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Uncovering the mock nature of single-centered BPS black holes

Valentin Reys

Black holes, BPS and quantum information

22-09-2021

Based mainly on [arXiv:1512.01553], [arXiv:1702.02755] and [arXiv:1912.06562]

with (some permutation of)

A. Chowdhury, A. Kidambi, F. Ferrari, S. Murthy, T. Wrase

and on some ongoing work with **A. Kidambi** and **M. Rosselló**.

Motivation

- String theory provides a description of the microscopic states making up supersymmetric black holes.
- In some favorable setting, can give an exact count.
- The counting exhibits fascinating connections to number theory. Lots of progress in recent years. [see (many) talks at this workshop]

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- Susy black holes can also be studied as solutions to the equations of motion of the low-energy effective supergravity description.
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- Semi-classically, black hole entropy is given by the Bekenstein-Hawking area-law $S_{\mathrm{BH}} = A_H/(4G_N) + \ldots$
- Can we compute corrections to this formula, obtain integer degeneracies, and compare to the string theory counting?

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- There are some cases – like with maximal supersymmetry – where the comparison can be made successfully, even including exponentially suppressed contributions.
- Besides checking numbers, it is interesting to try and uncover the number-theoretic structures predicted by string theory from the supergravity description.

Outline

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Dyons in type II string theory @ T^6

- Counting function for 1/8-BPS dyons:

[Maldacena,Moore,Strominger'99; Shih,Strominger,Yin'05]

$$F_{1/8}(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)}\,.$$

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- An exact convergent expansion for C: [Rademacher,Zuckerman'38]

$$C_{1/8}(\Delta) = 2\pi \sum_{k=1}^{+\infty} \frac{\mathrm{Kl}(k,\Delta)}{k} \left(\frac{1}{\Delta}\right)^{7/4} I_{7/2}\Big(\frac{\pi\sqrt{\Delta}}{k}\Big)\,.$$

Dyons in type II string theory @ $\mathrm{K3} \times T^2$

- Counting fct for 1/4-BPS dyons: [Dijkgraaf,Verlinde,Verlinde'96]

$$F_{1/4}(\tau, \sigma, z) = \left(\Phi_{10}(\tau, \sigma, z)\right)^{-1}.$$

- $F_{1/4}$ is a meromorphic Siegel modular form of weight -10, with double poles at $z = 0$ and its $Sp(2, \mathbb{Z})$ images.

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- The contour depends on the moduli. At fixed charges, $d_{1/4}$ "jumps" when $\mathcal{C}$ crosses a pole. [Cheng,Verlinde'07]
- Manifestation of wall-crossing where a 1/4-BPS bound state of two 1/2-BPS states (dis-)appears when crossing codim-1 walls.

Single-centered dyons in type II string theory @ $\mathrm{K3} \times T^2$

- Single-centered black holes exist everywhere in the moduli space, provided $\Delta = 4mn - \ell^2 > 0$. [Sen'09]
- Near the BH, massless moduli are attracted to values that are fully determined by the charges. [Ferrara,Kallosh,Strominger'95]

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- The single-centered degeneracies $d^*_{1/4}$ depend only on charges.
- They are Fourier coefficients of certain mock Jacobi forms ψ^{F}_m. [Dabholkar,Murthy,Zagier'12]

$$d^*_{1/4}(n,m,\ell) = (-1)^{\ell+1}\, c^{\mathrm{F}}_m(n,\ell) \quad \text{for} \quad \Delta = 4mn - \ell^2 > 0\,.$$

An exact formula for c_m^{F} with $\Delta > 0$

- ψ_m^{F} admits a standard ϑ-decomposition into vector-valued mixed mock modular forms $\{h_\ell\}$ with shadow the unary theta series of weight $1/2$ times η^{-24}. [Dabholkar,Murthy,Zagier'12]
- There exists a generalization of the Rademacher expansion for mixed mock modular forms. [Bringmann,Manschot'10]

