Geometry of Krylov Complexity

Paweł Caputa

UNIVERSITY OF WARSAW

POLSKIE POWROTY
POLISH RETURNS

NATIONAL SCIENCE CENTRE

Black Holes, BPS and QI, Lisbon, 20.09.2021

Outline

- Introduction and Motivation
- Operator Growth and Krylov Complexity
- Symmetry
- Geometry of Krylov Complexity
- Conclusions/Open Questions

Based on:

"Geometry of Krylov Complexity" with J.M. Magan (U.Penn.) and D. Patramanis (UW)
arXiv:2109.03824 [hep-th]

Intro/Motivation

- How CFT states encode holographic geometry?
- Important hint: Entanglement (RT and HRT)!
- QI for AdS/CFT: Renyi, EoP/RE, Pseudo Entropy, Capacity…. [Talks: Tatsuma, Robert, Roberto Tokiro…]
- "Entanglement (entropy) is not (always) enough"?
- We need more fine-grained probes: "Complexity"?
- Holographic developments: CV, CA, Complexity/Momentum…. [Federico's Talk]
- QC for AdS/CFT:….?

Some ideas for "Complexity" in QFT (CFT)?

States

Geometric Approaches ("Nielsen")

Quantum circuit

$$|\Psi_T\rangle = U(t)\,|\Psi_R\rangle$$

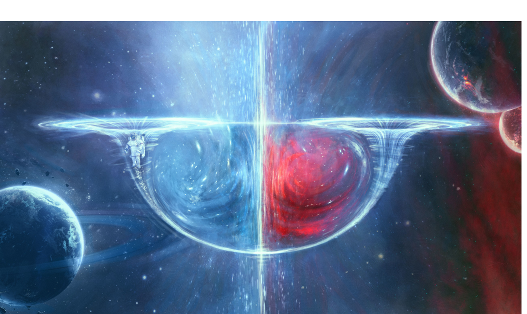
Complexity~ "Geodesic length"

Path Integral Complexity

PI Geometry ~ TN

Complexity ~ "Liouville action"

….



Growth of TFD…

Operators

This Talk!

Operator Size?

$$\mathcal{O}(t) = e^{iHt}\,\mathcal{O}(0)\,e^{-iHt}$$

"Operator Size" in SYK

$$n_j \equiv c_j^\dagger c_j = \frac{1}{2}\left(1 + i\psi_j^L \psi_j^R\right)$$

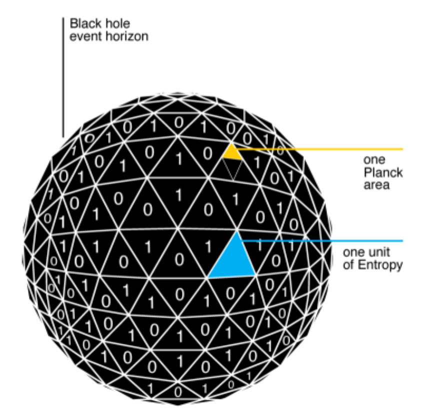
~Momentum of a particle in AdS2

OTOC

….

?

Near (behind?) horizon of BH….



This talk: focus on a definition of “operator complexity” called Krylov complexity

that can be universally defined in many-body systems (from QM to QFT).

I will discuss its geometric aspects in systems governed by symmetries (~CFT)

and interesting generalizations to the AdS/CFT contexts.

References:

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]

[Barbon, Rabinovici, Shir, Sinha ’19]

[Dymarsky, Gorsky ’19]

[Rabinovici, Sanchez-Garrido, Shir, Sonner ’20]

[Magan, Simon’20]

[Dymarsky, Smolkin ’21]

[Kar, Lamprou, Rozali, Sully ’21]

Operator Growth

Heisenberg evolution

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \qquad\qquad \mathcal{O}(t) = e^{iHt}\,\mathcal{O}(0)\,e^{-iHt}$$

Formally, we can write the operator as

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\tilde{\mathcal{O}}_n \qquad\qquad \tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [H, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [H, [H, \mathcal{O}]], ...$$

“Simple” operator evolves/spreads in the space of “Complex” operators.

