Generalized Siegel-Weil formula & Holography

Abhiram M Kidambi (Kavli IPMU, U. Tokyo)

Based on 2104.14710 (JHEP 2021, 44 (2021)) + WIP

w/ M. Ashwinkumar, M. Dodelson, J. Leedom & M. Yamazaki

Workshop on Black Holes, BPS and Quantum Information

IST Lisbon

The Lisbon feel

Elevator Pitch

1. "Canonical" idea of holography: CFT $\longleftrightarrow$ Gravity
2. When do you consider an ensemble average of CFT's? What is its holographic dual?
3. When the CFT moduli space is a locally symmetric space

$$\mathcal{M}_{p,q} = O(p,q;\mathbb{Z})\backslash O(p,q;\mathbb{R})/(O(p;\mathbb{R}) \times O(q;\mathbb{R})),$$

   the average over CFT ensembles is an exotic Abelian Chern-Simons gauge theory coupled to topological gravity
4. The CFT ensemble average computes $3d$ Chern-Simons invariants

Overview

1. A physical motivation: Wormholes and ensemble averages?
2. Number theory: Lattices, quadratic forms, theta functions, Eisenstein series
3. The (generalized) Siegel-Weil formula
4. Averaging over CFT's associated to indefinite quadratic forms
5. Averaging over CFT's associated to positive definite quadratic forms
6. Averaging over fermionic CFT's and spin Chern-Simons invariants

Wormholes and ensembles

The study of ensembles in gravity is not new.

Topological fluctuations (wormholes) lead to quantum decoherence
[Hawking; Giddings, Strominger; Lavreshvili, Rubakov, Tinyakov]

Wormholes and ensembles

The study of ensembles in gravity is not new.

Topological fluctuations (wormholes) lead to quantum decoherence [Hawking; Giddings, Strominger; Lavreshvili, Rubakov, Tinyakov]

This can be avoided if one considers "bounce" wormholes in Eudliean path integral and integrates over them like in an ensemble [Coleman; Strominger, Giddings]

The factorization problem

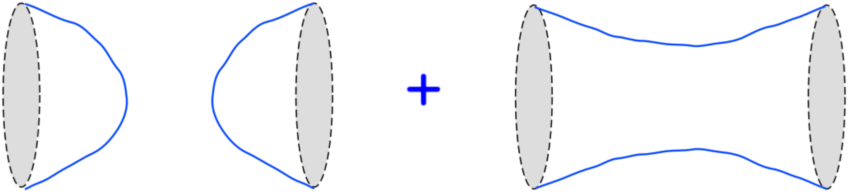
Consider two decoupled left/right moving CFT's ($\mathrm{CFT}_{L/R}$).

Partition functions are expected to holomorphically factorize.

$$Z_{tot} = Z_L \times Z_R.$$

The factorization problem

However in the case of the bulk dual, we expect to see wormhole contributions. [Maldacena, Maoz]



Wormholes break hol. factorization, maybe don't include them?

The factorization problem

But wormholes are also valid gravitational solutions whose inclusion in the gravitational path integral is useful.

[Maldacena,Qi; Saad, Shenker, Stanford + Page curve papers]

So what do we do?

This basically boils down to a deep issue in holography - we don't understand the rules of the duality.

The factorization problem

But wormholes are also valid gravitational solutions whose inclusion in the gravitational path integral is useful.

[Maldacena,Qi; Saad, Shenker, Stanford + Page curve papers]

So what do we do?

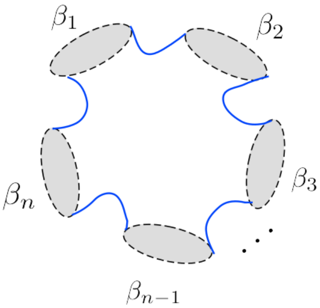
This basically boils down to a deep issue in holography - we don't understand the rules of the duality.

When do we need to consider ensembles?

In the case of JT gravity/SYK models:

There is no factorization problem if one considers the **ensemble average** of the boundary CFT.

[Saad, Shenker, Stanford; Stanford, Witten]



$$Z_{JT}(\beta_1, \cdots, \beta_n) = \left\langle \prod_{i=1}^{n} \mathrm{Tr}\, e^{-\beta_i H} \right\rangle ,$$

β_i: lengths of the geodesics on the hyperbolic Riemann surface.

Averaging in three dimensions?

Gravitational path integral with T^2 boundaries

[Cotler, Jensen]

Averaging in three dimensions?

Gravitational path integral with T^2 boundaries

[Cotler, Jensen]

But what moduli space of CFT's do you average over?

Most non-supersymmetric 2d CFT's do not admit a moduli space.

Averaging in three dimensions?

Gravitational path integral with T^2 boundaries

[Cotler, Jensen]

But what moduli space of CFT's do you average over?

Most non-supersymmetric 2d CFT's do not admit a moduli space.

However, if one considers the space of CFT's whose target space is toroidal (Abuse of notation for clarity: $T^d \times T^d$), there is a moduli space.

This is moduli space of toriodal conformal field theories (Narain Moduli Space)

[Narain; Narain, Sarmadi, Witten]

Averaging in three dimensions?

Averaging over moduli spaces whose target is toroidal has been studied extensively with many different generalizations. [Afkhami-Jeddi et.al; Maloney, Witten; Maloney, Datta et.al; Hartman et.al; Maloney, Collier]

Averaging in three dimensions?

Averaging over moduli spaces whose target is toroidal has been studied extensively with many different generalizations. [Afkhami-Jeddi et.al; Maloney, Witten; Maloney, Datta et.al; Hartman et.al; Maloney, Collier]

By considering moduli spaces of indefinite lattices (Abuse of notation again: $T^d \times T^{d'}$), we expect richer phenomena to enter from due to the number theory of CFT partition functions and in the bulk. [ADKLY]

Physically, this is due to gravitational anomalies.

But before we proceed further, let us recap the following:

1. What is... *a Narain Moduli Space*?

   [Giveon, Porrati, Rabinovici; Wendland (PhD Thesis)]

2. Integer lattices and quadratic forms

   [Andrianov (Quadratic forms and Hecke Operators)]

3. Modular forms associated to lattices and quadratic forms

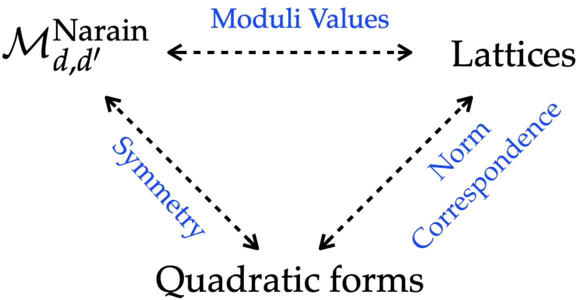
But before we proceed further, let us recap the following:

1. What is... *a Narain Moduli Space*?
   [Giveon, Porrati, Rabinovici; Wendland (PhD Thesis)]
2. Integer lattices and quadratic forms
   [Andrianov (Quadratic forms and Hecke Operators)]
3. Modular forms associated to lattices and quadratic forms

Disclaimer: I should point out that these mathematical techniqes discussed in this talk are not new to string theorists

[c.f. String field theory literature (2-loop string perturbation by D'Hokker, Phong), Papers by Green, Vanhove et.al; Obers, Pioline, $\cdots +$ Kachru, Tripathy for Siegel-Weil theorem]

Key point to remember



One can talk about Narain moduli spaces, lattices and quadratic forms equivalently.

