are interested in n-center bound states with non-vanishing D4-brane charge P_j, $j = 1, \ldots, n$ with associated lattices Λ_j.

This gives rise to an $(n\, b_2)$-dimensional lattice $\boldsymbol{\Lambda} = \Lambda_1 \oplus \cdots \oplus \Lambda_n$, with quadratic form $\vec{D} = \mathrm{d}iag(D_1, \ldots, D_n)$

The total electric charge $Q = \sum_j Q_j$ is distributed over the n constituents. We therefore want to decompose $\boldsymbol{\Lambda}$ in terms of a lattice $\overline{\boldsymbol{\Lambda}}$ associated to the total charge, and a lattice $\underline{\boldsymbol{\Lambda}}$ associated to the charge distribution. Let $\overline{\boldsymbol{\Lambda}} \subset \boldsymbol{\Lambda}$ be defined by

$$\overline{\boldsymbol{\Lambda}} = \{\vec{k} = (k, k, \ldots, k) \in \boldsymbol{\Lambda} \,|\, k \in \mathbb{Z}^{b_2}\},$$

and $\underline{\boldsymbol{\Lambda}} \subset \boldsymbol{\Lambda}$

$$\underline{\boldsymbol{\Lambda}} = \left\{ \vec{k} \in \boldsymbol{\Lambda} \,\middle|\, \sum_{j=1}^{n} D_j k_j = 0 \right\}.$$

Charge Lattices for Bound States

The glue group is the coset $\mathbf{\Lambda}/(\overline{\mathbf{\Lambda}} \oplus \underline{\mathbf{\Lambda}})$. Its number of elements is

$$N_g = \sqrt{\frac{\det(\overline{D})\,\det(\underline{D})}{\prod_{j=1}^{n}\det(D_j)}}.$$

The order of the quotient group $(\underline{\mathbf{\Lambda}}^*/\underline{\mathbf{\Lambda}})/\underline{h}(G)$ is

$$N_q = \frac{\det(\underline{D})}{N_g}.$$

The quadratic form on $\underline{\mathbf{\Lambda}}^*$ is

$$\boldsymbol{Q}^2 = -Q^2 + \sum_j (Q_j)_j^2$$

$\underline{\mathbf{\Lambda}}$ has signature $(n-1)\,(1, b_2 - 1)$

Partition function for scaling black holes

$\Omega(\gamma, t_\gamma^\lambda) = \ldots \prod \Omega_T(\gamma_j)$

Partition function for total invariants:

$$h^T_{P,\mu}(\tau) = \sum_{Q_0} \bar{\Omega}_T(\gamma, t^\lambda_\gamma)\, q^{-Q_0+(\mu+P/2)^2/2}$$

The relation between attractor and total invariants leads to the decomposition

$$h_{P,\mu}(\tau) = h^T_{P,\mu}(\tau) + \sum_{n>1} \sum_{\sum_{j=1}^n P_j = P} \frac{g_C(\{\gamma_j\}, \{c_j^\lambda\})}{|\,\mathrm{Aut}(\{\gamma_j\})|}\, q^{Q^2/2 - \sum_j (Q_j)_j^2/2} \prod_{j=1}^n h^T_{P_j,\mu_j}(\tau)$$

Then, the partition functions for scaling solutions reads:

$$h^{3\,T}_{\{P_j\},\mu}(\tau) = \sum_{\substack{\mu_j \in \Lambda_j^*/\Lambda_j,\ j=1,2,3, \\ \mu_1+\mu_2+\mu_3=\mu}} h^{T}_{P_1,\mu_1}(\tau)\, h^{T}_{P_2,\mu_2}(\tau)\, h^{T}_{P_3,\mu_3}(\tau)\, \Psi_{\boldsymbol{\mu}}(\tau)\,,$$

with

$$\Psi_{\boldsymbol{\mu}}(\tau) = \sum_{\boldsymbol{Q} \in \underline{\boldsymbol{\Lambda}}^*_{\boldsymbol{\mu}}} g_C(\{\gamma_j\}, \{c_j^\lambda\})\, q^{-\boldsymbol{Q}^2/2},$$

with $\underline{\boldsymbol{\Lambda}}^*_{\boldsymbol{\mu}} = \boldsymbol{\mu} + \boldsymbol{P}/2 + \underline{\boldsymbol{\Lambda}}$ with $\boldsymbol{\mu} \in \underline{\boldsymbol{\Lambda}}^*$

Partition function for scaling solutions

The number of terms in the sum for $h^{3T}_{\{P_j\},\mu}$ is N_q.