Common Lore: The more “chaotic” H the faster the operator grows.

How to quantify this?

Krylov Basis

[Recursion Method: Viswanath,Muller '63]

Liouvillian (super)operator

$$\mathcal{L} = [H, \cdot], \qquad \mathcal{O}(t) \equiv e^{i\mathcal{L}t}\mathcal{O} \qquad \tilde{\mathcal{O}}_n \equiv \mathcal{L}^n \mathcal{O}$$

Given $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, ...\}$ we need a basis $|\mathcal{O}), |\mathcal{O}_1), |\mathcal{O}_2), ...$

First, we must pick an inner product (freedom):

$$(A|B) = \langle e^{H\beta/2} A^\dagger e^{-H\beta/2} B \rangle_\beta \qquad \langle A \rangle_\beta = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} A\right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

Then the orthonormal basis is constructed using Lanczos algorithm (G-S)

1) $|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) = |\mathcal{O}), \qquad |\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0), \qquad b_1 = (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$

2) $|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2}) \qquad b_0 = 0$

3) $|\mathcal{O}_n) = b_n^{-1}|A_n), \qquad b_n = (A_n|A_n)^{1/2} \qquad (\mathcal{O}_n|\mathcal{O}_m) = \delta_{n,m}$

Lanczos algorithm gives us $|\mathcal{O}_n)$ and b_n

Schrodinger equation

Now we expand the operator in the Krylov basis

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \equiv \sum_n i^n \varphi_n(t)|\mathcal{O}_n)$$

And derive equation for $\varphi_n(t)$

$$\partial_t|\mathcal{O}(t)) = \sum_n i^n \partial_t \varphi_n(t)|\mathcal{O}_n) = i\mathcal{L}|\mathcal{O}(t)) = \sum_n i^n \varphi_n(t)\mathcal{L}|\mathcal{O}_n)$$

From Lanczos algorithm

$$\mathcal{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

Comparing the coefficients and shifting the summation we derive

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n0}$$

Once we know Lanczos coefficients b_n we can find the “amplitudes”!

Comment: Auto-correlators

Lanczos coefficients are also “encoded” in the auto-correlator

$$C(t) = (\mathcal{O}|\mathcal{O}(t)) = (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O}) = \varphi_0(t)$$

Moments of C(t) can give us   in some recursive algorithm

$$\mu_{2n} := (\mathcal{O}|\mathcal{L}^{2n}|\mathcal{O}) = \frac{d^{2n}}{dt^{2n}}C(t)|_{t=0} \qquad b_1^2 \ldots b_n^2 = \det\left(\mu_{i+j}\right)_{0\leq i,j\leq n}.$$

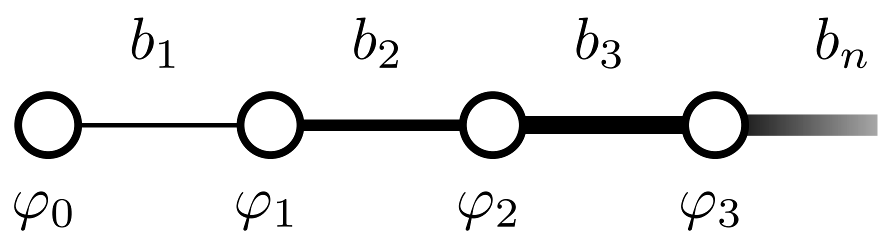
Usually C(t) are difficult to obtain but in some cases they are known explicitly (2d CFT on a line, integrable models, SYK, RM…). They are also related to Green’s functions or spectral functions.