What is a Narain moduli space?

- It is important to distinguish the moduli space of tori ($\mathcal{M}^d_{tori}$ from the moduli space of toroidal CFT's of central charge $c = \bar{c} = d$ ($\mathcal{M}^d_{Narain} = \mathcal{M}_{d,d}$).

What is a Narain moduli space?

- It is important to distinguish the moduli space of tori ($\mathcal{M}^d_{tori}$ from the moduli space of toroidal CFT's of central charge $c = \bar{c} = d$ ($\mathcal{M}^d_{Narain} = \mathcal{M}_{d,d}$).
- However, there exists a surjection between the moduli space of $d-$tori and the moduli space of $T^d \times T^d$ toroidal CFTs.

$$\exists \, \mathcal{M}^d_{tori} \twoheadrightarrow \mathcal{M}^d_{Narain}$$

[Huybrechts - Kaiserslautern Lecture]

What is a Narain moduli space?

- It is important to distinguish the moduli space of tori ($\mathcal{M}^d_{tori}$ from the moduli space of toroidal CFT's of central charge $c = \bar{c} = d$ ($\mathcal{M}^d_{Narain} = \mathcal{M}_{d,d}$).
- However, there exists a surjection between the moduli space of $d-$tori and the moduli space of $T^d \times T^d$ toroidal CFTs.

$$\exists\ \mathcal{M}^d_{tori} \twoheadrightarrow \mathcal{M}^d_{Narain}$$

[Huybrechts - Kaiserslautern Lecture]

- An analogous surjection also exists for moduli space of $T^p \times T^q$ tori and CFT's with these tori as target.

What is a Narain moduli space?

- It is important to distinguish the moduli space of tori ($\mathcal{M}^d_{tori}$ from the moduli space of toroidal CFT's of central charge $c = \bar{c} = d$ ($\mathcal{M}^d_{Narain} = \mathcal{M}_{d,d}$).
- However, there exists a surjection between the moduli space of $d-$tori and the moduli space of $T^d \times T^d$ toroidal CFTs.

$$\exists\ \mathcal{M}^d_{tori} \twoheadrightarrow \mathcal{M}^d_{Narain}$$

[Huybrechts - Kaiserslautern Lecture]

- An analogous surjection also exists for moduli space of $T^p \times T^q$ tori and CFT's with these tori as target.
- The Narain moduli space is physically the moduli space of a CFT whose field content take values in some integral (unimodular) (even) lattice, known as the Narain lattice.

What is a Narain moduli space?

- The Narain moduli space generically has the form of a the locally symmetric space:

$$\mathcal{M}_{p,q} = \underbrace{O(p,q;\mathbb{Z})}_{\mathrm{Aut}(\Lambda^{p,q})} \backslash O(p,q;\mathbb{R}) / (O(p;\mathbb{R}) \times O(q;\mathbb{R}))$$

What is a Narain moduli space?

- The Narain moduli space generically has the form of a the locally symmetric space:

$$\mathcal{M}_{p,q} = \underbrace{O(p,q;\mathbb{Z})}_{\mathrm{Aut}(\Lambda^{p,q})} \backslash O(p,q;\mathbb{R}) / (O(p;\mathbb{R}) \times O(q;\mathbb{R}))$$

- This is the space of CFT's with $U(1)^p \times U(1)^q$ current algebra, with central charge $(c, \bar{c}) = (p, q)$.

What is a Narain moduli space?

- The Narain moduli space generically has the form of a the locally symmetric space:

$$\mathcal{M}_{p,q} = \underbrace{O(p,q;\mathbb{Z})}_{\mathrm{Aut}(\Lambda^{p,q})} \backslash O(p,q;\mathbb{R}) / (O(p;\mathbb{R}) \times O(q;\mathbb{R}))$$

- This is the space of CFT's with $U(1)^p \times U(1)^q$ current algebra, with central charge $(c,\bar{c}) = (p,q)$.
- The moduli take values in an integer lattice $\Lambda^{p,q}$.

Ex: In string compactification, these lattices are unimodular lattices due to the requirement of modular invariance of CFT.

Modular forms

A modular form of weight k is a holomorphic function $f(\tau) : \mathbb{H} \to \mathbb{C}$, $\mathbb{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$, which transforms as

$$f(\tau) \to f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ N \geq 1$$

Modular forms

A modular form of weight k is a holomorphic function $f(\tau) : \mathbb{H} \to \mathbb{C}$, $\mathbb{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$, which transforms as

$$f(\tau) \to f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$
$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ N \geq 1$$

Example: The Holomorphic Eisenstein Series (wt. k)

$$E_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m\tau + n)^k}.$$

Non-holomorphic modular forms

Often, we have CFT's with independent left and right moving sectors so their partition functions are not expected to be holomorphic.

Non-holomorphic modular forms

Often, we have CFT's with independent left and right moving sectors so their partition functions are not expected to be holomorphic.

A non-holomorphic modular form of weight (k, k') is a function $f(\tau, \bar{\tau}) : \mathbb{H} \to \mathbb{C}$ which transforms as

$$f(\tau, \bar{\tau}) \to f\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) = (c\tau + d)^k (c\bar{\tau} + d)^{k'} f(\tau, \bar{\tau}),$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ N \geq 1$$

Non-holomorphic modular forms

Often, we have CFT's with independent left and right moving sectors so their partition functions are not expected to be holomorphic.

A non-holomorphic modular form of weight (k, k') is a function $f(\tau, \bar{\tau}) : \mathbb{H} \to \mathbb{C}$ which transforms as

$$f(\tau, \bar{\tau}) \to f\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) = (c\tau + d)^k (c\bar{\tau} + d)^{k'} f(\tau, \bar{\tau}),$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ N \geq 1$$

Example: Non-holomorphic Eisenstein Series (wt. (k, k))

$$E_{2k}(\tau, \bar{\tau})\Big|_{s=0} = \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{\Im\tau^k}{|m\tau + n|^{2k}}$$

Lattices

- Consider a lattice $\Lambda^{p,q} \subset \mathbb{Z}^{p,q} \subset \mathbb{R}^{p,q}$ of dimension $p+q$. It is a free Abelian group of rank $p-q$.

Lattices

- Consider a lattice $\Lambda^{p,q} \subset \mathbb{Z}^{p,q} \subset \mathbb{R}^{p,q}$ of dimension $p+q$. It is a free Abelian group of rank $p-q$.
- The lattice is equipped with the symmetric bilinear form $(\cdot,\cdot)$. For even (resp. odd) lattices, the norm of any element in the lattice wrt the symmetric bilinear is even (resp. odd).