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- There exists a generalization of the Rademacher expansion for mixed mock modular forms. [Bringmann,Manschot'10]
- Gives an analytic expression for $c^{\mathrm{F}}_m(n,\ell)$ with $\Delta > 0$ in terms of the polar coefficients $c^{\mathrm{F}}_m(n,\ell)$ with $\Delta < 0$, [Ferrari,VR'17]

$$c^{\mathrm{F}}_m(n,\ell) \sim \sum_{k=1}^{+\infty} \Bigg[\sum_{\substack{\tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ \tilde{\Delta}<0}} c^{\mathrm{F}}_m(\tilde{n},\tilde{\ell})\, \frac{\mathrm{Kl}(k,\widetilde{\Delta},\Delta)}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi\sqrt{|\widetilde{\Delta}|\Delta}}{mk}\right) + \frac{\mathrm{Kl}(k,m,\Delta)}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi\sqrt{\Delta}}{\sqrt{mk}}\right) + \mathcal{I}(k,m,\Delta) \Bigg].$$

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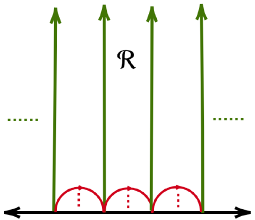

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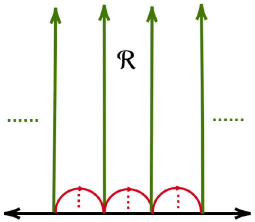

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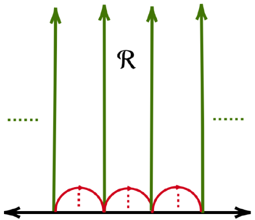

- Degeneracies of decay products are Fourier coeffs of η^{-24}.
- In the region $\mathcal{R}$, walls lie at the bottom of the strip and form Farey arcs (in red).
- Sum over all decay channels: [Chowdhury,Kidambi,Murthy,VR,Wrase'19]

$$c_m^{\mathrm{F}}(n,\ell) = \sum_{\gamma \in W(n,\ell,m)} (-1)^{\ell_\gamma+1} \, |\ell_\gamma| \, d(m_\gamma) \, d(n_\gamma) \quad \text{for} \quad \Delta < 0\,.$$

- W is a set of $SL(2,\mathbb{Z})$ matrices encoding the relevant walls.

The set W

- Since $d(p) = 0$ for $p < -1$, the set W of walls giving a non-zero contribution to the polar coefficients c_m^{F} with $\Delta < 0$ is finite.
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- It has an elegant characterization in terms of the continued fraction expansion of $\ell/(2m)$. [Cardoso,Nampuri,Rosselló'20]
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- While there is an efficient algorithmic way to construct the set for any values of the charges, we currently lack a proper group-theoretic definition of W as a subset of $SL(2, \mathbb{Z})$.
- Still, the previous result for c_m^{F} with $\Delta < 0$ implies that the single-centered degeneracies c_m^{F} with $\Delta > 0$ are completely determined from the Fourier coefficients of η^{-24}.

Outline

The view from supergravity

- At strong coupling, the systems gravitate and form black holes.
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- How much of the number-theoretic structure can we recover from the supergravity description?
- Not a priori obvious: supergravity is a low-energy effective description of the UV complete string theory.
- Some encouraging results can be obtained using the framework of the Quantum Entropy Function.

The Quantum Entropy Function

- The quantum entropy of 1/2-BPS BHs in $4d\ \mathcal{N}=2$ sugra is defined as the expectation value of a Wilson line: [Sen'08]

$$e^{S_{\mathsf{BH}}(p,q)} = \left\langle \exp\left[-\mathrm{i}\, q_I \oint \mathrm{d}\tau\, A_\tau^I\,\right] \right\rangle_{\mathsf{AdS}_2}^{\mathsf{finite}} .$$

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- This is achieved using supersymmetric localization.
- Use a supercharge Q to constrain the field configurations that contribute to the QEF – the localization manifold $\mathcal{M}_Q$.

Supergravity localization

- When the metric fluctuates, the definition of Q is unclear.
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- The manifold $\mathcal{M}_Q$ is most conveniently analyzed in a formalism where the susy transformations close off-shell
  $\rightarrow 4d\ \mathcal{N}=2$ conformal supergravity.
- In the gravitational sector, find all bosonic configurations solving

$$\delta_{\text{eq}}\psi_\mu = D_\mu(\mathring{\epsilon}+\epsilon) + (T\cdot\gamma)\gamma_\mu(\mathring{\epsilon}+\epsilon) + \gamma_\mu(\mathring{\eta}+\eta)\,.$$

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- Imposing $\text{AdS}_2 \times S^2$ boundary conditions, the only solution is to set the bosonic fields to their attractor values and $\epsilon = \eta = 0$.