In 2d CFT they can be interpreted in terms of geodesic between two sides of TFD at 0 and t.

$$C(t) \sim \cosh^{-2h}\left(\frac{\pi t}{\beta}\right)$$

“Krylov Complexity” (K-Complexity)

The physics of the growth can be understood as a motion of a particle on a chain



$$\sum_n |\varphi_n(t)|^2 = 1$$

The further in the chain the particle is, the more complex state in the Krylov basis is employed

This motivates a natural definition of complexity as average position on the chain:

$$K_{\mathcal{O}} = \sum_n n|\varphi_n(t)|^2$$

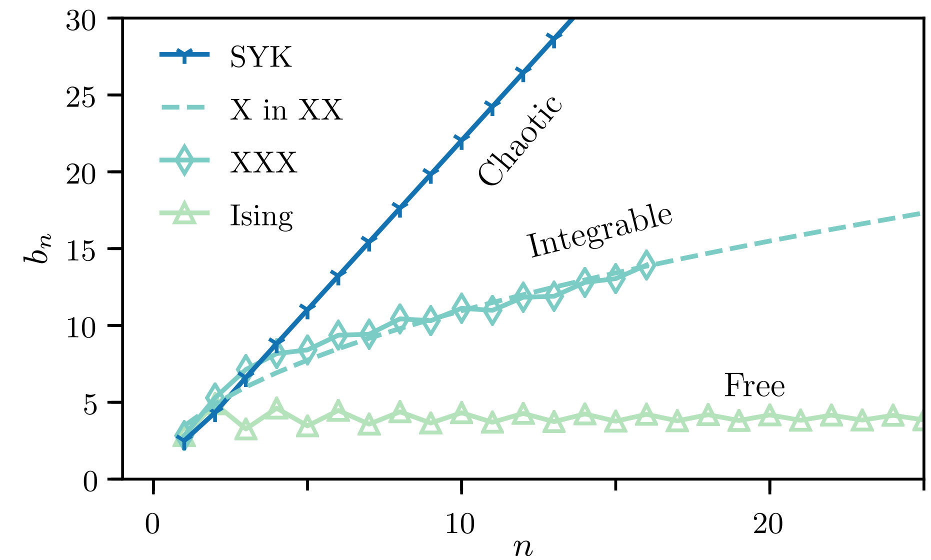
One can also think about the “Complexity Operator”

$$\hat{K}_{\mathcal{O}} = \sum_n n|\mathcal{O}_n)(\mathcal{O}_n| \qquad K_{\mathcal{O}} = (\mathcal{O}(t)|\hat{K}_{\mathcal{O}}|\mathcal{O}(t))$$

Universal Operator Growth Hypothesis

[Parker et al. ’19]

“Maximal growth Lanczos coefficients”



$$b_n \leq \alpha n + \gamma + O(1)$$

Saturated for “maximally chaotic” systems (OTOC)

Saturation is related to the exponential growth to Krylov Complexity

$$K_{\mathcal{O}} \sim e^{\lambda t} \qquad \lambda = 2\alpha$$

Example: SYK model (low T, $\eta \sim 2/q$)

$$b_n = \frac{\pi}{\beta}\sqrt{n(\eta + n - 1)} \qquad K_{\mathcal{O}} = \eta \sinh^2(\alpha t) \sim \frac{\eta}{4} e^{2\alpha t}$$

$$\varphi_n(t) = \sqrt{\frac{\Gamma(\eta + n)}{n!\Gamma(\eta)}} \frac{\tanh^n(\alpha t)}{\cosh^\eta(\alpha t)} \qquad \lambda = 2\alpha = \frac{2\pi}{\beta}$$

Questions:

Operational meaning of “Krylov Complexity” and Lanczos b_n ?

What determines the exponential growth (chaotic vs integrable)?

Symmetry? SYK and SL(2,R)?

Classification and generalizations? What can we say analytically?

Is it related to other approaches (geom. Nielsen, circuit)? $|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O})$

Features of QFT complexity? Holographic discussions?

Krylov Complexity and Symmetry?

[PC, J.M.Magan, D.Patramanis '21]

Liouvillian in the Krylov basis

$$\mathcal{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

It may be natural to think about it in terms of "Ladder Operators"

$$\mathcal{L} = \alpha\left(L_+ + L_-\right)$$

Such that

$$\alpha L_+|\mathcal{O}_n) = b_{n+1}|\mathcal{O}_{n+1}), \quad \alpha L_-|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1})$$

If these ladder operators belong to some Lie algebra then this would give a lot of predictive power! We could easily read of b_n !