Lattices

- Consider a lattice $\Lambda^{p,q} \subset \mathbb{Z}^{p,q} \subset \mathbb{R}^{p,q}$ of dimension $p+q$. It is a free Abelian group of rank $p-q$.
- The lattice is equipped with the symmetric bilinear form $(\cdot,\cdot)$. For even (resp. odd) lattices, the norm of any element in the lattice wrt the symmetric bilinear is even (resp. odd).
- The inner product of any two vectors $V, W \in \Lambda^{p,q}$ is given by

$$(V,W) = V^T Q_{ij} W,$$

where Q_{ij} is either the Hessian or the Gram matrix of the lattice. Ex: E_8, Niemeier lattices

Lattices

- Consider a lattice $\Lambda^{p,q} \subset \mathbb{Z}^{p,q} \subset \mathbb{R}^{p,q}$ of dimension $p+q$. It is a free Abelian group of rank $p-q$.
- The lattice is equipped with the symmetric bilinear form $(\cdot,\cdot)$. For even (resp. odd) lattices, the norm of any element in the lattice wrt the symmetric bilinear is even (resp. odd).
- The inner product of any two vectors $V, W \in \Lambda^{p,q}$ is given by

$$(V,W) = V^T Q_{ij} W,$$

where Q_{ij} is either the Hessian or the Gram matrix of the lattice. Ex: E_8, Niemeier lattices

- If $\det Q_{ij} = \pm 1$, the lattice is said to be unimodular or self-dual.

Here, $\Lambda^{p,q}$ is known as an indefinite lattice if $p, q \neq 0,\ p \neq q$.

Lattices

- Consider a positive definite lattice ($\Lambda^{p,0}$, norm of all non-zero vectors is even).

Lattices

- Consider a positive definite lattice ($\Lambda^{p,0}$, norm of all non-zero vectors is even). One can associate to this lattice the following function:

$$\theta(\tau) = \sum_{x \in \Lambda} q^{(x,x)/2} = \sum_{n \in \mathbb{Z}_+} c_n q^n, \; q := e^{2\pi i \tau}.$$

Lattices

- Consider a positive definite lattice ($\Lambda^{p,0}$, norm of all non-zero vectors is even). One can associate to this lattice the following function:

$$\theta(\tau) = \sum_{x \in \Lambda} q^{(x,x)/2} = \sum_{n \in \mathbb{Z}_+} c_n q^n, \; q := e^{2\pi i \tau}.$$

- This function is known as the theta function of the lattice and tell you that there are c_n vectors of norm $n = (x,x)/2$.

Lattices

- Consider a positive definite lattice ($\Lambda^{p,0}$, norm of all non-zero vectors is even). One can associate to this lattice the following function:

$$\theta(\tau) = \sum_{x\in\Lambda} q^{(x,x)/2} = \sum_{n\in\mathbb{Z}_+} c_n q^n, \; q := e^{2\pi i\tau}.$$

- This function is known as the theta function of the lattice and tell you that there are c_n vectors of norm $n = (x,x)/2$.
- This is a definite theta function which is a modular form of weight $p/2$ on $\Gamma_0(N)$ where N is the level of the lattice i.e., it is the smallest integer for which $\Lambda = N\Lambda^*$.

Lattices

- Consider a positive definite lattice ($\Lambda^{p,0}$, norm of all non-zero vectors is even). One can associate to this lattice the following function:

$$\theta(\tau) = \sum_{x \in \Lambda} q^{(x,x)/2} = \sum_{n \in \mathbb{Z}_+} c_n q^n, \; q := e^{2\pi i \tau}.$$

- This function is known as the theta function of the lattice and tell you that there are c_n vectors of norm $n = (x,x)/2$.
- This is a definite theta function which is a modular form of weight $p/2$ on $\Gamma_0(N)$ where N is the level of the lattice i.e., it is the smallest integer for which $\Lambda = N\Lambda^*$.
- For indefinite lattices $\Lambda^{p,p'}$, we can define something analogous (known as an indefinite theta function) which is a non-holomorphic modular form of weight $(\frac{p}{2}, \frac{p'}{2})$ on $\Gamma_0(N)$ where N is again the level. [Vigneras]

Quadratic forms and Lattices

- The symmetric bilinear form of a lattice looks like a quadratic form: $Q(x,y) = ax^2 + bxy + cy^2$.

Quadratic forms and Lattices

- The symmetric bilinear form of a lattice looks like a quadratic form: $Q(x,y) = ax^2 + bxy + cy^2$.
- One can associate to every lattice Λ a binary quadratic form, which can be identified with the Gram matrix of the Lattice.

[Jonathan Hanke's Arizona Winter School notes (Quadratic Forms and Automorphic Forms)]

Quadratic forms and Lattices

- The symmetric bilinear form of a lattice looks like a quadratic form: $Q(x,y) = ax^2 + bxy + cy^2$.
- One can associate to every lattice Λ a binary quadratic form, which can be identified with the Gram matrix of the Lattice.

  [Jonathan Hanke's Arizona Winter School notes (Quadratic Forms and Automorphic Forms)]
- A BQF of level $N \longleftrightarrow$ A lattice whose theta function is modular on $\Gamma_0(N)$.

Quadratic forms and Lattices

- The symmetric bilinear form of a lattice looks like a quadratic form: $Q(x, y) = ax^2 + bxy + cy^2$.
- One can associate to every lattice Λ a binary quadratic form, which can be identified with the Gram matrix of the Lattice.

[Jonathan Hanke's Arizona Winter School notes (Quadratic Forms and Automorphic Forms)]

- A BQF of level $N \longleftrightarrow$ A lattice whose theta function is modular on $\Gamma_0(N)$.
- By choosing higer representations of the quadratic form, one can construct not just $\theta-$functions but also Siegel-Theta functions that are modular under $Sp(2g, \mathbb{Z})$. These represent higher genus analogues. (For $g = 1, Sp(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})$).

Indefinite theta functions and lattices

- While theta functions tell us about the lattice, there is a way to define them directly from the quadratic form.

Indefinite theta functions and lattices

- While theta functions tell us about the lattice, there is a way to define them directly from the quadratic form.
- Let Q be associated to $\Lambda^{p,q}$.

$$Q(x) = \sum_{i,j=1}^{p+q} Q_{ij} x^i x^j, \; Q(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$$

Let $\Lambda^{p,q} = \Lambda_L^p \oplus \Lambda_R^q$. $Q_L(x) = Q(x)\big|_{x\in\Lambda_L^p}$ and $Q_R(x) = Q(x)\big|_{x\in\Lambda_R^q}$

Naturally: $Q(x) = Q_L(x) - Q_R(x)$

Indefinite theta functions and lattices

- While theta functions tell us about the lattice, there is a way to define them directly from the quadratic form.
- Let Q be associated to $\Lambda^{p,q}$.

$$Q(x) = \sum_{i,j=1}^{p+q} Q_{ij} x^i x^j, \; Q(x,y) = \frac{1}{2} \left(Q(x+y) - Q(x) - Q(y) \right)$$

Let $\Lambda^{p,q} = \Lambda_L^p \oplus \Lambda_R^q$. $Q_L(x) = Q(x)\big|_{x \in \Lambda_L^p}$ and $Q_R(x) = Q(x)\big|_{x \in \Lambda_R^q}$

Naturally: $Q(x) = Q_L(x) - Q_R(x)$

Here, I seem to always work with an orthonormal basis. This is becasue $\exists g \in GL(p+q, \mathbb{F})$ that allows conjugation to this basis.