  [Gupta,Murthy'12]

The localized QEF

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- The localized QEF is a finite-dimensional integral:

$$e^{S_{\mathsf{BH}}(p,q)} = \int_{\mathcal{M}_Q} \mathrm{d}\phi^I \, e^{-\pi q_I \phi^I + \mathcal{S}(\phi + \mathrm{i}p)} \, Z_{\text{1-loop}}(\phi) \, Z_{\text{measure}}(\phi)$$

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- $Z_{\text{1-loop}}$ comes from Gaussian integration around $\mathcal{M}_Q$.
- Contribution $\log Z^{\bullet}_{\text{1-loop}}(\phi) = a_\bullet \log K(\phi)$ from each multiplet, [Murthy,VR‘15; Gupta,Ito,Jeon‘15; Jeon,Murthy‘18]

$$a_2 = 2\,, \quad a_{3/2} = -\frac{11}{12}\,, \quad a_1 = -\frac{1}{12}\,, \quad a_0 = \frac{1}{12}\,.$$

The action and measure

- The $\mathcal{N} = 2$ sugra action contains “F-terms” (chiral superspace integrals) and “D-terms” (full superspace integrals).
- A large class of D-terms can be constructed using the $\mathcal{N} = 2$ kinetic multiplet. [de Wit,Katmadas,van Zalk‘10]

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- The F-terms and the remaining measure factor are expressed in terms of the prepotential F of the $\mathcal{N}=2$ theory.

Outline

Maximal supergravity

- Cast the $\mathcal{N} = 8$ graviton multiplet in an $\mathcal{N} = 2$ language:

$$n_2 = 1\,, \quad n_{3/2} = 6\,, \quad n_1 = 15\,, \quad n_0 = 10\,.$$

- One-loop determinant is $K(\phi)^{-4}$ with $K(Y) = \mathrm{i}(\bar{Y}^I F_I - Y^I \bar{F}_I)$.

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- In the truncated theory,

$$\mathcal{S} = 4\pi\,\mathrm{Im}F(\phi + \mathrm{i}p)\,, \quad \text{with} \quad F(Y) = -\frac{Y^1}{Y^0}\,Y^a C_{ab} Y^b\,.$$

- C_{ab} is the intersection matrix of the 6 two-cycles on $T^4 \subset T^6$.

Entropy of 1/8-BPS black holes

- The measure is expressed in terms of the second derivatives of F,

  [Cardoso,de Wit,Mahapatra'08; Dabholkar,Gomes,Murthy'11]

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- Using all the above in the localized QEF,

$$e^{S_{\text{BH}}(p,q)} = \sqrt{2}\,\pi\left(\frac{1}{\Delta}\right)^{7/4} I_{7/2}(\pi\sqrt{\Delta})\,.$$

- Precisely matches the $k = 1$ term of $d_{1/8}$ obtained from the Rademacher expansion of the microscopic counting function.

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- Precisely matches the $k=1$ term of $d_{1/8}$ obtained from the Rademacher expansion of the microscopic counting function.
- Higher-k terms can also be recovered from localization in an orbifold of the near-horizon geometry. [Dabholkar,Gomes,Murthy'14]

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Half-maximal supergravity

- Cast the $\mathcal{N}=4$ graviton multiplet and $N_v = 22$ $\mathcal{N}=4$ vector multiplets in an $\mathcal{N}=2$ language:

$$n_2 = 1\,, \quad n_{3/2} = 2\,, \quad n_1 = 23\,, \quad n_0 = 22\,.$$

- Resulting one-loop determinant is unity. Cancellations between the gravitini and the hypers, and between the graviton and the vectors, at all order in perturbation theory.

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- Resulting one-loop determinant is unity. Cancellations between the gravitini and the hypers, and between the graviton and the vectors, at all order in perturbation theory.
- This is equivalent to a truncated $\mathcal{N}=2$ theory with $n_1 = 23$.
- The prepotential F is modified beyond the $(Y^1/Y^0)YCY$ term:
  - Higher-derivative corrections,
  - Non-holomorphic corrections.