Examples: SL(2,R)

[PC, J.M.Magan, D.Patramanis '21]

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_1, L_{-1}] = 2L_0$$

"Ladder Operators":

$$\mathcal{L} = \alpha \left(L_{-1} + L_1 \right)$$

Representation:

$$L_0 \left| h, n \right\rangle = (h + n) \left| h, n \right\rangle ,$$
$$L_{-1} \left| h, n \right\rangle = \sqrt{(n+1)(2h+n)} \left| h, n+1 \right\rangle ,$$
$$L_1 \left| h, n \right\rangle = \sqrt{n(2h+n-1)} \left| h, n-1 \right\rangle ,$$

$$\left| h, n \right\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}} L_{-1}^{n} \left| h \right\rangle$$

$|h,n\rangle \to |\mathcal{O}_n)$ $\qquad$ $\sqrt{n(2h+n-1)} \to \sim b_n$

SYK:

$$b_n = \frac{\pi}{\beta} \sqrt{n(\eta + n - 1)} \qquad \eta = 2h$$

Operator Growth and Coherent States

[PC, J.M.Magan, D.Patramanis '21]

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O})$$

$$\mathcal{L} = \alpha\,(L_{-1} + L_1)$$

Recall coherent states for SL(2,R) (SU(1,1))

[Perelomov'72]

$$|z,h\rangle \equiv D(\xi)\,|h\rangle\,, \qquad D(\xi) = e^{\xi L_{-1} - \bar{\xi} L_1},$$

$$\xi = \tfrac{1}{2}\rho e^{i\phi}$$

$$z = \tanh\left(\frac{\rho}{2}\right)e^{i\phi}, \qquad |z| < 1.$$

More explicitly:

$$|z,h\rangle = \sum_{n=0}^{\infty} e^{in\phi}\sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\frac{\tanh^n(\rho/2)}{\cosh^{2h}(\rho/2)}|k,h\rangle$$

Compare SYK:

$$\varphi_n(t) = \sqrt{\frac{\Gamma(\eta+n)}{n!\Gamma(\eta)}}\frac{\tanh^n(\alpha t)}{\cosh^{\eta}(\alpha t)}$$

"Trajectory in Phase Space": $\rho = 2\alpha t, \qquad \phi = \pi/2$

$$|\mathcal{O}(t)) = |z = i\tanh(\alpha t), h = \eta/2\rangle$$

Geometry of Krylov Complexity

[PC, J.M.Magan, D.Patramanis '21]

With coherent states we can associate a natural "information metric"

$$ds^2_{FS} = \langle dz|dz\rangle - \langle dz|z\rangle\langle z|dz\rangle \qquad \text{(Fubini-Study)}$$

E.g. for SL(2,R) this becomes a hyperbolic disc metric

$$ds^2_{FS} = \frac{2h\,dz\,d\bar{z}}{(1-z\bar{z})^2} = \frac{h}{2}\left(d\rho^2 + \sinh^2(\rho)d\phi^2\right) \qquad R = -\frac{4}{h}$$

Operator growth is a geodesic in this manifold (phase space):