Indefinite theta functions and lattices

Majorize the quadratic form: $H(x) = Q_L(x) + Q_R(x)$.

Indefinite theta functions and lattices

Majorize the quadratic form: $H(x) = Q_L(x) + Q_R(x)$.

The Siegel-Narain Theta (Riemann theta)

$$\theta(\tau, \bar{\tau}; m) = \sum_{x \in \Lambda} e^{i\pi\tau_1 Q(x) - \pi\tau_2 H(x)}$$
$$= \sum_{x \in \Lambda} q^{Q_L(x)/2} \bar{q}^{Q_R(x)/2}, \ q := e^{2\pi i \tau}$$

m is the point in moduli space.

More generically, we can also shift the lattice element by an element of the dual lattice Λ^*

Indefinite theta functions and lattices

On the level of quadratic forms: $\Lambda^* = \{y | \, Q(x,y) \in \mathbb{Z}, \, \forall \, x \in \Lambda\}$.

Easily see that $\Lambda \subset \Lambda^*$ (unless $\Lambda = \Lambda^*$ i.e., unimodular/self-dual)

Discriminant group $D = \Lambda^*/\Lambda$

The generic Siegel-Narain Theta

$$\theta_h(\tau, \bar{\tau}; m) = \sum_{x \in \Lambda} e^{i\pi\tau_1 Q(x+h) - \pi\tau_2 H(x+h)}$$

m is the point in moduli space, $h \in D$.

Modularity of indefinite theta functions

(Combining all hol. and anti-hol. periods)

$$T : \theta_h(\tau+1; m) = e^{i\pi Q(h,h)}\theta_h(\tau; m)$$
$$S : \theta_h\left(\frac{-1}{\tau}; m\right) = \frac{e^{-i\pi(p-q)/4}}{\sqrt{|\det Q|}}\tau^{p/q}\bar{\tau}^{q/2}\sum_{h'\in D} e^(- 2\pi i Q(h,h'))\theta_{h'}(\tau; m)$$

Also remind ourselves of the modularity properties of the Dedekind eta

$$\eta(\tau) = q^{1/24}\sum_{n=1}^{\infty}(1-q^n)$$
$$T : \eta(\tau+1) = e^{2\pi i/24}\,\eta(\tau),\;\; S : \eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

The modularity of $\eta^p\bar{\eta}^q$ is what gives us the gravitational anomaly. Rank $24\mathbb{Z}$ lattices have no such anomaly.

Lets recap

- There is a correspondence between quadratic forms and lattices.

Lets recap

- There is a correspondence between quadratic forms and lattices.
- For every integral, indefinite lattice $\Lambda^{p,q}$, there exists a Q.

Lets recap

- There is a correspondence between quadratic forms and lattices.
- For every integral, indefinite lattice $\Lambda^{p,q}$, there exists a Q.
- The Siegel-Narain theta function of the lattice is non-holomorphic modular form of weight $(p/2, q/2)$ and it is modular under $\Gamma_0(N)$ where N is the level of the lattice/quadratic form.

Lets recap

- There is a correspondence between quadratic forms and lattices.
- For every integral, indefinite lattice $\Lambda^{p,q}$, there exists a Q.
- The Siegel-Narain theta function of the lattice is non-holomorphic modular form of weight $(p/2, q/2)$ and it is modular under $\Gamma_0(N)$ where N is the level of the lattice/quadratic form.
- How does moduli dependence enter here?

Lets recap

- There is a correspondence between quadratic forms and lattices.
- For every integral, indefinite lattice $\Lambda^{p,q}$, there exists a Q.
- The Siegel-Narain theta function of the lattice is non-holomorphic modular form of weight $(p/2, q/2)$ and it is modular under $\Gamma_0(N)$ where N is the level of the lattice/quadratic form.
- How does moduli dependence enter here?

> Now, we let $\Lambda^{p,q}$ to be the Narain lattice.
>
> The associated quadratic form/ norm/ theta function becomes a function of the moduli now.

Averaging over a moduli space

One can average over the moduli parametrized by $\mathcal{M}_{d,d}$
[Maloney, Witten; Afhkami-Jeddi et.al; Maloney, Datta et.al; Maloney Collier; Hartman et.al; ADKLY]

But what does it mean to consider the “ensemble average"?

Integrate a function with moduli dependence over moduli space and divide by the volume of moduli space

Averaging over Narain moduli spaces

We want to compute the average partition function over the Narain moduli space.

Averaging over Narain moduli spaces

We want to compute the average partition function over the Narain moduli space.

$Z_{CFT}(\Omega, \bar{\Omega})$: The non-holomorphic partition function,

where $\Omega, \bar{\Omega}$ correspond to the period matrices of the genus g Riemann surfaces. Ex: $g = 1 \Rightarrow (\Omega, \bar{\Omega}) = (\tau, \bar{\tau})$

@ genus $g = 1$, the moduli "m" dependent partition function is given by:

$$Z_{CFT}(\tau, \bar{\tau}; m) = \frac{\theta(\tau, \bar{\tau}; m)}{\eta(\tau)^p \bar{\eta}(\bar{\tau})^q},$$

Therefore, averaging the partition function is a problem of averaging the theta function.

The Siegel-Weil Formula

The modulus m takes values in
$G/H = O(p,q;\mathbb{R})/(O(p;\mathbb{R}) \times O(q;\mathbb{R}))$.

There is a G-invariant Haar measure $[dm]$ which is precisely the Zamolodchikov metric.

The Siegel-Weil Formula

The modulus m takes values in $G/H = O(p,q;\mathbb{R})/(O(p;\mathbb{R}) \times O(q;\mathbb{R}))$.

There is a G-invariant Haar measure $[dm]$ which is precisely the Zamolodchikov metric. Set $\tau = const$.

$$\langle Z_{CFT}(\tau) \rangle = \frac{1}{\text{vol}(\mathcal{M})} \int_{\mathcal{M}} [dm]\, Z_{CFT}(\tau, \bar{\tau}; m)$$

$$\text{vol}(\mathcal{M}) = \int_{\mathcal{M}} [dm]$$

[More on Zamolodchikov metrics & volumes: Moore]

The Siegel-Weil Formula

The modulus m takes values in
$G/H = O(p,q;\mathbb{R})/(O(p;\mathbb{R}) \times O(q;\mathbb{R}))$.