We can assume that $h^{T}_{P_j,\mu_j}$ transform as earlier stated, by requiring that the P_j are irreducible.

Thus we need to understand the transformations of $\Psi_{\boldsymbol{\mu}}$, to determine those of $h^{3T}_{\{P_j\},\mu}$

We also introduce

$$\Phi_{\boldsymbol{\mu}}(\tau) = \sum_{\boldsymbol{Q}\in\underline{\boldsymbol{\Lambda}}^*_{\boldsymbol{\mu}}} f_C(\{\gamma_j\},\{c_j^\lambda\})\, q^{-\boldsymbol{Q}^2/2}$$

i.e. the generating function of scaling configurations.

Convergence

For three centers, $\underline{\mathbf{\Lambda}}$ has signature $(2, 2b_2 - 2)$. There is a general approach for the convergence and modular completion of functions such as $\Phi_{\boldsymbol{\mu}}$ and $\Psi_{\boldsymbol{\mu}}$

Consider

$$\Theta_\mu[\mathcal{K}](\tau; L) = \sum_{x \in L+\mu} \mathcal{K}(x)\, q^{-B(x)/2}$$

Let $\mathcal{V} = \{V_1, \dots, V_N\}$ be a collection of positive vectors. The kernel reads

$$\mathcal{K}(x, \mathcal{V}) = \frac{1}{4} \left(w(\mathcal{V}) + \sum_{j=1}^{N} \mathrm{sgn}(B(x, V_j))\mathrm{sgn}(B(x, V_{j+1})) \right) ,$$

with

$$w(\mathcal{V}) = -\sum_{j=1}^{N} \mathrm{sgn}(B(v, V_j))\mathrm{sgn}(B(v, V_{j+1})),$$

Alexandrov, Banerjee, JM, Pioline (2016); Funke, Kudla (2017),

Convergence

Conditions for convergence:

$$
\begin{aligned}
&B(V_j, V_j) > 0, \\
&B(V_j, V_j)\, B(V_{j+1}, V_{j+1}) - B(V_j, V_{j+1})^2 > 0, \\
&B(V_j, V_j)\, B(V_{j-1}, V_{j+1}) - B(V_j, V_{j-1})\, B(V_j, V_{j+1}) < 0.
\end{aligned}
$$

Convergence

The functions Φ_μ and Ψ_μ are of the right form to apply these general results.
To apply this general formalism to Φ_μ, we determine the vector C_a such that $(C_a, \boldsymbol{Q}) = a$ for all $\boldsymbol{Q} \in \underline{\Lambda}$ and similarly for C_b and C_c,

$$C_a = (-P_2, P_1, 0),$$
$$C_b = (0, -P_3, P_2),$$
$$C_c = (P_3, 0, -P_1),$$

We then have for Φ_μ, $C_1 = C_a + C_b - C_c$, $C_2 = C_a - C_b + C_c$ and $C_3 = -C_a + C_b + C_c$, and convergence follows. The proof for Ψ_μ is similar

Generalized error functions

Let E_2 be the 2-dimensional generalization of the error function defined by:

$$E_2(\alpha; u_1, u_2) = \int_{\mathbb{R}^2} e^{-\pi(u_1-u_1')^2-\pi(u_2-u_2')^2}\,\mathrm{sgn}(u_2')\,\mathrm{sgn}(u_1'+\alpha\, u_2')\,du_1'\,du_2'$$

If we rescale the arguments, it satisfies

$$\lim_{\lambda\to\infty} E_2(\alpha; \lambda\, u_1, \lambda\, u_2) = \begin{cases} \mathrm{sgn}(u_1)\,\mathrm{sgn}(u_1+\alpha u_2), & (u_1, u_2) \neq (0,0) \\ \frac{2}{\pi}\arctan(\alpha), & (u_1, u_2) = (0,0) \end{cases}$$

E_2 satisfies the Vignéras equation ensuring modular properties, when used in the kernel of a theta series.