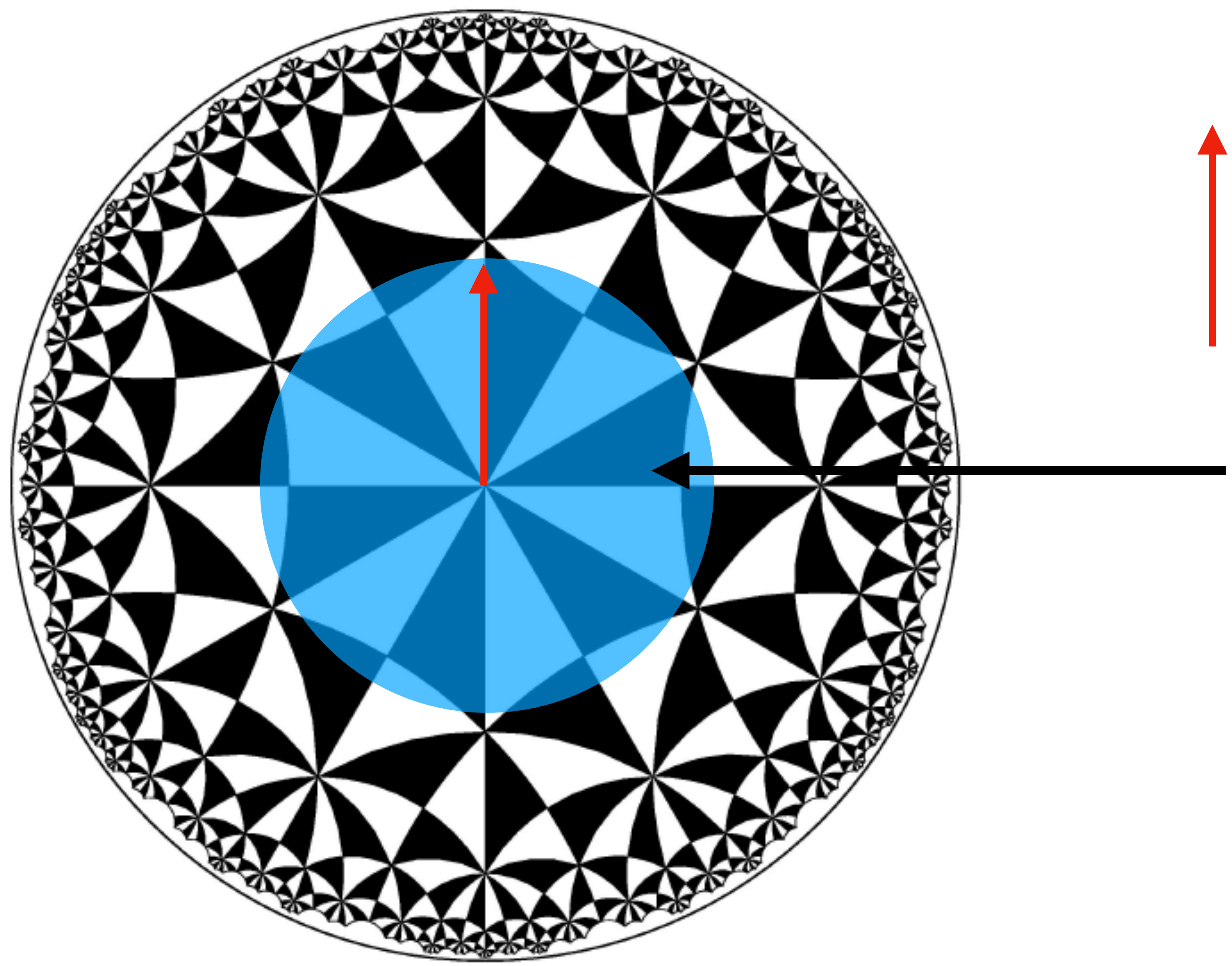
$$\rho = 2\alpha t, \qquad \phi = \pi/2$$

Observe a universal relation between the Volume and Krylov complexity

$$V_t = \int_0^{2\alpha t} d\rho \int_0^{2\pi} d\phi \sqrt{g} = 2\pi h \sinh^2(\alpha t) = \pi K_{\mathcal{O}}$$

This relation holds in all examples that we studied

Cartoon:



Phase Space Information Geometry

Comments:

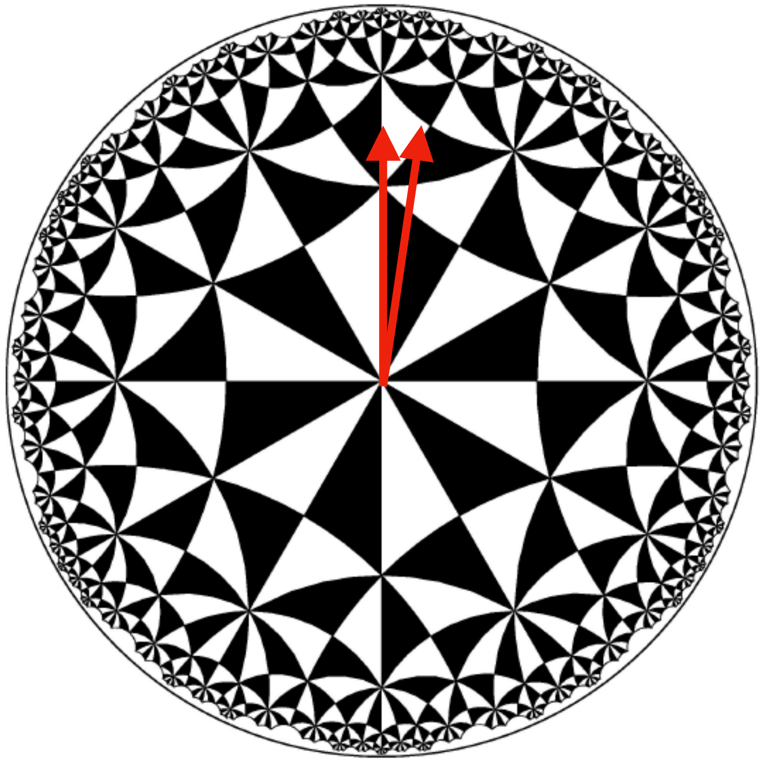
1. Geodesic length ~ at most linear

$$\cosh(L/l) = \cosh(\rho_f)\cosh(\rho_i) - \cos(\Delta\phi)\sinh(\rho_f)\sinh(\rho_i)$$

2. Observation: explicit computation (in all our examples)

$$\mathcal{F}_1 = |\langle z|\delta z\rangle| = K_{\mathcal{O}} d\phi$$

[PC, J.M.Magan '19]



Phase Space Information Geometry

F1 distance between

$(\rho = 2\alpha t, \phi = \pi/2)$ $(\rho = 2\alpha t, \phi = \pi/2 + \delta\phi)$

~"Classical Chaos"

Berry?

Example: SU(2)

[PC, J.M.Magan, D.Patramanis '21]

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad J_\pm = J_1 \pm iJ_2 \qquad [J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0$$

Liouvillian:

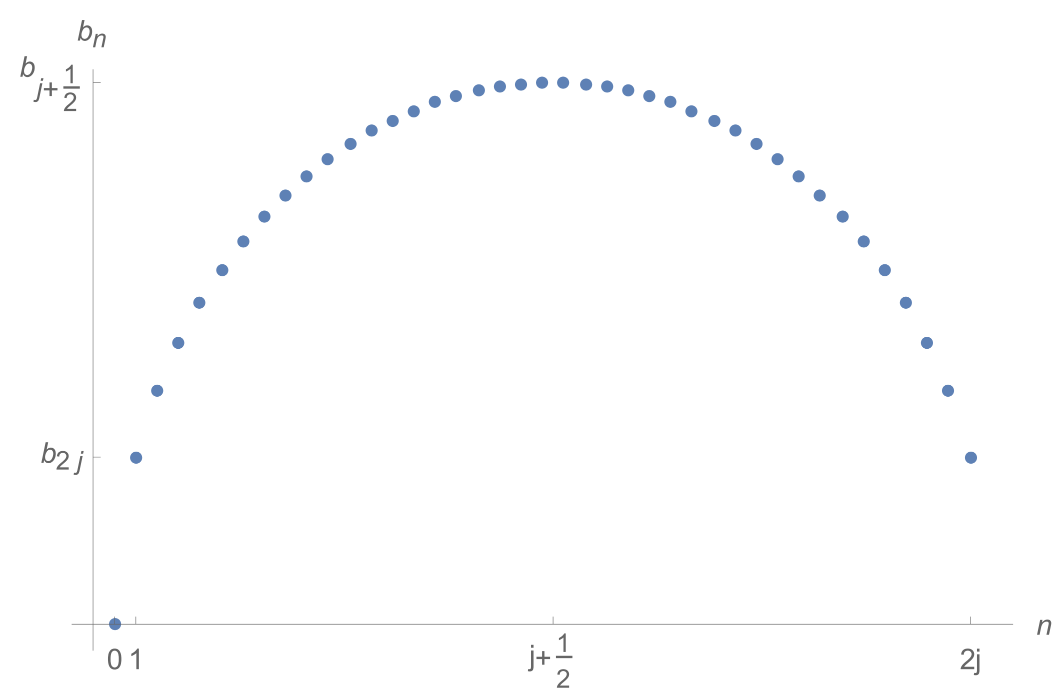
$$\mathcal{L} = \alpha(J_+ + J_-)$$

Representation:

$$\begin{aligned} J_0 \left|j, -j+n\right\rangle &= (-j+n)\left|j, -j+n\right\rangle, \\ J_+ \left|j, -j+n\right\rangle &= \sqrt{(n+1)(2j-n)}\left|j, -j+n+1\right\rangle \\ J_- \left|j, -j+n\right\rangle &= \sqrt{n(2j-n+1)}\left|j, -j+n-1\right\rangle. \end{aligned}$$

$$b_n = \alpha\sqrt{n(2j-n+1)}.$$

$$n = 0, .., 2j$$



Example: SU(2)

[PC, J.M.Magan, D.Patramanis '21]

Spin coherent states:

$$|z, j\rangle = (1 + z\bar{z})^{-j} \sum_{n=0}^{2j} z^n \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}} |j, -j+n\rangle \qquad z = \tan\left(\frac{\theta}{2}\right) e^{i\phi}$$

Trajectory: $\theta = 2\alpha t$ and $\phi = \pi/2$

$$\varphi_n(t) = \frac{\tan^n(\alpha t)}{\cos^{-2j}(\alpha t)} \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}$$

Krylov complexity:

$$K_{\mathcal{O}} = \sum_{n=0}^{2j} n|\varphi_n(t)|^2 = 2j \sin^2(\alpha t)$$

Information Geometry

$$ds^2 = \frac{2j dz d\bar{z}}{(1+|z|^2)^2} = \frac{j}{2}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad V_t = \int_0^{2\alpha t} d\theta \int_0^{2\pi} d\phi \sqrt{g} = 2\pi j \sin^2(\alpha t) = \pi K_{\mathcal{O}}$$

“Complexity Algebra”

[PC, J.M.Magan, D.Patramanis ’21]

More generally lessons from the symmetry approach

$$\mathcal{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1}) \qquad \mathcal{L} = \tilde{L}_+ + \tilde{L}_-$$

$$\mathcal{B}|\mathcal{O}_n) = -b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1}) \qquad \mathcal{B} = \tilde{L}_+ - \tilde{L}_-$$

Lets commute: From these definitions

$$\tilde{K} \equiv [\mathcal{L}, B]|\mathcal{O}_n) = 2(b_{n+1}^2 - b_n^2)|\mathcal{O}_n)$$

We can demand that the algebra closes at this first step. This gives

$$2(b_{n+1}^2 - b_n^2) = An + B \qquad b_n = \sqrt{\frac{1}{4}An(n-1) + \frac{1}{2}Bn + C}$$

What if it doesn't? Number of steps to the closure? Classification?

“Complexity Algebra”

[PC, J.M.Magan, D.Patramanis ’21]

For SL(2,R)

$$\mathcal{L} = \alpha(L_{-1} + L_1), \quad \mathcal{B} = \alpha(L_{-1} - L_1), \quad \tilde{K} = 4\alpha^2 L_0,$$

Geometrically, these are simply combinations of the isometry generators

$$ds^2 = \frac{h}{2}\left(d\rho^2 + \sinh^2(\rho) d\phi^2\right)$$

$$\begin{aligned} L_0 &= i\partial_\phi, \\ L_{-1} &= -ie^{-i\phi}\left[\coth(\rho)\partial_\phi + i\partial_\rho\right], \\ L_1 &= -ie^{i\phi}\left[\coth(\rho)\partial_\phi - i\partial_\rho\right]. \end{aligned}$$