There is a G-invariant Haar measure $[dm]$ which is precisely the Zamolodchikov metric. Set $\tau = const$.

$$\langle Z_{CFT}(\tau)\rangle = \frac{1}{\mathrm{vol}(\mathcal{M})}\int_{\mathcal{M}}[dm]\, Z_{CFT}(\tau,\bar{\tau};m)$$

$$\mathrm{vol}(\mathcal{M}) = \int_{\mathcal{M}}[dm]$$

[More on Zamolodchikov metrics & volumes: Moore]

Since the Dedekind eta is moduli independent:

$$\langle\theta(\tau,\bar{\tau})\rangle = \frac{1}{\mathrm{vol}(\mathcal{M})}\int_{\mathcal{M}}[dm]\, \theta(\tau,\bar{\tau};m)$$

(Above expression also holds for shifted Siegel-Narain theta functions.)

So what is the average of the theta function?

The Siegel-Weil Formula

Theorem ("Generalized Siegel-Weil"):

The average of an indefinite theta function associated to an indefinite lattice of signature (p, q) is the non-holomorphic Eisenstein series of weight $\left(\frac{p}{2}, \frac{q}{2}\right)$ that is modular on $\Gamma_0(N)$, where N is the level of the lattice/quadratic form. [Siegel; Weil]

$$\left\langle \theta_{Q,h}(\tau, \bar{\tau}) \right\rangle = \frac{1}{\mathrm{vol}(\mathcal{M})} \int_{\mathcal{M}} [dm]\, \theta_{Q,h}(\tau, \bar{\tau}; m) = E_{Q,h}(\tau, \bar{\tau})$$

where $E_{Q,h}(\tau, \bar{\tau}) = \delta_{h \in \Lambda} + \displaystyle\sum_{(c,d)=1, c>0} \frac{\gamma_{Q,h}(c,d)}{(c\tau + d)^{\frac{p}{2}} (c\bar{\tau} + d)^{\frac{q}{2}}},$

$$\underbrace{\gamma_{Q,h}(c,d)}_{\text{Quadratic Gauss Sum}} = e^{i\pi \frac{p-q}{4}} |\det Q|^{-\frac{1}{2}} c^{-\frac{p+q}{2}} \sum_{x \in \Lambda / c\Lambda} \exp\left(-\pi i \frac{d}{c} Q(x+h) \right)$$

[ADKLY; For QGS: Turaev, Deloup]

Average CFT partition function

So, the average partition function of toroidal CFT's

$$\langle Z^{CFT}_{Q,h}(\tau,\bar\tau)\rangle = \frac{\langle \theta_{Q,h}(\tau,\bar\tau)\rangle}{\eta(\tau)^p\bar\eta(\bar\tau)^q} = \frac{E_{Q,h}(\tau,\bar\tau)}{\eta(\tau)^p\bar\eta(\bar\tau)^q},$$

Modularity properties of this averaged partition function are easy to deduce from below. [ADKLY]

$$T : E_{Q,h}(\tau+1,\bar\tau+1) = e^{i\pi Q(h,h)} E_{Q,h}(\tau,\bar\tau)$$

$$S : E_{Q,h}\left(\frac{-1}{\tau},\frac{-1}{\bar\tau}\right) = \frac{e^{i\pi\frac{p-q}{4}}}{\sqrt{|\det Q|}}\tau^{p/2}\bar\tau^{q/2}\sum_{h'\in\Lambda^*/\Lambda} e^{2\pi i Q(h,h')} E_{Q,h'}(\tau,\bar\tau)$$

Average CFT partition function

So, the average partition function of toroidal CFT's

$$\langle Z^{CFT}_{Q,h}(\tau,\bar\tau)\rangle = \frac{\langle\theta_{Q,h}(\tau,\bar\tau)\rangle}{\eta(\tau)^p\bar\eta(\bar\tau)^q} = \frac{E_{Q,h}(\tau,\bar\tau)}{\eta(\tau)^p\bar\eta(\bar\tau)^q},$$

Modularity properties of this averaged partition function are easy to deduce from below. [ADKLY]

$$T : E_{Q,h}(\tau+1,\bar\tau+1) = e^{i\pi Q(h,h)}E_{Q,h}(\tau,\bar\tau)$$

$$S : E_{Q,h}\left(\frac{-1}{\tau},\frac{-1}{\bar\tau}\right) = \frac{e^{i\pi\frac{p-q}{4}}}{\sqrt{|\det Q|}}\tau^{p/2}\bar\tau^{q/2}\sum_{h'\in\Lambda^*/\Lambda} e^{2\pi i Q(h,h')}E_{Q,h'}(\tau,\bar\tau)$$

Non-trivialities: In considering an indefinite lattice, we have gravitational anomalies. The presence of these anomalies makes the averaged PF more intricate.

Holographic interpretation

- We can also match symmetries. The $U(1)^p \times U(1)^q$ boundary symmetry becomes a gauge symmetry in the bulk.

Holographic interpretation

- We can also match symmetries. The $U(1)^p \times U(1)^q$ boundary symmetry becomes a gauge symmetry in the bulk.
- Using the quadratic form, we can write a bulk $U(1)^{p+q}$ Chern-Simons as

$$S_{CS} = \frac{i}{8\pi} \int_M Q_{ij} \, A^i \wedge A^j$$

Holographic interpretation

- We can also match symmetries. The $U(1)^p \times U(1)^q$ boundary symmetry becomes a gauge symmetry in the bulk.
- Using the quadratic form, we can write a bulk $U(1)^{p+q}$ Chern-Simons as

$$S_{CS} = \frac{i}{8\pi} \int_M Q_{ij} \, A^i \wedge A^j$$

- The lattice elements will label the states in the bulk Chern-Simons theory and correspond to Wilson lines wrapping non-contractible cycles in the bulk.

Holographic interpretation

- We can also match symmetries. The $U(1)^p \times U(1)^q$ boundary symmetry becomes a gauge symmetry in the bulk.
- Using the quadratic form, we can write a bulk $U(1)^{p+q}$ Chern-Simons as

$$S_{CS} = \frac{i}{8\pi} \int_M Q_{ij} \, A^i \wedge A^j$$

- The lattice elements will label the states in the bulk Chern-Simons theory and correspond to Wilson lines wrapping non-contractible cycles in the bulk.

In classification of 2d interacting topological phases, the Q_{ij} is indeed the $K-$ matrix. It is remarkable that it can be derived from averaging.

Holographic interpretation

- Let us consider even, indefinite lattices for the moment.
- Naïvely, we expect the bulk partition function to be a sum over geometries ($PSL(2,\mathbb{Z})$ black holes). [Maloney, Witten (2008)]

Holographic interpretation

- Let us consider even, indefinite lattices for the moment.
- Naïvely, we expect the bulk partition function to be a sum over geometries ($PSL(2,\mathbb{Z})$ black holes). [Maloney, Witten (2008)]
- These geometries are solid tori with torus boundaries with $PSL(2,\mathbb{Z})$ the mapping class group of the boundary torus.