Corrected prepotential

- The corrections are encoded in a real homogeneous function Ω,

[Cardoso,de Wit,Mahapatra'08]

$$F = -\frac{Y^1}{Y^0}\, Y^a C_{ab} Y^b + 2\mathrm{i}\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\,.$$

- For $\mathcal{N}=4$ compactifications and with $S=-\mathrm{i}Y^1/Y^0$,

$$\Omega = \frac{1}{256\pi}\Big[\Upsilon\log\eta^{24}(S) + \bar{\Upsilon}\log\eta^{24}(\bar{S}) + \frac{1}{2}(\Upsilon+\bar{\Upsilon})\log(S+\bar{S})^{12}\Big]\,.$$

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- The non-holomorphic term is required so that Ω_S transforms under S-duality as a modular form of weight 2.

  Similar to the Eisenstein series $G_2^*(z) = G_2(z) - \pi/(2y)$.

  [Cardoso,de Wit,Mohaupt'99]

- Signals the departure from a Wilsonian effective action.

Previous approximations

- Due to Ω, the measure is also corrected,

$$Z_{\mathsf{measure}} = \sqrt{\left|\mathsf{det}\left[\mathsf{Im}(F_{IJ} - F_{I\bar{J}})\right]\right|}\,.$$

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- Previously, dealt with by neglecting non-holomorphic corrections and approximating [Murthy,VR'15]

$$Z_{\text{measure}} \approx (\phi^0)\, K^{\text{hol}}(\phi)\,, \quad \text{with} \quad K^{\text{hol}}(Y) = \mathrm{i}(\bar{Y}^I F_I^{\text{hol}} - Y^I \bar{F}_I^{\text{hol}})\,.$$

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- Agrees with a saddle-point evaluation of the $d_{1/4}$ from DVV.
  [David,Sen'06], see also talk by M. Rosselló
- Related to the measure factor in the refined OSV conjecture.
  [Denef,Moore'07]

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A partial supergravity result

- With the above approximations, we find [Murthy,VR'15]

$$e^{S_{\mathsf{BH}}} \approx \sum_{\substack{\tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ \tilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell})\left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi\sqrt{|\widetilde{\Delta}|\Delta}}{m}\right)$$

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  with $\alpha_m(n,\ell) = (\ell-2n)\, d(m+n-\ell)\, d(n)$.

- Compare with the previous formula for $c_m^{\mathrm{F}}(n,\ell)$ with $\Delta < 0$.

  We see that α_m equals a single term in the sum over W, corresponding to $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, *i.e.* the first S-wall from 0 to 1.

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- We recovered the leading part of the $k=1$ term in the Rademacher contribution to the microscopic degeneracies.

Completing the partial result

- Compared to the general structure of $d_{1/4}$ in the first part of the talk, we are missing
  - the other terms in the sum over W,
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- For the latter, a hint for progress is in the fact that mock-modularity can be traded for non-holomorphicity.
- The completion of a mock-modular form h with shadow g,

$$\widehat{h}(\tau) = h(\tau) + \int_{-\bar{\tau}}^{\infty} \frac{\overline{g(-\bar{z})}}{(z+\tau)^{w}}\,\mathrm{d}z\,,$$

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  is modular but non-holomorphic, $\partial\widehat{h}/\partial\bar{\tau} \propto \bar{g}$.
- To recover the shadow contributions, reinstate the non-holomorphic terms $\log(S+\bar{S})$ in the prepotential.

Completing the partial result

- The non-holomorphic term does not involve the full η function.
- Produces a single additional term in $K = \mathrm{i}(\bar{Y}^I F_I - Y^I \bar{F}_I)$.
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- Has the right structure to generate the I_{12} Bessel function when using the approximate measure. `[work in progress]`
- We should also remove the approximation on the measure.
- Will modify the Rademacher-type contribution, possibly generating the full sum over W in the polar coefficients?
- Generates the $\mathcal{I}$ term? The structure there is less clear... Understand the terms with double derivatives of the η function.

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Outline

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Conclusion

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- Understand the origin of the "mock pieces" in $d_{1/4}$ from sugra $\rightarrow$ non-holomorphic corrections to F should play a role.
- The exact result will involve proper treatment of the measure.
- Once established, turn to $\mathcal{N} = 2$ compactifications. Beautiful results available, more to explore! [Cardoso,Nampuri,Polini'19]

Thank you for your attention.