In particular

$$\tilde{K} = 4\alpha^2(\hat{K}_{\mathcal{O}} + h) \;\sim \partial_\phi$$

Relation between complexity and Isometries (Momentum/Boost)

[Lin,Maldacena,Zhao’19]

Generalisations

[PC, J.M.Magan, D.Patramanis '21]

SL(2,R)xSL(2,R)

$$\mathcal{L} = \alpha_+ \left(L_{-1} + L_1\right) + \alpha_- \left(\bar{L}_{-1} + \bar{L}_1\right)$$

$$K_{\mathcal{O}} = \Delta \left[\sinh^2\left(\frac{\pi t}{\beta_+}\right) + \sinh^2\left(\frac{\pi t}{\beta_-}\right)\right] + s\left[\sinh^2\left(\frac{\pi t}{\beta_+}\right) - \sinh^2\left(\frac{\pi t}{\beta_-}\right)\right].$$

"Towards Virasoro": $\{L_{-k}, L_0, L_k\}$

$$\mathcal{L}_k = \alpha(L_{-k} + L_k), \qquad |\mathcal{O}_n) = |h, nk\rangle$$

$$K_{\mathcal{O}} = 2h_k \sinh^2(\alpha_k t)$$

$$b_n = k\alpha\sqrt{n(2h_k + n - 1)}.$$

$$h_k = \frac{c}{24}\left(k - \frac{1}{k} + \frac{24h}{ck}\right), \qquad \alpha_k = k\alpha$$

Non-universal: Composite operators, more general initial states

$$|\mathcal{O}_1(0)\mathcal{O}_2(t)), \qquad |[\mathcal{O}_1(0), \mathcal{O}_2(t)])$$

$$e^{-iH_L t}|\Psi_{TFD}\rangle$$

Auto-correlator becomes a 4pt function -> OTOC, Spectral Form Factors

[work in progress]

Quantum Optics for Operator Growth

[PC, J.M.Magan, D.Patramanis '21]

In quantum optics it is useful (physical) to work with two-mode representation of

$$L_{-1} = a_1^\dagger a_2^\dagger, \quad L_1 = a_1 a_2, \quad L_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \qquad |\mathcal{O}_n) = |n+k, n\rangle = \frac{(a_1^\dagger)^{n+k}}{\sqrt{(n+k)!}} \frac{(a_2^\dagger)^n}{\sqrt{n!}} |0,0\rangle$$

In this representation we can derive "density matrix of the operator" (?)

$$\rho_1^{(k)} = Tr_2\left(|z,k\rangle\langle z,k|\right) = \sum_{n=0}^{\infty} \lambda_n |n+k\rangle\langle n+k| \qquad \lambda_n = |\varphi_n(t)|^2$$

This allows to study and compare more conventional QI tools

$$S_{\mathcal{O}} = -\sum_n |\varphi_n|^2 \log(|\varphi_n|^2), \qquad S_{\mathcal{O}}^{(q)} = \frac{1}{1-q}\log\left(\sum_n |\varphi_n|^{2q}\right)$$

$$E_{\mathcal{N}}(\rho) = 2\log\left(\sum_n |\varphi_n|\right) \qquad \mathcal{C}_{\mathcal{O}} = \lim_{q\to 1} q^2 \partial_q^2 \left[(1-q) S_{\mathcal{O}}^{(q)}\right]$$

K-Entropies and negativity show a linear growth with time
Capacity saturates to 1 at late times (all sensitive to the rate α)

Conclusions and Open Problems

- Krylov Complexity is a new (good) candidate for operator complexity in QFTs
- Symmetry: New angle on the Liouvillian and Lanczos coefficients
- Geometric interpretation with many “desired” features of “complexity”
- Growth of Lanczos Coefficients? Math Proof? Bieberbach?
- Generalized Coherent States and other Lie groups? (Integrable/Chaotic?)
- Higher and lower dimensional CFT? Virasoro, Matrix Models, LLM?
- QI tools for the operator growth? Two-mode representation (ER=EPR)?
- Connection with Holography? First Law? Bulk Momentum? Near Horizon Geom?

Thank You! Stay Tuned!