Holographic interpretation

- Let us consider even, indefinite lattices for the moment.
- Naïvely, we expect the bulk partition function to be a sum over geometries ($PSL(2,\mathbb{Z})$ black holes). [Maloney, Witten (2008)]
- These geometries are solid tori with torus boundaries with $PSL(2,\mathbb{Z})$ the mapping class group of the boundary torus.
- A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$ labels each geometry as $M_{(c,d)}$. (Ex: $M_{(1,0)}$: BTZ black hole, $M_{(0,1)}$: Thermal AdS_3)

Holographic interpretation

- Let us consider even, indefinite lattices for the moment.
- Naïvely, we expect the bulk partition function to be a sum over geometries ($PSL(2,\mathbb{Z})$ black holes). [Maloney, Witten (2008)]
- These geometries are solid tori with torus boundaries with $PSL(2,\mathbb{Z})$ the mapping class group of the boundary torus.
- A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$ labels each geometry as $M_{(c,d)}$. (Ex: $M_{(1,0)}$: BTZ black hole, $M_{(0,1)}$: Thermal AdS_3)
- So the sum over (c,d) in the Eisenstein series can be interpreted as a sum over geometries in the bulk.

Holographic interpretation

- For each geometry $M_{(c,d)}$, we need a bulk calculation of the quadratic Gauss sum, $\gamma_{Q,h}(c,d)$.

Holographic interpretation

- For each geometry $M_{(c,d)}$, we need a bulk calculation of the quadratic Gauss sum, $\gamma_{Q,h}(c,d)$.
- In the canonical quantization of $U(1)_k$ Chern-Simons theory, the space of states at level k is spanned by $|h\rangle$, where $h = 0, \frac{1}{k}, \cdots, \frac{(k-1)}{k}$

Holographic interpretation

- For each geometry $M_{(c,d)}$, we need a bulk calculation of the quadratic Gauss sum, $\gamma_{Q,h}(c,d)$.
- In the canonical quantization of $U(1)_k$ Chern-Simons theory, the space of states at level k is spanned by $|h\rangle$, where $h = 0, \frac{1}{k}, \cdots, \frac{(k-1)}{k}$
- This corresponds to a path integral over a solid torus with a kh Wilson line insertion in the bulk.
- The modular group acts on these states as

$$T|h\rangle = e^{i\pi kh^2} e^{-2\pi i/24}|h\rangle$$
$$S|h\rangle = \frac{1}{\sqrt{k}} \sum_{h' \in \Lambda^*/\Lambda} e^{-2\pi ikhh'}|h\rangle$$

Holographic interpretation

- For the case of the $U(1)^q \times U(1)^q$ Abelian Chern-Simons theory, this generalizes to

$$T|h;m\rangle = e^{i\pi Q(h,h)} e^{-2\pi i(p-q)/24}|h;m\rangle$$
$$S|h;m\rangle = \frac{1}{\sqrt{|\det Q|}} \sum_{h' \in \Lambda^*/\Lambda} e^{-2\pi i Q(h,h')}|h;m\rangle$$

- Since the S and T matrices generate the group $SL(2,\mathbb{Z})$, we can generalize the action of any element $g \in SL(2,\mathbb{Z}$ on $|h;m\rangle$ as

$$U(g)|h;m\rangle = \frac{1}{\sqrt{|\det Q|}} \sum_{h' \in \Lambda^*/\Lambda} U(g)_{h,h'}|h;m\rangle$$

Holographic computation of averaged partition function

- Now, we want a state without a Wilson line insertion ($|h = 0\rangle$).

Holographic computation of averaged partition function

- Now, we want a state without a Wilson line insertion ($|h = 0\rangle$).
- So we want to compute $U(g)_{0,h'}$. It turns out that its complex conjugate is the quadratic Gauss sum $\gamma_{Q,h}(c,d)$.

$$\langle 0|U(g)|h'\rangle^* = e^{2\pi i(p-q)\Phi(g)/24 - i\pi(p-q)/4}\gamma_{Q,h}(c,d),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

and $\Phi(g)$ is the Rademacher-Phi function: phase picked up by the Dedekind eta under modular transformation of an arbitrary element $g \in SL(2,\mathbb{Z})$.

[More on $\Phi(g)$: Dedekind's book; Dabholkar, Murthy, Gomes (2014)]

Holographic computation of averaged partition function

- Now, we want a state without a Wilson line insertion ($|h = 0\rangle$).
- So we want to compute $U(g)_{0,h'}$. It turns out that its complex conjugate is the quadratic Gauss sum $\gamma_{Q,h}(c,d)$.

$$\langle 0|U(g)|h'\rangle^* = e^{2\pi i(p-q)\Phi(g)/24 - i\pi(p-q)/4}\gamma_{Q,h}(c,d),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

and $\Phi(g)$ is the Rademacher-Phi function: phase picked up by the Dedekind eta under modular transformation of an arbitrary element $g \in SL(2,\mathbb{Z})$.

[More on $\Phi(g)$: Dedekind's book; Dabholkar, Murthy, Gomes (2014)]

- These are Lens spaces $L(c,d)$ computations and one can compute invariants of the three manifold invariants using these techniques. Ex: $U(g)_{0,0}$ computes the $\eta-$invariant of the 3-manifold.

Holographic computation of averaged partition function

For the case of an even lattice, the bulk partition is given by

$$\langle Z_{bulk} \rangle = \sum_{g \in \Gamma_\infty \backslash PSL(2,\mathbb{Z})} \frac{U(g)^*_{0,h}}{\eta(g \cdot \tau)^p \bar{\eta}(g \cdot \bar{\tau})^q}$$

Holographic computation of averaged partition function

For the case of an even lattice, the bulk partition is given by

$$\langle Z_{bulk} \rangle = \sum_{g \in \Gamma_\infty \backslash PSL(2,\mathbb{Z})} \frac{U(g)^*_{0,h}}{\eta(g \cdot \tau)^p \bar{\eta}(g \cdot \bar{\tau})^q}$$

- This agrees with the average of the CFT partition function.

Holographic computation of averaged partition function

For the case of an even lattice, the bulk partition is given by

$$\langle Z_{bulk} \rangle = \sum_{g \in \Gamma_\infty \backslash PSL(2,\mathbb{Z})} \frac{U(g)^*_{0,h}}{\eta(g \cdot \tau)^p \bar{\eta}(g \cdot \bar{\tau})^q}$$

- This agrees with the average of the CFT partition function.
- There are some subtleties here that need to be better understood.

Holographic computation of averaged partition function

For the case of an even lattice, the bulk partition is given by

$$\langle Z_{bulk} \rangle = \sum_{g \in \Gamma_\infty \backslash PSL(2,\mathbb{Z})} \frac{U(g)^*_{0,h}}{\eta(g\cdot\tau)^p \bar{\eta}(g\cdot\bar{\tau})^q}$$

- This agrees with the average of the CFT partition function.
- There are some subtleties here that need to be better understood.
- In particular, the holographic match demands that the gauge group of the Chern-Simons theory is $U(1)^{p+q}$ and not $\mathbb{R}^{p+q}$.

[Maloney, Witten]

Positive definite cases

- When we consider a lattice of the form $\Lambda^{p,0}$, the moduli space of CFT's is trivial.