Modular completion

Thus the modular completions, $\widehat{\Phi}_{\boldsymbol{\mu}}$ and $\widehat{\Psi}_{\boldsymbol{\mu}}$, of $\Phi_{\boldsymbol{\mu}}$ and $\Psi_{\boldsymbol{\mu}}$ is obtained by replacing

$$\mathrm{sgn}(C_1.x)\,\mathrm{sgn}(C_2.x) + \mathcal{A}\,\delta_{(C_1.x)}\,\delta_{(C_2.x)}$$

by $E_2(\alpha; u_1, u_2)$, with

$$\alpha = \frac{(C_1.C_2)}{\sqrt{C_1^2\,C_2^2 - (C_1.C_2)^2}}$$

$$u_1 = \sqrt{2\tau_2}\,\frac{(C_{1\perp 2}.x)}{|C_{1\perp 2}|}$$

$$u_2 = \sqrt{2\tau_2}\,\frac{(C_2.x)}{|C_2|}$$

We can fix the constants A, A_ℓ introduced earlier, by requiring that the added non-holomorphic terms are subleading, i.e. vanish in the $\tau_2 \to \infty$ limit.

- For $\widehat{\Psi}_{\boldsymbol{\mu}}$, we find that $A = 1$ generically, in agreement with the physically preferred value.
- For $\widehat{\Phi}_{\boldsymbol{\mu}}$, we find that A_ℓ can be irrational. While peculiar, this is maybe not so worrisome since $\Phi_{\boldsymbol{\mu}}$ is not a proper physical partition function.

Completion of $\Phi_{\boldsymbol{\mu}}$

Split holomorphic and non-holomorphic part:

$$\widehat{\Phi}_{\boldsymbol{\mu}}(\tau,\bar\tau)=\Phi_{\boldsymbol{\mu}}(\tau)+R^{\Phi}_{\boldsymbol{\mu}}(\tau,\bar\tau)$$

with

$$R^{\Phi}_{\boldsymbol{\mu}}(\tau,\bar\tau)=\sum_{\ell=1,2,3} i\int_{-\bar\tau}^{i\infty} dw\,\frac{\widehat{\Theta}_{\boldsymbol{\mu}+\rho}(\tau,-w;L_\ell^{\perp},\{C_{\ell-1}C_{\ell+1}\})\,\Upsilon_{\boldsymbol{\mu}+\rho}(w;C_\ell^2,\boldsymbol{K}\cdot C_\ell)}{\sqrt{-i(w+\tau)}}.$$

Modular completion

As a result, we find that the completion

$$\widehat{h}^{3T}_{\{P_j\},\mu}$$

transforms identically $\widehat{h}_{P,\mu}$.

Therefore, in the decomposition

$$\widehat{\mathcal{Z}}^{\lambda}_{P} = \widehat{\mathcal{Z}}^{T}_{P} + \widehat{\mathcal{Z}}^{3T}_{P} + \dots$$

each term has the same modular properties.

Case study

Let us consider a concrete example: let X be the K3 fibration with intersection numbers with $h^{1,1} = 2$ and $h^{2,1} = 86$, and intersection numbers

$$d_{111} = d_{112} = 0, \qquad d_{122} = 4, \qquad d_{222} = 2,$$

Choose charges:

$$P_1 = P_2 = (0,1), \qquad P_3 = (1,1)$$

q-series for $\Phi_{\boldsymbol{\mu}}$

Let $\boldsymbol{\mu} = (0,0)$, then

$$\begin{aligned}\Phi_{\boldsymbol{\mu}} &= 2q^6 + 4q^{20} + 6q^{24} + 4q^{30} + 4q^{44} + 4q^{50} + 8q^{52} + 2q^{54} + 12q^{56} + 4q^{60} \\ &+ 4q^{64} + 4q^{68} + 4q^{70} + 12q^{80} + 2q^{88} + 8q^{90} + 8q^{92} + 8q^{94} + 14q^{96} + 16q^{100} \\ &\quad + \frac{(A_1 + A_2 + A_3 + 1)}{2} + (A_1 + A_2 + A_3 + 3)(q^8 + q^{32} + q^{72}) + \dots\end{aligned}$$

with

$$A_1 = \frac{2}{\pi}\arctan(-5/\sqrt{11}),$$

$$A_2 = A_3 = \frac{2}{\pi}\arctan(-1/\sqrt{8}),$$

q-series for $\Psi_{\boldsymbol{\mu}}$

Let $\boldsymbol{\mu} = (0, 0)$:

$$\Psi_{\boldsymbol{\mu}} = 16q^{30}\left(1 + 2q^{22} + 4q^{34} + q^{40} + 2q^{60} + 8q^{62} + 2q^{64} + 4q^{70}\right) + \dots$$

More centers?

While technically involved, we expect that these results can be generalized to scaling black holes with more centers. This would lead to higher depth mock modular forms.

Thank you!