Positive definite cases

- When we consider a lattice of the form $\Lambda^{p,0}$, the moduli space of CFT's is trivial.
- However, one case still has a Siegel-Weil formula [Siegel]

$$\langle\langle\theta_Q(\tau)\rangle\rangle = E_Q(\tau)$$

but the average is now over the class of quadratic forms.

Positive definite cases

- When we consider a lattice of the form $\Lambda^{p,0}$, the moduli space of CFT's is trivial.
- However, one case still has a Siegel-Weil formula [Siegel]

$$\langle\langle \theta_Q(\tau) \rangle\rangle = E_Q(\tau)$$

but the average is now over the class of quadratic forms.

- Two quadratic forms Q, Q' are equivalent in a field $\mathbb{F}$ if $\exists g \in GL(p+q, \mathbb{F})$ such that $Q' = g^T Q g$. If $\mathbb{F} = \mathbb{Z}(\mathbb{R})$, we say that Q, Q' are in the same class(genus). The number of equivalence classes of Q is called is class number $h(Q)$.

Positive definite cases

- When we consider a lattice of the form $\Lambda^{p,0}$, the moduli space of CFT's is trivial.
- However, one case still has a Siegel-Weil formula [Siegel]

$$\langle\langle\theta_Q(\tau)\rangle\rangle = E_Q(\tau)$$

but the average is now over the class of quadratic forms.

- Two quadratic forms Q, Q' are equivalent in a field $\mathbb{F}$ if $\exists g \in GL(p+q, \mathbb{F})$ such that $Q' = g^T Q g$. If $\mathbb{F} = \mathbb{Z}(\mathbb{R})$, we say that Q, Q' are in the same class(genus). The number of equivalence classes of Q is called is class number $h(Q)$.
- The class number is finite. [Lagrange]

Positive definite cases

The ensemble average of CFT's whose Narain lattice is a positive definite lattice is given by

$$\langle\langle\theta_Q(\tau)\rangle\rangle = \frac{1}{M(Q)}\sum_{j=1}^{h(Q)}\frac{\theta_{Q_j}}{|\mathrm{Aut}(\Lambda)|},\ M(Q) = \sum_{Q'\in\mathrm{Genus}(Q)}\frac{1}{|\mathrm{Aut}(\Lambda'_Q)|}$$

[Siegel; Smith; Minkowski]

From this, one can compute the partition function of chiral theories as

$$\langle\langle Z_{chiral}(\tau)\rangle\rangle = \frac{\langle\langle\theta_Q(\tau)\rangle\rangle}{\eta(\tau)^p}.$$

Positive definite cases

The holography of such theories has also been studied

[Dymarsky, Shapere]

Positive definite cases

The holography of such theories has also been studied

[Dymarsky, Shapere]

Remarkable if something like ensemble averages predicts mass formulae for unimodular lattices with no roots. [King, 2003]

Positive definite cases

The holography of such theories has also been studied

[Dymarsky, Shapere]

Remarkable if something like ensemble averages predicts mass formulae for unimodular lattices with no roots. [King, 2003]

Still unclear what the physical motivation for considering ensemble averages for postive definite theories is.

Spin Chern-Simons invariants and Fermionic CFT's

- When considering the case odd lattices, one must consider fermionic CFT's.

Spin Chern-Simons invariants and Fermionic CFT's

- When considering the case odd lattices, one must consider fermionic CFT's.
- Depending on the genus g of the Riemann surface, you get 2^{2g} spin structures.

Spin Chern-Simons invariants and Fermionic CFT's

- When considering the case odd lattices, one must consider fermionic CFT's.
- Depending on the genus g of the Riemann surface, you get 2^{2g} spin structures.
- This generalizes the theta function to 2^{2g} theta functions labelled by choice of spin structure. Spin structures often transform into one another under modular transformations.

Spin Chern-Simons invariants and Fermionic CFT's

- When considering the case odd lattices, one must consider fermionic CFT's.
- Depending on the genus g of the Riemann surface, you get 2^{2g} spin structures.
- This generalizes the theta function to 2^{2g} theta functions labelled by choice of spin structure. Spin structures often transform into one another under modular transformations.
- The key idea here is to average each spin structre independently.

Spin Chern-Simons invariants and Fermionic CFT's

- Consider $g = 1$. We have 4 spin structures labelled by $(\epsilon_1, \epsilon_2) = (0,0), (0,1), (1,0), (1,1)$.

Spin Chern-Simons invariants and Fermionic CFT's

- Consider $g = 1$. We have 4 spin structures labelled by $(\epsilon_1, \epsilon_2) = (0,0), (0,1), (1,0), (1,1)$.
- The theta function generalizes as

$$\theta^{\epsilon_1,\epsilon_2}_{Q,h}(\tau; m) = \sum_{x \in \Lambda + h + \epsilon_1 W/2} e^{i\pi\tau Q_L(x) - i\pi\bar{\tau} Q_R(x)} (-1)^{\epsilon_2 (W,x)},$$

where $Q_{L/R}$ were as defined previously, W is the a characteristic class of the dual lattice known as the integral Wu class, $(W, x) \equiv Q(x)$ mod 2, $x \in \Lambda$, and $h \in \Lambda^* / \Lambda$.

Spin Chern-Simons invariants and Fermionic CFT's

- Consider $g = 1$. We have 4 spin structures labelled by $(\epsilon_1, \epsilon_2) = (0,0), (0,1), (1,0), (1,1)$.
- The theta function generalizes as

$$\theta^{\epsilon_1,\epsilon_2}_{Q,h}(\tau; m) = \sum_{x \in \Lambda + h + \epsilon_1 W/2} e^{i\pi\tau Q_L(x) - i\pi\bar{\tau} Q_R(x)} (-1)^{\epsilon_2 (W,x)},$$

where $Q_{L/R}$ were as defined previously, W is the a characteristic class of the dual lattice known as the integral Wu class, $(W, x) \equiv Q(x)$ mod 2, $x \in \Lambda$, and $h \in \Lambda^*/\Lambda$.

- The Siegel-Weil theorem doen't care about $\mathbb{Z}_2$ refinements:

$$E^{\epsilon_1,\epsilon_2}_{Q,h}(\tau, \bar{\tau}; m) = \langle \theta^{\epsilon_1,\epsilon_2}_{Q,h}(\tau, \bar{\tau}; m) \rangle$$

Computing the Eisenstein series for spin CFT's: Start with $E^{0,0}$

$$E^{0,0}_{Q,h}(\tau;m) = \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

Computing the Eisenstein series for spin CFT's: Start with $E^{0,0}$

$$E^{0,0}_{Q,h}(\tau;m) = \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1 \\ cd\in 2\mathbb{Z} \\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$E^{0,1}_{Q,h}(\tau;m) = E^{0,0}_{Q,h}(\tau+1;m) = 1 + \sum_{\substack{(c,d)=1 \\ cd\in 2\mathbb{Z} \\ c>0}} \frac{\gamma_{Q,0}(c,d-c)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

Computing the Eisenstein series for spin CFT's: Start with $E^{0,0}$

$$E^{0,0}_{Q,h}(\tau;m) = \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$E^{0,1}_{Q,h}(\tau;m) = E^{0,0}_{Q,h}(\tau+1;m) = 1 + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,0}(c,d-c)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$\begin{aligned} E^{1,0}_{Q,h}(\tau;m) &= \frac{e^{i\pi(p-q)/4}E^{0,0}_{Q,h}\left(\frac{\tau-1}{\tau}\right)}{\tau^{p/2}\bar{\tau}^{q/2}} \\ &= \frac{e^{i\pi(p-q)/4}}{\tau^{p/2}\bar{\tau}^{q/2}} + e^{i\pi(p-q)/4} \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ d<0}} \frac{\gamma_{Q,0}(-d,c+d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}} \end{aligned}$$

Computing the Eisenstein series for spin CFT's: Start with $E^{0,0}$

$$E^{0,0}_{Q,h}(\tau;m) = \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$E^{0,1}_{Q,h}(\tau;m) = E^{0,0}_{Q,h}(\tau+1;m) = 1 + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,0}(c,d-c)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$E^{1,0}_{Q,h}(\tau;m) = \frac{e^{i\pi(p-q)/4}E^{0,0}_{Q,h}\left(\frac{\tau-1}{\tau}\right)}{\tau^{p/2}\bar{\tau}^{q/2}}$$

$$= \frac{e^{i\pi(p-q)/4}}{\tau^{p/2}\bar{\tau}^{q/2}} + e^{i\pi(p-q)/4}\sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ d<0}} \frac{\gamma_{Q,0}(-d,c+d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$E^{1,1}_{Q,h}(\tau;m) \equiv 0$$

Computing the Eisenstein series for spin CFT's: Start with $E^{0,0}$

$$E^{0,0}_{Q,h}(\tau;m) = \delta_{h\in\Lambda} + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,h}(c,d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$E^{0,1}_{Q,h}(\tau;m) = E^{0,0}_{Q,h}(\tau+1;m) = 1 + \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ c>0}} \frac{\gamma_{Q,0}(c,d-c)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}}$$

$$\begin{aligned} E^{1,0}_{Q,h}(\tau;m) &= \frac{e^{i\pi(p-q)/4} E^{0,0}_{Q,h}\left(\frac{\tau-1}{\tau}\right)}{\tau^{p/2}\bar{\tau}^{q/2}} \\ &= \frac{e^{i\pi(p-q)/4}}{\tau^{p/2}\bar{\tau}^{q/2}} + e^{i\pi(p-q)/4} \sum_{\substack{(c,d)=1\\ cd\in 2\mathbb{Z}\\ d<0}} \frac{\gamma_{Q,0}(-d,c+d)}{(c\tau+d)^{p/2}(c\bar{\tau}+p)^{q/2}} \end{aligned}$$

$$E^{1,1}_{Q,h}(\tau;m) \equiv 0$$

Non-trivialities in the computation of $\gamma_{Q,h}(c,d)$ here. [ADKLY]

The Bulk Interpretation: Fermionic CFT's

- Analogous to the previous case, we are looking for a set of operators U that compute the Lens space partition functions for spin Chern-Simons invariants. [Moore, Belov]

The Bulk Interpretation: Fermionic CFT's

- Analogous to the previous case, we are looking for a set of operators U that compute the Lens space partition functions for spin Chern-Simons invariants. [Moore, Belov]

- Now, there is explicit spin-structure dependence as well. This gives us not a basis as in $U(g)_{h,h'}$ but rather a gluing matrix relating the spin structres (ϵ_1, ϵ_2) of h to $(\epsilon'_1, \epsilon'_2)$ of h'.

$$O_{h,h'} \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon'_1 & \epsilon'_2 \end{bmatrix}$$

We know how T and S matrix elements act on a matrix $\begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon'_1 & \epsilon'_2 \end{bmatrix}$.
[ADKLY]

The Bulk Interpretation: Fermionic CFT's

- Analogous to the previous case, we are looking for a set of operators U that compute the Lens space partition functions for spin Chern-Simons invariants. [Moore, Belov]

- Now, there is explicit spin-structure dependence as well. This gives us not a basis as in $U(g)_{h,h'}$ but rather a gluing matrix relating the spin structres (ϵ_1, ϵ_2) of h to $(\epsilon_1', \epsilon_2')$ of h'.

$$O_{h,h'} \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_1' & \epsilon_2' \end{bmatrix}$$

We know how T and S matrix elements act on a matrix $\begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_1' & \epsilon_2' \end{bmatrix}$. [ADKLY]

- Repeat exactly as before to compute the partition functions.

The Bulk Interpretation: Fermionic CFT's

- The partition functions for various spin structures obtained match the expectation from the ensemble averages of spin structures [ADKLY].

The Bulk Interpretation: Fermionic CFT's

- The partition functions for various spin structures obtained match the expectation from the ensemble averages of spin structures [ADKLY].
- These also give us spin Chern-Simons invariants, analogous to the WRT invariants computed by Lisa Jeffrey. [Jeffrey]

The Bulk Interpretation: Fermionic CFT's

- The partition functions for various spin structures obtained match the expectation from the ensemble averages of spin structures [ADKLY].
- These also give us spin Chern-Simons invariants, analogous to the WRT invariants computed by Lisa Jeffrey. [Jeffrey]
- It is quite interesting that topological invariants can be computed from the ensemble averages of field theories that are not topological.

Take home messages

- By considering ensemble averages of CFTs associated to indefinite lattices, you can compute topological invariants of 3-manifolds
- It seems that once you take an ensemble average of the CFT, the sum of geometries is automatically incorporated.

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

(Quasi)-Realistically:

1. Orbifolds of indefinite lattices [WIP]

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

(Quasi)-Realistically:

1. Orbifolds of indefinite lattices [WIP]
2. Averaging over other kinds of moduli space?

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

(Quasi)-Realistically:

1. Orbifolds of indefinite lattices [WIP]
2. Averaging over other kinds of moduli space?
3. Higher dimensions? (Very interesting mathematics expected!)

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

(Quasi)-Realistically:

1. Orbifolds of indefinite lattices [WIP]
2. Averaging over other kinds of moduli space?
3. Higher dimensions? (Very interesting mathematics expected!)
4. Gravitational dual construction

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

(Quasi)-Realistically:

1. Orbifolds of indefinite lattices [WIP]

2. Averaging over other kinds of moduli space?

3. Higher dimensions? (Very interesting mathematics expected!)

4. Gravitational dual construction

5. Many more number theoretic connections that arise in CFT whose interpretation in the bulk is unclear.

Open problems

Philosophically: What are the rules of averaging? Are there universal features to averaged CFT's? How much number theory does $3d$ non-supersymmetric gravity actually know?

(Quasi)-Realistically:

1. Orbifolds of indefinite lattices [WIP]

2. Averaging over other kinds of moduli space?

3. Higher dimensions? (Very interesting mathematics expected!)

4. Gravitational dual construction

5. Many more number theoretic connections that arise in CFT whose interpretation in the bulk is unclear.

There seems to be a deeper connection between holography and number theory, even at a non-supersymmetric level. Perhaps this requires more elaborate and careful analysis.

Obrigado!